Quantitative convergence of Games of Moderate Interaction

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Introduction

We consider the setting studied in [2] of N-player games, in which the following SDE gives the motion of the i-th player

$$dX_t^{i,N} = \Big(\alpha^{i,N}(t) + b\big(X_t^{i,N}, \frac{1}{N}\sum_{j=1}^{N} V^N(X_t^{i,N} - X_t^{j,N})\big)\Big)\,dt + dW_t^{i,N},\ \ t\in[0,T],$$

where b is a deterministic field, $W^{i,N}$ are independent Wiener processes and V^N is a reference function to be defined in the next section. In addition, $X_0^{i,N}$, $i = 1, ..., N$, are $\mathbb{R}^d$ -valued (i.i.d) random variables, independent of the Wiener processes, such that $X_0^{i,N}$ is distributed as a random variable with law μ_0.

Also, $\alpha^{i,N}$ is a strategy vector for each player i acts to minimize the expected total cost, given by the following functional

$$J_i^N(\alpha^N) \doteq \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\left|\alpha^{i,N}(s)\right|^2 + f\Big(X_s^{i,N}, \frac{1}{N}\sum_{j=1}^{N} V^N(X_s^{i,N} - X_s^{j,N})\Big)\right) ds + g(X_T^{i,N})\right]$$

where X_t^N is the solution of (1) under $\alpha^N = (\alpha^{i,N})_{i=1,...,N}$.

Hypothesis and Motivation

We assume the following.

(H1) b and f are bounded Borel measurable functions, continuous and such that there exist $L > 0$ for which it holds that

$$|b(x,p) - b(y,q)| + |f(x,p) - f(y,q)| \le L(|x-y| + |p-q|)$$

for all $x, y \in \mathbb{R}^d, p, q \in \mathbb{R}_+$.

(H2) g is a Borel measurable function s.t. $g, \partial_{x_i} g \in C_b(\mathbb{R}^d)$, $i = 1, ..., d$.

(H3) The reference function $V \in C_c^1(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$, i.e., V is a smooth density and has compact support.

(H4) For all L^q estimates, $\beta \in (0, \frac{1}{2})$ works. More precisely, for $q \in [1, 2]$ is sufficient to take $\beta \in (0, 1)$, while $q > 2$ needs $\beta \in (0, \frac{q}{2q-2})$.

(H5) The law μ_0 has a density $p_0 \in C_b(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} e^{\lambda|x|}\, p_0(x)\, dx < \infty, \quad \forall \lambda > 0.$$

The goal of this work is to quantify the rate of convergence of a mollified empirical measure towards a density that appears as the solution of a limiting equation.

We define now a central object: the mollified empirical measure.

Consider for each $N \in \mathbb{N}$, $\quad V^N(z) \doteq N^\beta V(N^{\frac{\beta}{d}} z)$

and let the empirical measure

$$S_t^N \doteq \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$$

We choose the reference function V^N to regularize the above:

$$p_t^N \doteq S_t^N * V^N, \qquad p_t^N(x) \doteq \frac{1}{N}\sum_{i=1}^{N} V^N(x - X_t^{i,N}), \quad \text{for } t \in [0,T].$$

p_t^N is also accounting for the mean-field interaction. Let's say V has support on the unit ball. For large N, the player i interacts, via $N^{-1}V^N(X_t^{i,N} - X_t^{j,N})$ only with few players, those of distance less than $N^{-\frac{\beta}{d}}$:

$$N^{-1}V^N(X_t^{i,N} - X_t^{j,N}) = N^{\beta-1} V\big(N^{\frac{\beta}{d}}(X_t^{i,N} - X_t^{j,N})\big)$$

The idea behind the *moderate* interaction is that $\beta = 0$ and $\beta = 1$ represent *strong* and *weak* interactions regimes (range of interaction 1 and $N^{-\frac{1}{d}}$ respect.).

For a given representative player, the other players' positions are encoded in $p^N(t,x)$ which, for N large approximates a limit density $p(t,x)$. So, in [2] they studied the "isolated" SDE where $p(t,x)$ is a given density:

$$dX_t = (\alpha(t) + b(X_t, p(t, X_t)))\,dt + dW_t$$

under cost functional

$$J(\alpha) = \mathbb{E}\Big[\int_0^T \big(\frac{1}{2}|\alpha(s)|^2 + f(X_s, p(s, X_s))\big) ds + g(X_T)\Big]$$

and concluded that the optimal strategy for this problem is $\alpha^*(t,x) = -\nabla u(t,x)$, for (u,p) solution of the MFG System of equations

$$\begin{cases} -\partial_t u - \frac{1}{2}\Delta u - b(x, p(t,x))\cdot\nabla u + \frac{1}{2}|\nabla u|^2 = f(x, p(t,x)), \\ \partial_t p - \frac{1}{2}\Delta p + \nabla\cdot[p(t,x)(-\nabla u(t,x) + b(x, p(t,x)))] = 0, \\ p(0,\cdot) = p_0(\cdot), \quad u(T,\cdot) = g(\cdot). \end{cases} \tag{2}$$

Moreover, $\alpha = -\nabla u$ is an ε-Nash equilibrium for the N-player game, for large N. This way, we proceed supposing that the strategies are fixed.

