Moduli space of principal G-bundles

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Abstract

In this poster we explain how to construct a weakly normal projective moduli spaces of torsion free sheaves and of singular principal G—bundles over a complex projective manifold X. We restrict our problem to families over weakly normal parameter spaces.

- We will use slope semistability for bounding both moduli problems. We calculate the slope w.r.t. **a multipolarization $(H_1, H_2, ...H_{n-1})$ where each H_i is an integral ample divisor class**.
- The existence of a semi ample line bundle on the respective **Quot** scheme will give us the existence of each moduli space, see [1].
- We will describe the points of the moduli space using filtered objects and the "Quot to Chow"morphism.

Introduction: Moduli via GIT adjacent methods

The construction of many moduli spaces follows the same general pattern.

1. Fix some **discrete invariants** which are compatible with the equivalence relation.
2. Restrict to a reasonably sized class of objects. Usually one restricts to a class of **semistable objets**.
3. Find a family $\mathscr{F}$ over a scheme P which has **the local universal property**, i.e. locally any family parametrized by a scheme B is isomorphic to a pullback of the family $\mathscr{F}$.
4. Find a group G acting on P such that for any $p, q \in P$, that lie in the same G-orbit if and only if there is an isomorphism $\mathscr{F}_p \cong \mathscr{F}_q$. Hence we have a **bijection $\{$Orbits of $P\} \cong \{$equivalence classes of objects$\}$** . This is not always possible! See **S-equivalence or Uhlenbeck compactification**.
5. This group action should be algebraic and reductive. **Taking the quotient produces the moduli space**, this should be taken in the category of schemes. This is achieved by using **Mumford 's Geometric Invariant Theory** .
6. When 4. is not possible one has to linearize the action with an ample line bundle instead of a very ample one (chapter 4 [2]).

Moduli of torsion free sheaves

Now let B be a scheme of finite type over $\mathbb{C}$ and $\mathscr{E}$ a B—flat family of coherent sheaves on X with class $c \in K(X)_{\text{num}}$ and fixed determinant bundle Λ. Let the following composition

$$K(X) \xrightarrow{p^*} K^0(B \times X) \xrightarrow{\otimes[\mathscr{E}]} K^0(B \times X) \xrightarrow{q_*} \mathrm{Pic}(B)$$

be denoted by $\lambda_{\mathscr{E}}$ [2], where $B \xleftarrow{q} X \times B \xrightarrow{p} X$. In particular for each $c \in K(X)_{num}$ we let $\lambda_{\mathscr{E}}(c) \in \mathrm{Pic}(B)$ be the evaluation of the above composition in c.

Let $R^{\mu ss} \subset \mathrm{Quot}(\mathscr{H}, P)$ be the locally closed subscheme of all quotients $[q: \mathscr{H} \to F]$ (3.11[1]). Let $S := (R^{\mu ss}_{\text{red}})^{\text{wn}}$.

Let $\mathscr{O}_S \otimes \mathscr{H} \to \mathscr{F}$ be the pullback of the **universal quotient** from $\mathrm{Quot}(\mathscr{H}, P(m))$ to S via the inclusion. Choosing a fixed base point $x \in X$ and a class $c \in K(X)_{num}$, consider the class $u_{n-1}(c) = -rh_{n-1} \cdot ... \cdot h_1 + \chi(c \cdot h_{n-1} \cdot ... \cdot h_1)[\mathscr{O}_x]$ in $K(X)$, where h_i is the class of $\mathscr{O}_{H_i}$ in $K(X)$, then define the line bundle

$$\mathscr{L}_{n-1} = \lambda_{\mathscr{F}}(u_{n-1}(c)) \in \mathrm{Pic}(S).$$

(**Equivariant semiampleness**)[1] There exists a positive integer $\nu \in \mathbb{N}$ such that $\mathscr{L}_{n-1}^{\otimes \nu}$ **is generated** by $\mathrm{SL}(V)$—invariant sections.

(**Finite generation**)[1] Hence we can build the moduli space in the following way: There exists a natural number $N \geq 1$ such that the graded ring $\oplus_{k \geq 0} H^0(S, \mathscr{L}_{n-1}^{\otimes kN})$ is generated over $k = 0$ by finitely many elements of degree one. Then we define the moduli space by

$$M^{\mu ss} = \mathrm{Proj}(\oplus_{k \geq 0} H^0(S, \mathscr{L}_{n-1}^{\otimes Nk})^{\mathrm{SL}(V)}).$$

