Lefschetz properties and the Togliatti Surface

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Abstract

This work aims to present the Togliatti Surface, the first to relate the existence of non-homogeneous Artinian ideals that fail the Weak Lefschetz Property to the existence of projective varieties that satisfy at least one Laplace equation.

Laplace Equations

Let $\mathcal{X}$ be a quasi-projective variety of dimension n contained in $\mathbb{P}^N$, and let x be a smooth point on $\mathcal{X}$. We can choose a system of N affine coordinates around x, together with a local parameterization $\mathcal{U}$ of x, such that $f : \Delta \to \mathcal{U}$, where Δ is a multidisk. This allows us to define the tangent space $T_x(\mathcal{X})$ as a vector space generated by the n linearly independent vectors associated with the affine coordinates. Additionally, we can define the osculating space $T_x^s(\mathcal{X})$ as the space spanned by all partial derivatives of order less than or equal to s.

Although the expected dimension of $T_x^s(\mathcal{X})$ is given by $\binom{n+s}{s} - 1$, in general, the dimension is $\dim(T_x^{(s)}(\mathcal{X}))$ is less than or equal to this value. If, for all smooth points on $\mathcal{X}$, the actual dimension of $T_x^s(X)$ is strictly less than $\binom{n+s}{s} - 1 - \delta$ for some δ, we say that $\mathcal{X}$ satisfies δ Laplace equations of order s.

Example: The Togliatti surface $\mathcal{X}$ is given by the closure of the image of the projection map:

$$\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5$$
$$(x : y : z) \mapsto (x^2y : x^2z : xy^2 : xz^2 : y^2z : yz^2).$$

We will see that $\mathcal{X}$ satisfies a Laplace equation of order 2 given by:

$$x^2\frac{\partial^2\varphi}{\partial x^2} - xy\frac{\partial^2\varphi}{\partial x\partial y} - xz\frac{\partial^2\varphi}{\partial x\partial z} + y^2\frac{\partial^2\varphi}{\partial y^2} - yz\frac{\partial^2\varphi}{\partial y\partial z} + z^2\frac{\partial^2\varphi}{\partial z^2} = 0$$

Weak Lefschetz Property

Consider the ring $R = \kappa[x_0, \cdots, x_n]$, where κ is an algebraically closed field of characteristic 0, and let $A = \bigoplus_{i=0} A_i$ be a κ-algebra. We define the Hilbert function as $\mathrm{Hilb}(A)(i) = \dim A_i$. If A is an Artinian algebra, the Hilbert function has a finite number of nonzero entries and is commonly referred to as the Hilbert vector of the algebra, denoted by $h(A)$.

Let A be a standard graded algebra. We say that such an algebra has the Weak Lefschetz Property (WLP) if there exists a linear form L in A such that, for every integer i, the multiplicative map $\times L : A_i \to A_{i+1}$. It has maximum rank, that is, it is surjective or injective. In this case, we say that the linear map L is called a Lefschetz element of A.

Example: Consider the ring $R = \kappa[x, y, z]$, the ideal $I = (x^3, y^3, z^3, xyz)$, the linear form $L = ax + by + cz$, and $A = R/I$. The Hilbert vector of A is: $h(A) = (1, 3, 6, 6, 3)$. Note that a possible fail in the WLP lies in the map $\times L : A_2 \to A_3$. Note that, in fact, there exists a nontrivial element in the kernel of $\times L$, which is $f = a^2x^2 + b^2y^2 + c^2z^2 + abxy + acxz + bcyz$. Therefore, the map does not have maximum rank and, consequently, A fails the WLP.

Macaulay's Inverse System

Let V be a κ-vector space of dimension $n+1$, and let $R = \bigoplus_{i\geq 0} \mathrm{Sym}^i V^*$ and $Q = \bigoplus_{i\geq 0} \mathrm{Sym}^i V$. Consider $x_0, \cdots, x_n$ and $X_0, \cdots, X_n$ as dual bases of V^* and V, respectively. Thus, we can express R and Q as $R = \kappa[x_0, \cdots, x_n]$ and $Q = \kappa[X_0, \cdots, X_n]$. In this way, the following action guarantees the existence of the product that gives R the structure of a Q-module:

$$\begin{array}{ccc} \mathrm{Sym}^i V \otimes \mathrm{Sym}^j V^* & \to & \mathrm{Sym}^{j-i} V \\ \alpha \otimes f & & \alpha(f). \end{array}$$

Let $I \subset Q$ be a homogeneous ideal. We define $I^{-1} := \{f \in R \mid \alpha(f) = 0,$ for all $\alpha\}$ as the Macaulay inverse system of I. Such I^{-1} is also a Q-submodule of R that inherits the grading of R.

Let $I \subset Q$ be a homogeneous ideal and $I^{-1} \subset R$ be a Macaulay inverse system. Consider a positive integer k and assume that $\dim I_k = r > 0$, where $I_k = \langle F_1, \cdots, F_r \rangle$.

