Stabilization of the Kawahara-Kadomtsev-Petviashvili equation with time-delayed feedback

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Setting of the problem

One of the main interest in control theory is to understand how some feedback mechanism acts in the asymptotic behavior for water waves systems which are modeled by PDEs.

Here, we deal with the stabilization problem for two dispersive systems with localized damping and delay terms posed on a bounded domain $\Omega = (0,L)\times(0,L)\subset\mathbb{R}^2$:

1. The Kawahara-Kadomtsev-Petviashvili (K-KP-II) equation deduced in [3, 4]

$$\begin{aligned}\partial_t u + \alpha\partial_x^3 u + \beta\partial_x^5 u + \gamma\partial_x^{-1}\partial_y^2 u + \frac{1}{2}\partial_x(u^2)\\ + a(x,y)u + b(x,y)u(x,y,t-h) = 0\end{aligned} \tag{1}$$

2. Motivated by [1] we study the so-called μ_i-system

$$\begin{aligned}\partial_t u + \alpha\partial_x^3 u + \beta\partial_x^5 u + \gamma\partial_x^{-1}\partial_y^2 u + \frac{1}{2}\partial_x(u^2)\\ + a(x,y)\left(\mu_1 u + \mu_2 u(x,y,t-h)\right) = 0\end{aligned} \tag{2}$$

where $\mu_1 > \mu_2$ are positive real numbers.

Here, $h > 0$ is the time delay, $\alpha > 0, \gamma > 0$ and $\beta < 0$ are real constants. Additionally, define the operator ∂_x^{-1} as follows $\partial_x^{-1}\varphi(x,y,t) = \psi(x,y,t)$ such that $\psi(L,y,t) = 0$ and $\partial_x\psi(x,y,t) = \varphi(x,y,t)$ and let us consider the following assumption

Assumption 1. *Consider $a, b \in L^\infty(\Omega)$ non-negative real functions. Moreover, $a(x,y)\ge a_0 > 0$ a.e. in a nonempty open subset $\omega\subset\Omega$.*

Both systems (1) and (2) are equipped with boundary conditions

$$\begin{cases} u(0,y,t) = u(L,y,t) = 0, & y\in(0,L),\\ \partial_x u(L,y,t) = \partial_x u(0,y,t) = 0, & y\in(0,L),\\ \partial_x^2 u(L,y,t) = 0, & y\in(0,L),\\ u(x,L,t) = u(x,0,t) = 0, & x\in(0,L)\end{cases} \tag{3}$$

and initial data

$$\begin{cases} u(x,y,0) = u_0(x,y), \\ u(x,y,t) = z_0(x,y,t), \quad t\in(-h,0).\end{cases} \tag{4}$$

The energy associated to (1) and (2) with boundary conditions (3) are given respectively by

$$\begin{aligned}E_u(t) &= \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy\\ &+ \frac{h}{2}\int_0^L\int_0^L\int_0^1 b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy\end{aligned} \tag{5}$$

and

$$\begin{aligned}E_u(t) &= \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy\\ &+ \frac{\xi}{2}\int_0^L\int_0^L\int_0^1 a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy\end{aligned} \tag{6}$$

where $\xi > 0$ satisfies

$$h\mu_2 < \xi < h(2\mu_1 - \mu_2). \tag{7}$$

We are mainly concerned to solve the next question: Does $E_u(t)\to 0$ as $t\to\infty$? If it is the case, can we give the decay rate?

Stabilization results

Throughout this section, we assume that Assumption 1 is satisfied.

Theorem 1 (Optimal local stabilization). *Let $L > 0$, $\xi > 1$, $0 < \mu < 1$ and T_0 given by*

$$T_0 = \frac{1}{2\theta}\ln\left(\frac{2\xi\kappa}{\mu}\right) + 1, \tag{8}$$

with $\theta = \frac{3\alpha\eta}{(1+2\eta L)L^2}$, $\kappa = 1 + \max\left\{2\eta L, \frac{\sigma}{\xi}\right\}$ and $\eta\in\left(0, \frac{\xi - 1}{2L(1+2\xi)}\right)$ satisfying

$$\frac{2\alpha\eta}{(2+2\eta L)L^2} = \frac{\sigma}{2h(\xi+\sigma)}$$

where $\sigma = \xi - 1 - 2L\eta(1+2\xi)$. Let $T_{\min} > 0$ given by

$$T_{\min} := -\frac{1}{\nu}\ln\left(\frac{\mu}{2}\right) + \left(\frac{2\|b\|_\infty}{\nu} + 1\right)T_0,$$

