On minimal coverings and pairwise generation of some primitive groups

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1 Introduction

The covering number of a finite noncyclic group G, denoted $\sigma(G)$, is the smallest positive integer k such that G is a union of k proper subgroups. If G is 2-generated, the generating graph of G is the simple graph whose vertices are the elements of G and two vertices are connected by an edge if together they generate G. A clique of a simple graph is a complete subgraph and its clique number is the maximal size of a clique. We denote by $\omega(G)$ the clique number of the generating graph of G, in other words $\omega(G)$ is the maximal size of a subset S of G with the property that $\langle x, y\rangle = G$ whenever $x, y \in S$ and $x \neq y$. Since any proper subgroup of G can contain at most one element of such a set S, we have $\omega(G) \leqslant \sigma(G)$. It is very natural to ask whether equality occurs for some families of groups, at least asymptotically.

An example of equality is:

$$\sigma(S_n) = \omega(S_n) = 2^{n-1},$$

for all odd $n \geqslant 17$ and $n \in \{7, 11, 13\}$.

2 Main Results

Let $G = G_{n,m}$ be the semidirect product $A_n^m \rtimes \langle\gamma\rangle$ where $\gamma = (1, \ldots, 1, \tau)\delta \in S_n \wr S_m$, with $\tau = (1\ 2)$ and $\delta = (1 \ldots m)$. If $x_1, \ldots, x_m \in A_n$, we have $(x_1, \ldots, x_m)^\gamma = (x_m{}^\tau, x_1, \ldots, x_{m-1})$.

We establish the following result, generalizing the main result of [3] about $\sigma(S_n)$, which corresponds to the case $m = 1$.

Theorem 1 (J. Almeida, M. Garonzi)**.** *Let $G = G_{n,m}$, for $n \geqslant 30$ divisible by 6 and $m \geqslant 2$. Denote by $\alpha(x)$ the number of distinct prime factors of the positive integer x. Then*

$$\sigma(G) = \alpha(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{n/3-1}\binom{n}{i}^m.$$

Moreover, G has a unique minimal covering consisting of maximal subgroups.

We also prove that:

Theorem 2 (J. Almeida, M. Garonzi)**.** *Set $G := G_{n,m}$. For fixed $m \geqslant 2$, $\omega(G)$ is asymptotically equal to*

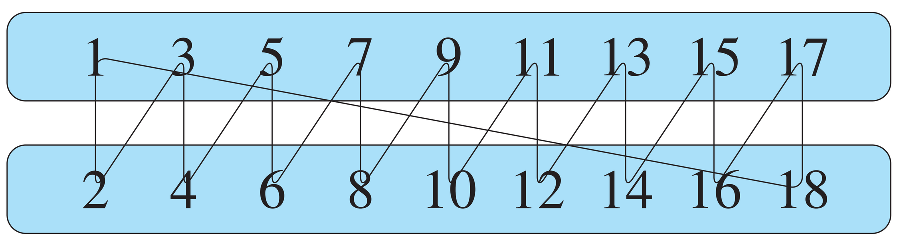
$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m$$

for $n \to \infty$, n even, and $\omega(G)/\sigma(G)$ tends to 1 as $n \to \infty$, n even.

3 Strategy of the proof

3.1 The covering when $m = 1$

In this case $G = S_n$. If $n \geqslant 30$ and n is divisible by 6, the collection $\mathcal{C}_n$ that consist of all maximal subgroups of S_n isomorphic to one of the following: $S_{n/2} \wr S_2$, A_n, or $S_i \times S_{n-i}$, $i = 1, \ldots, n/3 - 1$, is a cover of the elements of S_n. To prove this, consider an element $g \in S_n$. If g is an n-cycle, then g preserves a decomposition of $\{1, \ldots, n\}$ into two sets of size $n/2$, and hence g is contained in a subgroup isomorphic to $S_{n/2} \wr S_2$.



If g has cycle structure $(j, n-j)$ for some $1 \leqslant j \leqslant n/2$, then g is contained in A_n. If g fixes any element in $\{1, \ldots, n\}$, then g is contained in a subgroup isomorphic to S_{n-1}. If the cycle structure of g contains an i-cycle, where $2 \leqslant i \leqslant n/3 - 1$, then g is contained in a subgroup isomorphic to $S_i \times S_{n-i}$. Finally, if g has cycle structure $(n/3, n/3, n/3)$, then since $n/3$ is even, g stabilizes a decomposition of $\{1, \ldots, n\}$ into two sets of size $n/2$ and is contained in a subgroup isomorphic $S_{n/2} \wr S_2$.

