The Wecken property for n-valued maps of $\mathbb{S}^2$

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Introduction

An *n-valued map* $\phi : X \multimap Y$ is a continuous multivalued function that associates to each $x \in X$ an unordered subset of exactly n point of Y.
Given an n-valued map $\phi : X \multimap Y$, there is a natural association with a single-valued map $\Phi : X \longrightarrow D_n(Y)$, so, the set of homotopy classes of n-valued maps from X to Y metric spaces is in one-to-one correspondence with the set $[X, D_n(Y)]$ of homotopy classes of maps from X to $D_n(Y)$. We say that ϕ is *split* if there are single-valued maps $f_1, \ldots, f_n : X \longrightarrow Y$ such that $\phi(x) = \{f_1(x), \ldots, f_n(x)\}$ for all $x \in X$. In this case, we can associate ϕ with a single valued map $\hat{\Phi} : X \longrightarrow F_n(Y)$. It is well known that $\pi_1(D_n(Y))$ is the braid group $B_n(Y)$ and $\pi_1(F_n(Y))$ is the pure braid group $P_n(Y)$. Recall that $\pi : F_n(Y) \longrightarrow D_n(Y)$ is an $n!$-fold covering space, then $\pi_k(F_n(Y)) \cong \pi_k(D_n(Y))$ for $k \geq 2$.
A space X is *Wecken for n-valued maps* if every n-valued map $\phi : X \multimap X$ has the *Wecken property* i.e. there exists an n-valued map $\psi : X \multimap X$ homotopic to ϕ that has exactly $N(\phi)$ fixed points, where $N(\phi)$ stands for Nielsen number of ϕ. If $n = 1$, it was proved that the surfaces with non-negative Euler characteristic are all Wecken, so $\mathbb{S}^2$ is Wecken for single valued-maps. Recall that for a single-valued map $f : \mathbb{S}^2 \to \mathbb{S}^2$, if $L(f) \neq 0$ thus $N(f) = 1$ and if $L(f) = 0$ thus $N(f) = 0$, where $L(f)$ is the Lefschetz index of f given by $L(f) = 1 + \deg(f)$.
In this work, we will verify that the Wecken property holds for n-valued maps of $\mathbb{S}^2$ when $n \geq 2$. We firstly present this proof analysing the case $n > 2$, followed by the case $n = 2$.
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The case $n > 2$

With the purpose of classifiyng the homotopy classes of n-valued maps of $\mathbb{S}^2$, we must consider the following result ([2], *Theorem 3.1*):

Theorem 1. *(Splitting Characterization Theorem) Let $\phi : X \multimap Y$ a n-valued map, where X is connected and locally path-connected. Then ϕ is split if and only if the image of $\Phi_\# : \pi_1(X) \longrightarrow B_n(Y)$ is contained in the image of the homomorphism $\pi_\# : P_n(Y) \longrightarrow B_n(Y)$ induced by the projection $\pi : F_n(Y) \longrightarrow D_n(Y)$. In particular, if X is simply connected, then all n-valued maps $\phi : X \longrightarrow Y$ are split.*

So for every n-valued map $\phi : \mathbb{S}^2 \multimap \mathbb{S}^2$ there are single-valued maps $f_0, \ldots, f_n : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ such that $\phi(x) = \{f_1(x), \ldots, f_n(x)\}$. Furthermore, by Schirmer ([5], *Corollary 7.2*), we have $N(\phi) = N(f_1) + \ldots, N(f_n)$.

Theorem 2. *([5], Corollary 7.2) Let $\phi = \{f_1, \ldots, f_n\} : X \longrightarrow Y$ be a split n-valued map, where X is a compact polyhedron. Then $N(\phi) = N(f_1) + \cdots + N(f_n)$.*

Lemma 1. *If $n > 2$, the set of homotopy classes of n-valued maps of $\mathbb{S}^2$ possesses exactly one element.*

Proof. It suffices to determine the set $[\mathbb{S}^2, D_n(\mathbb{S}^2)]$, which in turn is the orbit space of $[\mathbb{S}^2, D_n(\mathbb{S}^2)]_0 \cong \pi_2(D_n(\mathbb{S}^2))$ under the action of $\pi_1(D_n(\mathbb{S}^2))$. By [4], *Proposition 10 (b)*, for $n > 2$, the universal covering of $F_n(\mathbb{S}^2)$ has the homotopy type of $\mathbb{S}^3$, which implies that $\pi_2(D_n(\mathbb{S}^2)) \cong \pi_2(F_n(\mathbb{S}^2)) \cong \pi_2(\mathbb{S}^3) \cong 1$, so $[\mathbb{S}^2, D_n(\mathbb{S}^2)]$ has only one element, which is the class of the constant map. □

Proposition 1. *The Wecken property holds for n-valued maps of $\mathbb{S}^2$ for all $n > 2$.*