with $\nu = \frac{1}{T_0}\ln\left(\frac{1}{(\mu+\varepsilon)}\right)$. Then, there exists $\delta > 0$, $r > 0$, $C > 0$ and γ, depending on $T_{\min}, \xi, L, h$, such that if $\|b\|_\infty \le \delta$, then for every $(u_0, z_0)\in\mathcal{H} = L^2(\Omega)\times L^2(\Omega\times(0,1))$ satisfying $\|(u_0,z_0)\|_{\mathcal{H}}\le r$, the energy (5) satisfies

$$E_u(t) \le Ce^{-\gamma t}E_u(0), \text{ for all } t > T_{\min}.$$

Theorem 2 (Local stabilization). *Let $L > 0$. Assume that (7) holds and $\beta < -\frac{1}{30}$. Then, there exists*

$$0 < r < \frac{\sqrt[4]{216\alpha^3}}{CL^{\frac{5}{2}}}$$

such that for every $(u_0, z_0(\cdot,\cdot,-h(\cdot)))\in\mathcal{H}$ satisfying $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}\le r$, the energy defined in (6) decays exponentially. More precisely, there exists two positives constants θ and κ such that $E_u(t)\le\kappa E_u(0)e^{-2\theta t}$ for all $t > 0$. Here,

$$\theta < \min\left\{\frac{\eta}{(1+2\eta L)L^2}\left[3\alpha - \frac{1}{2}C^{\frac{4}{3}}r^{\frac{4}{3}}L^{\frac{10}{3}}\right], \frac{\xi\sigma}{2h(\xi+\sigma\xi)}\right\},$$
$$\kappa = 1 + \max\{2\eta L, \sigma\}$$

and η and σ are positive constants such that

$$\sigma < \frac{2h}{\xi}\left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2h}\right)$$
$$\eta < \min\left\{\frac{1}{2L\mu_2}\left[\frac{\xi}{h} - \mu_2\right], \frac{\Lambda}{2L\mu_1 + L\mu_2}\right\}$$

where

$$\Lambda = \left[\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2h}(1+\sigma)\right].$$

Theorem 3 (Global stabilization). *Suppose that $\mu_1 > \mu_2$ satisfies (7). Let $R > 0$, then there exists $C = C(R) > 0$ and $\nu = \nu(R) > 0$ such that E_u, defined in (6) decays exponentially as t tends to infinity, when $\|(u_0, z_0)\|_{\mathcal{H}}\le R$.*

Brief resume of the proofs

- The well-posedness of the systems (1) and (2) are obtained using semigroup theory. Specifically, since the energy is not decreasing we use a perturbation argument to be able to use the Lummer-Phillips theorem and then a classical application of the Banach fixed point theorem guarantees the result. On the other hand, the regularity of the solutions follows from a standard application of the Morawetz multipliers.
- Theorems 1 and 2 are obtained choosing a suitable functional that is equivalent to the energy and therefore applying the Gronwall's inequality. We point up that the result is first obtained for $0 < t < T$ and then extended for every $t > 0$ using a boot-strap and induction arguments.
- Theorem 3 follows from the classical *compactness uniqueness* argument wich reduces our problem to prove an *observability inequality* and removes the hypotheses that the initial data are small enough.

Main remarks

- We highlight two important aspects for the Lyapunov's method:
  1. Due to nature of the nonlinearity we are able to apply directly for the nonlinear system
  2. It is possible to give an explicit (and optimal) decay rate, however, the initial data needs to be sufficiently small.
- In comparison with the one-dimensional version analyzed in [1], the absence of the drift term u_x allow us to get stabilization results without restriction on the length of the spatial domain.
- With a slightly different estimate we can obtain another result for exponential stability without restriction in the parameter β but with restriction in the length L of the domain.

  Theorem 4 (Local stabilization-*bis*). *Let $0 < L < \sqrt[4]{\frac{-30\beta}{C}}$. Assume that $a(x,y)\in L^\infty(\Omega)$ is a non-negative function and that the relation (7) holds. Then, there exists $0 < r < \frac{\sqrt[4]{216\alpha^3}}{CL^{\frac{5}{2}}}$ such that for every $(u_0, z_0(\cdot,\cdot,-h(\cdot)))\in\mathcal{H}$ satisfying $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}\le r$, the energy defined in (6) decays exponentially. More precisely, there exists two positives constants θ and κ such that $E_u(t)\le\kappa E_u(0)e^{-2\theta t}$ for all $t > 0$, where θ, κ, η and σ are positive constants defined as in Theorem 2.*
- It is possible to take a time-varying delay $u(x,y,t-h(t))$ for $h(t)$ a suitable real function and obtain asymptotic behavior results for $0 < t < T$ using the Lyapunov's approach, however, the extension to $t > 0$ is still an open problem.

This work contain recent results presented in [2] and is part of the Ph.D. thesis of Muñoz at the Department of Mathematics of the Universidade Federal de Pernambuco.

References

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