Associated with the linear system $|I_k^{-1}|$ of dimension $N = \binom{n+k}{k} - r - 1$, we have the rational map $\varphi_k : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ whose closure of the image, $\mathcal{X}_k = \overline{\varphi_{(I^{-1})_t}(\mathbb{P}^n)} \subset \mathbb{P}^N$, is the projection of the Veronese variety $V(k, n)$, of dimension n, from the linear system $I_k = \langle F_1, \cdots, F_r \rangle \subset \mid \mathcal{O}_{\mathbb{P}^n}(k) \mid$.

Theorem: Let $I \subset R$ be an artinian ideal generated by r homogeneous polynomials $F_1, \cdots, F_r$ of degree d, and let I^{-1} be a Macaulay inverse system. If $r \geq \binom{n+d-1}{n-1}$, the following statements are equivalent:

1. The ideal I fails the Weak Lefschetz Property (WLP) in degree $d-1$.
2. On a general hyperplane $H \subset \mathbb{P}^n$, the homogeneous forms are κ-linearly independent.
3. Given a rational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}$ associated with $(I^{-1})_d$, the variety $\mathcal{X} = \overline{\mathrm{Im}(\varphi)}$, of dimension n, satisfies at least one Laplace equation of order $d-1$.

Proof: See [3, Theorem 3.2].

Togliatti Systems

Let $I \subset R = \kappa[x_0, \cdots, x_n]$ be an Artinian ideal generated by r forms, with $r \leq \binom{n+d-1}{n-1}$. We define:

1. I is a Togliatti system if it satisfies one, and consequently all three, conditions of the above Theorem;
2. I is a monomial Togliatti system if it can be generated by monomials;
3. I is a smooth Togliatti system if the variety $\mathcal{X}$ is smooth;
4. I is a minimal Togliatti system if no proper subset of the set of generators defines a Togliatti system.

Let $I \subset R = \kappa[x_0, \cdots, x_n]$ be a monomial Artinian ideal generated by monomials of degree d, and let I^{-1} be the Macaulay inverse system. We denote by $\triangle_n$ the n-dimensional standard simplex in the lattice $\mathbb{Z}^{n+1}$. We consider $d\triangle_n$ and define the polytope P_I as the convex hull of a finite subset $A_I \subset \mathbb{Z}^{n+1}$ corresponding to the monomials of degree d in I^{-1}.

Proposition: Let $I \subset R = \kappa[x_0, \cdots, x_n]$ be a monomial Artinian ideal generated by r monomials of degree d. Assume $r \leq \binom{n+d-1}{n-1}$. Then, I is a Togliatti system if and only if there exists a hypersurface of degree $d-1$ containing $A_I \subset \mathbb{Z}^{n+1}$. Moreover, I is a minimal Togliatti system if and only if any other hypersurface F does not contain an integral point of $d\triangle_n$ except possibly at the vertices of $d\triangle_n$.

Proof: See [2, Proposition 3.4].

Example: Let $R = \kappa[x, y, z]$. Consider the Artinian ideal $I \subset R$ with $I = (x^3, y^3, z^3, xyz)$. Also, let

$$I^{-1} = (x^2y, x^2z, xy^2, xz^2, y^2z, yz^2)$$

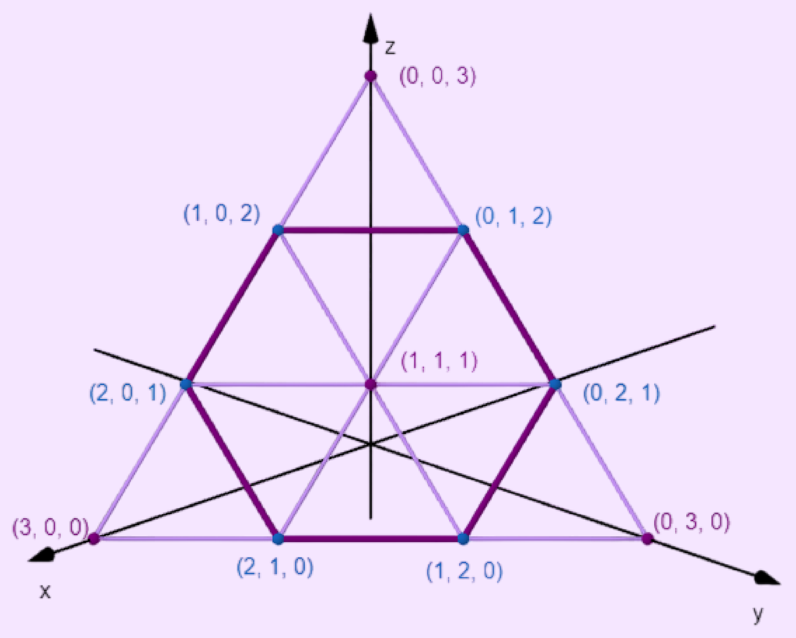
its Macaulay inverse system and $A_I \subset \mathbb{Z}^3$ given by

$$A_I = \{(2,1,0), (2,0,1), (1,2,0)(1,0,2), (0,2,1), (0,1,2)\}.$$

We have

$$3\triangle_2 = A_I \cup \{(3,0,0), (0,3,0), (0,0,3), (1,1,1).$$

and thus we construct P_I, the polytope associated with the Togliatti system I



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