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Main result

Using Itō's formula several times, we find the following mild formulation for $p^N(t)$:

$$p_t^N(x) = \mathcal{P}_t\, p_0^N(x) + \int_0^t \nabla\mathcal{P}_{t-s}\Big[V^N * \Big(\big(\alpha(s,\cdot) + b(\cdot, p_s^N(\cdot))\big) S_s^N\Big)\Big](x)\, ds + M_t^N(x)$$

where $M_t^N(x) = \frac{1}{N}\sum_{i=1}^N \int_0^t \nabla V^N(x - X_s^{i,N}) \cdot dW_s^{i,N}$ is a martingale-ish term. The essence of this work relies on the following *Propagation of Chaos* result

Theorem (Thm 5.1, [2]). Assume (H1), (H3) and (H5). Fix a common strategy $\alpha \in C_b([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ for all players. Then, the empirical measure S^N converges in probability to a flow $\mu \in C([0,T], \mathcal{P}(\mathbb{R}^d))$, each μ_t having density p_t, with $p \in C_b([0,T]\times\mathbb{R}^d)$ and being the unique solution in this space of the equation

$$p_t = \mathcal{P}_t p_0 + \int_0^t \nabla\mathcal{P}_{t-s}\left(p_s\left(\alpha(s,\cdot) + b(\cdot, p_s)\right)\right) ds$$

Our main result is the convergence rate of p_t^N to the p_t above.

Theorem 1. Suppose (H1)-(H5) are in place. Let $\alpha \in C_b^\gamma([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ be given. Then, $\forall\epsilon > 0$, $\forall m \ge 1$, we have for each $q \ge 1$

$$\Big\| \sup_{t\in[0,T]} \|p_t^N - p_t\|_{L^q(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} \le C \Big\| \|p_0^N - p_0\|_{L^q(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} + CN^{-\rho+\epsilon}$$

with

$$\rho \doteq \min\left(\frac{\beta}{d}(\gamma\wedge\gamma'), \frac{1}{2}(1 + \beta(1+\theta_q))\right)$$

where γ' is the Hölder regularity of p^N and $\theta_q \doteq d(1 - \frac{2}{q}) \vee 0$.

Here are a few key ingredients.

- First, the rate of convergence of the stochastic convolution integral M_t^N was already known in a similar setting [4, Prop. A.8]: Let $m \ge 1$, $q \ge 1$ and $\varepsilon > 0$. Then there exists $C > 0$ s.t. $\forall N \in \mathbb{N}^*$

$$\Big\| \sup_{s\in[0,t]} \big\| M_s^N \big\|_{L^q(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} \le CN^{-\frac{1}{2}(1+\beta(1+\theta_q))+\epsilon}, \qquad \forall t \in [0,T]$$

- semigroup estimates: $\|\nabla\mathcal{P}_t\|_{L^q\to L^q} \le \frac{C}{\sqrt{t}}$.
- Hölder estimates: $\|p_t^N\|_{\gamma'} \le C$ [2, Lemma 5.6]
- compact support of V providing part of the rate via

$$V^N(x-\cdot)|x-\cdot|^\gamma \overset{\text{def}}{=} N^\beta V(N^{\frac{\beta}{d}}(x-\cdot))|x-\cdot|^\gamma \le N^{-\frac{\beta}{d}\gamma} V^N(x-\cdot).$$

- a generalized version of *Grönwall's lemma* - Suppose $b \ge 0, \beta > 0$ and $a(t), b(t)$ are nonnegative functions locally integrable on $0 \le t \le T < \infty$ with

$$u(t) \le a(t) + b\int_0^t (t-s)^{\beta-1} u(s)\, ds$$

on this interval. Then

$$u(t) \le a(t) + \theta\int_0^t E_\beta'(\theta(t-s))\, a(s)\, ds, \quad 0 \le t < T,$$

where $\theta = (b\Gamma(\beta))^{1/\beta}$, $E_\beta(z) = \sum_{n=0}^\infty z^{n\beta}/\Gamma(n\beta+1)$.

$\rightsquigarrow$ We use the above with $\beta = \frac{1}{2}$ and

$$\|p^N(t) - p(t)\|_{L^q} \le a(t) + \int_0^t \frac{C}{\sqrt{t-s}} \|p^N(s) - p(s)\|_{L^q}\, ds$$

Work in Progress

We aim to generalize the model introducing a *common noise*, i.e., another Wiener process, B_t, independent of the previous one, inflicting at all players' motions. The SDE becomes

$$dX_t^{i,N} = F(X_t^{i,N}, p_t^N(X_t^{i,N}))\,dt + dW_t^{i,N} + \sigma(X_t^{i,N}) \circ dB_t$$

with σ sufficiently regular and the stochastic integral is a Stratonovich one. Now the (mollified) empirical measure should converge to a non-deterministic limit equation, a SPDE:

$$\partial_t p_t = -\nabla\cdot(p_t F(\cdot, p_t))\,dt + \frac{1}{2}\Delta p_t\,dt - \nabla p_t \circ \sigma\, dB_t$$

Semigroup techniques fail, but we expect to establish a similar convergence rate.

References

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