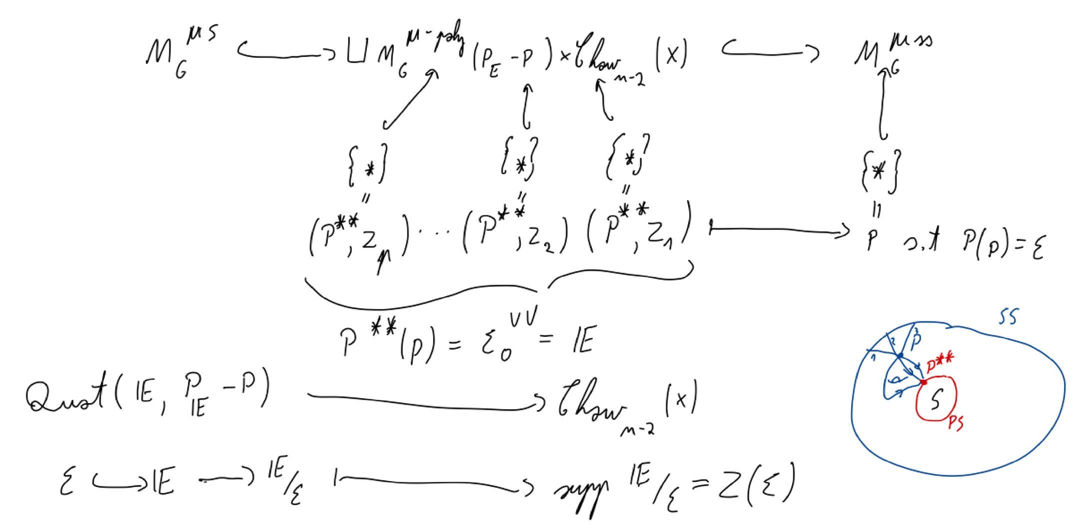
together with the ample line bundle $L := \mathscr{O}_{M^{\mu ss}}(1)$. Moreover, we let $\phi: S \to M^{\mu ss}$ be the induced invariant morphism with $\phi^*(L) = \mathscr{L}_{n-1}^{\otimes N}$.

(**Graded object**) Let X be a projective n-dimensional manifold, and let $(H_1, ..., H_n)$ be a multipolarization on X. For a semistable sheaf $\mathscr{E}$ on X, there exists a Jordan-Hölder filtration with torsion-free factors. Associated to the graded object of the filtration we let $\mathscr{E}^\sharp = (\mathrm{gr}^\bullet(\mathscr{E}))^{\vee\vee}$. The reflexive polystable sheaf $\mathscr{E}^\sharp$ depends only on $\mathscr{E}$ and not on the filtration.

(**Quot to Chow**) Consider the natural map $\iota: gr^\bullet(\mathscr{E}) \to \mathscr{E}^\sharp$. Since $gr^\bullet(\mathscr{E})$ is torsion-free and ι is injective, the quotient $\mathscr{E}^\sharp / \mathrm{gr}^\bullet(\mathscr{E})$ is supported in codimension at least two. We define the "Quot to Chow"morphism, denoted by $C_{\mathscr{E}}$, as the morphism that assigns to each $\mathscr{E}$ the support Chow cycle of the quotient $\mathscr{E}^\sharp / gr^\bullet(\mathscr{E})$.

(**Separating semistable sheaves in the moduli space**)[1] Let F_1 and F_2 be two $(H_1, \ldots, H_{n-1})$-semistable sheaves on the projective manifold X such that $F_1^\sharp \neq F_2^\sharp$ or $C_{F_1} \neq C_{F_2}$. Then, F_1, F_2 give rise to distinct points in $M^{\mu ss}$.

(**Non-separation**)[1] Let F_1, F_2 be two $(H_1, \ldots, H_{n-1})$-semistable sheaves with the same Hilbert polynomial P on X such that $F_1^\sharp \cong F_2^\sharp =: F$, and $C_{F_1} = C_{F_2} =: C$. Suppose in addition that the isomorphism classes of the sheaves $F/\mathrm{gr}^\mu F_1$ and $F/\mathrm{gr}^\mu F_2$ lie in the same connected component of the fibre over C of the canonical morphism from the seminormalisation of the Quot scheme $\mathrm{Quot}\,(F, P_F - P)$ to the Chow variety $\mathrm{Chow}_{n-2}(X)$ of cycles of codimension two on X. Then, F_1 and F_2 give rise to the same point in $M^{\mu ss}$.