The size of collection $\mathcal{C}_n$ is: $1 + \frac{1}{2}\binom{n}{n/2} + \sum_{i=1}^{n/3-1}\binom{n}{i}$.

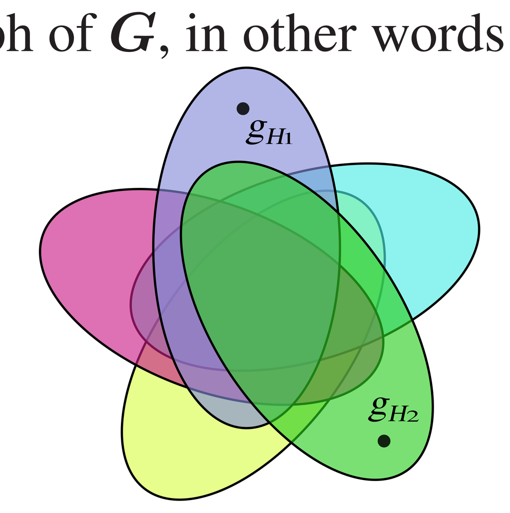
3.2 The covering when $m = 2$

In this case $G = A_n^2 \rtimes \langle\gamma\rangle$. If $n \geqslant 30$ and n is divisible by 6, the collection $\mathcal{C}$ that consist of all maximal subgroups of G isomorphic to one of the following: $H = N_G(M \times M^a)$, where $a \in A_n$ and $N_{S_n}(M) \cong S_{n/2} \wr S_2$ or $N_{S_n}(M) \cong S_i \times S_{n-i}$, $i = 1, \ldots, n/3 - 1$, or $H = A_n^2 \rtimes \langle\gamma^2\rangle$, is a cover of the elements of G. To prove this, consider an element $g = (x, y)\gamma^k \in G$ where $x, y \in A_n$, $k \in \{0, 1, 2, 3\}$. If $k = 0$ or 2, then g belongs to $A_n^2 \rtimes \langle\gamma^2\rangle$. Since $\langle g\rangle = \langle g^{-1}\rangle$, we can assume that $k = 1$. Since $\mathcal{C}_n$ is a covering of S_n, there exists $M \in \mathcal{C}_n$ such that the odd permutation $xy\tau$ belongs to $N_{S_n}(M)$ and $H = N_G(M \times M^x)$ is a member of $\mathcal{C}$ containing g.

If $H = N_G(M \times M^a)$, H is a maximal subgroup of G supplementing the socle $N = A_n^2$ of G, and $H \cap N$ is conjugate to M^2 in N. It follows that $|H| = |G/N||H \cap N| = 4 \cdot |M|^2$ and H has $|G : H| = |A_n : M|^2$ conjugates in G. Therefore, the size of $\mathcal{C}$ equals $1 + \left(\frac{1}{2}\binom{n}{n/2}\right)^2 + \sum_{i=1}^{n/3-1}\binom{n}{i}^2$.

3.3 Generation when $m = 2$

In this case $G = A_n^2 \rtimes \langle\gamma\rangle$. If n is even, we define $\mathcal{N} = \{N_G(M \times M^a) : M \in \mathcal{F}\}$, where $\mathcal{F}$ is the family of maximal imprimitive subgroups of A_n with 2 blocks, $(S_{n/2} \wr S_2) \cap A_n$, and $a \in A_n$. Let B be the set of n-cycles in S_n and let Π be the set of elements of G of the form $(x, y)\gamma$ with the property that $xy\tau \in B$. For $H \in \mathcal{N}$ define $C(H) = \Pi \cap H$. Using the Lovász Local Lemma, proved by Erdös and Lovász, which is a probabilistic result, we show that there exists a choice of g_H in each $C(H)$, $H \in \mathcal{N}$, with the property that $\langle g_{H_1}, g_{H_2}\rangle = G$ for all $H_1 \neq H_2$ in $\mathcal{N}$, therefore these elements form a clique of the generating graph of G, in other words $\omega(G) \geqslant |\mathcal{N}|$.



Referências

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