Laplacian coflow of G_2-structures

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Abstract

This poster presents some results regarding the Laplacian coflow of two papers [3, 4]. In one of them, we show that the Laplacian coflow collapses (under normalized volume) on contact Calabi-Yau 7-manifolds to a lower-dimensional limit respect to the natural initial condition. Furthermore, it exhibits an infinite-time singularity of type IIB. In the other paper, we characterise the conditions for a vector field as an infinitesimal symmetry of a coclosed G_2-structure, as well as the soliton condition for the Laplacian coflow.

Introduction

A G_2-structure is defined by a positive 3-form φ, which, in turn, defines the metric g_φ and the corresponding Hodge dual 4-form $\psi := *_\varphi\varphi$. The main goal in G_2-geometric is the study of *torsion free* G_2-structures, i.e. $\nabla\varphi = 0$, which is equivalent to the *closed* $d\varphi = 0$ and the *coclosed* condition $d\psi = 0$. In [2] was introduced the *Laplacian coflow* of coclosed G_2-structures given by

$$\frac{\partial\psi(t)}{\partial t} = \Delta_t\psi(t), \quad \psi(0) = \psi. \tag{1}$$

One immediate problem with the Laplacian coflow is that the 4-form $\psi = *_\varphi\varphi$ is generated by both the 3-forms φ and $-\varphi$: in particular, ψ does not determine the orientation on M. However, it is natural to fix an orientation throughout the flow, which is determined for example by a choice of G_2-structure dual to the initial 4-form.

As for many geometric flows, we are interested in considering *self-similar solutions* $\varphi(t) = \lambda(t)f(t)^*\varphi$ where $\lambda(t) \in C^\infty(M)$ and $f(t) \in \mathrm{Diff}(M)$, it means, solutions that evolves the initial data φ by diffeomorphisms and scalings, since these kind of solutions are expected to be related to singularities of the flow.

Main results

Soliton solution of Laplacian coflow

In [4], using the following Proposition which decomposes the Hodge Laplacian of ψ according to the G_2-irreducible decomposition of Ω^4.

Proposition 1. *[1] Let φ be a coclosed G_2-structure on a manifold M with associated metric g. Then,*

$$\Delta_\psi\psi = \frac{2}{7}((\operatorname{tr}T)^2 + |T|^2)\psi \oplus (d\operatorname{tr}T)\wedge\varphi$$
$$\oplus *_\varphi i_\varphi\Big(\operatorname{Ric} - \frac{1}{2}T\circ T - (\operatorname{tr}T)T + \frac{1}{14}\left((\operatorname{tr}T)^2 + |T|^2\right)g\Big).$$

and computing the decomposition of the Lie derivative with respect to any vector field.

Proposition 2. *Let φ be a coclosed G_2-structure on M^7, with associated metric g, and let X be a vector field on M. Then, if $\psi = *\varphi$,*

$$\mathcal{L}_X\psi = \frac{4}{7}(\operatorname{div}X)\psi \oplus (-\frac{1}{2}\operatorname{curl}X + X\lrcorner T)^\flat \wedge\varphi$$
$$\oplus *i_\varphi\Big(\frac{1}{7}(\operatorname{div}X)g - \frac{1}{2}(\mathcal{L}_X g)\Big) \in \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}.$$

In particular, X is an infinitesimal symmetry of ψ if and only if X is a Killing vector field of g and satisfies $\operatorname{curl}(X) = 2X\lrcorner T$.

We recall that the vector field X is called an *infinitesimal symmetry* of ψ, if $\mathcal{L}_X\psi = 0$. Then, we have the following proposition.

Proposition 3. *Let φ be a coclosed G_2-structure on M with associated metric g. If (φ, X, λ) is a soliton of the Laplacian coflow, then its full torsion tensor T satisfies*

$$\operatorname{div}T = -\frac{1}{2}(\operatorname{curl}X)^\flat + X\lrcorner T,$$
$$-\operatorname{Ric} + \frac{1}{2}T\circ T + (\operatorname{tr}T)T = \frac{\lambda}{4}g + \frac{1}{2}\mathcal{L}_X g. \tag{2}$$

Laplacian coflow on cCY manifold

In [3], we will consider the flows described above on a *contact Calabi–Yau (cCY)* 7-manifold $(M^7, g_0, \eta_0, \Upsilon_0)$, where (M^7, g_0) is a Sasakian 7-manifold with Riemannian metric g_0, contact form η_0 and transverse Kähler form $\omega_0 = d\eta_0 \in \Omega^{1,1}(M)$, and $\Upsilon_0 \in \Omega^{3,0}(M)$ is a transverse holomorphic volume form; here (p, q) denotes basic bidegree with respect to the horizontal distribution $\mathcal{D}_0 = \ker\eta_0$.