Principal G-bundles

(**Singular G-principal bundles**)[3] Let G be a fixed simple algebraic group and $\rho: G \to \mathrm{SL}(V)$ be a fixed faithful representation. A singular principal G—bundle on X is a pair $(\mathscr{E}, \sigma)$ where $\mathscr{E}$ is a torsion free sheaf with generic fiber of type V on X, σ is a reduction of the structure group given by a non constant section $\sigma: X \to S_{\mathscr{E}} // G = \underline{\mathrm{Spec}}\,\mathrm{Sym}^*(\mathscr{E} \otimes \mathcal{O}(V))^G$ where G acts on $S_{\mathscr{E}}$ via the representation ρ. The geometric realization of such an object is the fibration obtained via the pullback diagram:

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & S_{\mathscr{E}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & S_{\mathscr{E}}//G. \end{array}$$

0.1 The construction of the moduli

We remark that a reduction datum σ can be equivalently viewed as giving an $\mathscr{O}_X$—algebra morphism $\tau: (\mathrm{Sym}^*(\mathscr{E} \otimes V))^G \to \mathscr{O}_X$. The map τ is obtained by dualizing the map σ.

Let $R_G(\rho)$ **be the scheme which parametrizes pairs (q, τ)** where $q \in S$ and τ is a reduction datum giving a singular principal bundle and let $R := R_G(\rho)^{\text{wn}}_{\text{red}}$. The existence of total families for singular principal bundles follows by the general theory of Hilbert schemes and is shown in [3], Section 6.7.

By the **universal property** of the scheme $R_G(\rho)$ it follows that there exists a tautological family on $X \times R_G(\rho)$. Let $\mathscr{P}$ denote this $R_G(\rho)$–flat **family of G–singular principal bundles** on X.

Consider the $R_G(\rho)$–flat family of torsion free sheaves $\mathscr{P}(\rho) = \mathscr{F}'$ on $X \times R_G(\rho)$. There is a G–action on $R_G(\rho)$ such that the family $\mathscr{F}'$ carries a linearization with respect to the G–action.

Fixing the class $c \in K(X)_{num}$ as seen above. We then have the determinant line bundle

$$\mathscr{L}'_{n-1} = \lambda_{\mathscr{F}'}((u_{n-1}(c)))$$

on $R_G(\rho)$ induced by the family $\mathscr{F}'$.

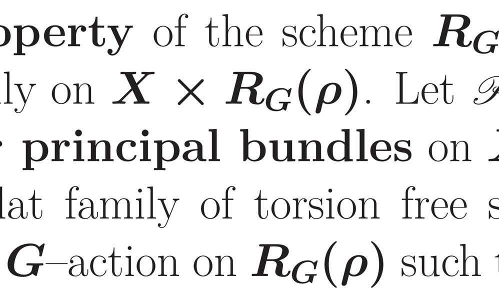
(**Semi Ampleness**) There exists an integer $\nu > 0$ such that the line bundle $\mathscr{L}'^{\otimes \nu}_{n-1}$ on R is **generated by G–invariant global sections** i.e., $\mathscr{L}'_{n-1}$ is G–semi–ample.

(**Equivariant semiampleness II**) There exists a natural number $M \geq 1$ such that the graded ring $\oplus_{k \geq 0} H^0(R_G(\rho), \mathscr{L}'^{\otimes kM}_{n-1})$ is generated over $k = 0$ by finitely many elements of degree one. Then we define the moduli space by

$$M_G^{\mu ss}(\rho) = \mathrm{Proj}(\oplus_{k \geq 0} H^0(R_G(\rho), \mathscr{L}'^{\otimes Mk}_{n-1})^{\mathrm{SL}(V)}).$$

together with the ample line bundle $L' := \mathscr{O}_{M_G^{\mu ss}}(1)$. Moreover, we let $\phi': R \to M_G^{\mu ss}(\rho)$ be the induced $\mathrm{SL}(V)$-invariant morphism with $\phi'^*(L') = \mathscr{L}'^{\otimes M}_{n-1}$.

(**Langton for G-bundles**)[4] Let P_K be a family of **semistable principal G—singular bundles** on $X \times \mathrm{Spec}(K)$, or equivalently, let P_K be a G_K—principal singular bundle on X_K. Then there exists a finite extension L/K, with integral closure B of R in L, such that P_K, after base change to $\mathrm{Spec}(B)$, **extends the G_K bundle** to a **G_R—singular principal bundles**.



M_G as a compactification

There exists a natural morphism

$$\phi: (M_G^{\mu s}(\rho)^\circ)^{wn} \to M_G^{\mu ss}(\rho)$$

that embeds the weak normalization of the moduli space of stable principal G-bundles as a Zariski open subset of $M_G^{\mu ss}(\rho)$.

We conclude that $M_G^{\mu ss}(\rho)$ contains the weak normalization of the moduli space of stable principal G-bundles with the previously specified invariants as a Zariski-open set. In particular, it compactifies the moduli.

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