On a cCY 7-manifold there exists a natural 1-parameter family of coclosed G_2-structures defined, for each $\epsilon > 0$, by

$$\varphi_0 = \epsilon\eta_0\wedge\omega_0 + Re\Upsilon_0, \tag{3}$$

with induced metric g_{φ_0} and corresponding dual 4-form

$$\psi_0 = *_{\varphi_0}\varphi_0 = \frac{1}{2}\omega_0^2 - \epsilon\eta_0\wedge Im\Upsilon_0. \tag{4}$$

Theorem 1 (Laplacian coflow on contact Calabi–Yau 7-manifolds)**.** *The Laplacian coflow* (1) *on M^7, with initial data determined by φ_0, is solved by the following family of coclosed G_2-structures φ_t, with associated metric g_t, volume form vol_t and dual 4-form ψ_t:*

$$\varphi_t = \epsilon p(t)^{-1}\eta_0\wedge\omega_0 + p(t)^3 Re\Upsilon_0;$$
$$\psi_t = \frac{1}{2}p(t)^4\omega_0^2 - \epsilon\eta_0\wedge Im\Upsilon_0;$$
$$g_t = \epsilon^2 p(t)^{-6}\eta_0^2 + p(t)^2 g_{\mathcal{D}_0};$$
$$\mathrm{vol}_t = \epsilon p(t)^3\eta_0\wedge\mathrm{vol}_{\mathcal{D}_0},$$

where $p(t) = (1 + 10\epsilon^2 t)^{1/10}$ and $t \in (-\frac{1}{10\epsilon^2}, \infty)$. Hence, the solution of the Laplacian coflow is immortal, with a finite time singularity (backwards in time) at $t = -\frac{1}{10\epsilon^2}$.

Let Rm_t denote the Riemann curvature tensor of g_t and let $Rm_0^{\mathcal{D}_0}$ denote the curvature of the transverse connection on $\mathcal{D}_0$ induced by the Levi-Civita connection of g_0. Then

$$|Rm_t|^2_{g_t} = (1 + 10\epsilon^2 t)^{-2/5}|Rm_0^{\mathcal{D}_0}|^2_{g_0} + c_0\epsilon^4(1 + 10\epsilon^2 t)^{-2}$$

for some constant $c_0 > 0$ and the associated metric g_t is uniformly continuous (in t) on any compact interval contained in $(-\frac{1}{10\epsilon^2}, \infty)$, but it is not uniformly continuous on $(-\frac{1}{10\epsilon^2}, S)$ or (S, ∞) for any S.

Theorem 2 (Singularities of the Laplacian flow and coflow)**.** *Let M^7 be compact contact calabi-Yau manifold. Then, the Laplacian coflow solution in the above Theorem has an infinite-time Type IIb singularity, unless the transverse metric on $\mathcal{D}_0$ is flat, in which case it has an infinite-time Type III singularity.*

The *full torsion tensor* T is defined locally by the formula $\nabla_i\varphi_{jkl} = T_i^m\psi_{mjkl}$. Therefore the full torsion tensor T_t of the solution to the Laplacian coflow

$$T_t == -\frac{3}{2}\epsilon^3(1 + 10\epsilon^2 t)^{-11/10}\eta_0^2 + \frac{1}{2}\epsilon(1 + 10\epsilon^2 t)^{-3/10}g_{\mathcal{D}_0}.$$

Then,

$$|T_t|^2_{g_t} = \frac{15}{4}\epsilon^2(1 + 10\epsilon^2 t)^{-1}, \quad |\nabla_t T_t|^2_{g_t} = c_0\epsilon^4(1 + 10\epsilon^2 t)^{-2}$$

Question: Are there any flows on cCY 7-manifolds that converge to a torsion-free G_2-structure?

References

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