Proof. Let $\phi : \mathbb{S}^2 \multimap \mathbb{S}^2$ an n-valued map with $n > 2$. By Lemma 1, ϕ is homotopic to the constant map $x \mapsto \{x_1, \ldots, x_n\}$. By Theorem 2, the Nielsen number of a constant map is n, since the constant single-valued map $x \mapsto x_i$ for $i = 1, \ldots, n$ has degree 0, which implies Lefschetz index 1 and Nielsen number equal to 1. To complete the proof, it suffices to observe that the constant map has exactly n fixed points, namely $x_1, \ldots, x_n$. □

The case $n = 2$

Lemma 2. *If $\phi = \{f_0, f_1\} : \mathbb{S}^2 \multimap \mathbb{S}^2$ is a 2-valued map, then deg$(f_0) = -$deg(f_1).*

Proof. Since $f_0(x) \neq f_1(x)$, for all $x \in \mathbb{S}^2$, we can consider the arc connecting the points $f_0(x)$ and $-f_1(x)$, so we have a homotopy between f_0 and $-f_1$, and consequently deg$(f_0) = -$deg(f_1). □

Lemma 2 allows us to define the *degree* of a 2-valued map $\phi = \{f_0, f_1\} : \mathbb{S}^2 \multimap \mathbb{S}^2$ as being deg$(\phi) = |d|$, where d is either deg(f_0) or deg(f_1). The following proposition states that, as in the case of single-valued maps, the degree of a 2-valued map defines which class it belongs to.

Proposition 2. *Two 2-valued maps $\phi, \psi : \mathbb{S}^2 \multimap \mathbb{S}^2$ are homotopic if and only if deg$(\phi) =$ deg(ψ). Thus, the homotopy classes of 2-valued maps of $\mathbb{S}^2$ are classified by degree.*

Proof. As previously mentioned, the 2-valued homotopy classes of 2-valued maps of $\mathbb{S}^2$ are $[\mathbb{S}^2, D_2(\mathbb{S}^2)]$. Notice that the map $f : \mathbb{S}^2 \to F_2(\mathbb{S}^2)$ given by $f(x) = (x, -x)$ is a homotopy equivalence that is $\mathbb{Z}_2$-equivariant with respect to the action of the antipodal map on $\mathbb{S}^2$ and the action on $F_2(\mathbb{S}^2)$ given by permutation of coordinates. This gives rise to a homotopy equivalence between the corresponding orbit spaces, namely $\mathbb{R}P^2$ and $D_2(\mathbb{S}^2)$. Now, given a map $a : \mathbb{S}^2 \to \mathbb{R}P^2$, it lifts to two maps $\tilde{a}, -\tilde{a} : \mathbb{S}^2 \to \mathbb{S}^2$. Suppose $H : \mathbb{S}^2 \times [0,1] \to \mathbb{R}P^2$ a homotopy such that $H(\cdot, 0) = a$ and $H(\cdot, 1) = b$, where $b : \mathbb{S}^2 \to \mathbb{R}P^2$ lifts to $\tilde{b}, -\tilde{b} : \mathbb{S}^2 \to \mathbb{S}^2$. If H is lifted to $\tilde{H} : \mathbb{S}^2 \times [0,1] \to \mathbb{S}^2$ such that $\tilde{H}(\cdot, 0) = \tilde{a}$, then either $\tilde{H}(\cdot, 0) = \tilde{b}$ or $\tilde{H}(\cdot, 0) = -\tilde{b}$. Since, by the Hopf Classification Theorem, the homotopy classes of single-valued maps of $\mathbb{S}^2$ are determined by degree, the homotopy class of a is determined by the natural number $|\deg(\tilde{a})| = |\deg(-\tilde{a})|$. □

Now that we know the homotopy classes of 2-valued maps are classified by degree, to verify the Wecken property for 2-valued maps of $\mathbb{S}^2$, it suffices to exhibit for each integer d a 2-valued map $\phi_d : \mathbb{S}^2 \multimap \mathbb{S}^2$ of degree $|d|$ that has $N(\phi_d)$ fixed points. By Theorem 2 and the observations made in section *Introduction*, $N(\phi) = 1$ if deg$(\phi) = 1$ and $N(\phi) = 2$, otherwise. For $|d| = 0$, we have a constant map $\phi_0 : \mathbb{S}^2 \multimap \mathbb{S}^2$ with 2 fixed points. For $|d| = 1$, consider $\phi_1 = \{\bar{f}, -\bar{f}\}$, where $\bar{f} : \mathbb{S}^2 \to \mathbb{S}^2$ is a small deformation of $Id_{\mathbb{S}^2}$ i.e. $\bar{f}$ satisfies $|x - \bar{f}(x)| < \pi/2$ for all $x \in \mathbb{S}^2$, and which has exactly one fixed point. Such map may be constructed using a vector field on $\mathbb{S}^2$ that possesses just one singular point (see for example Figure 1).

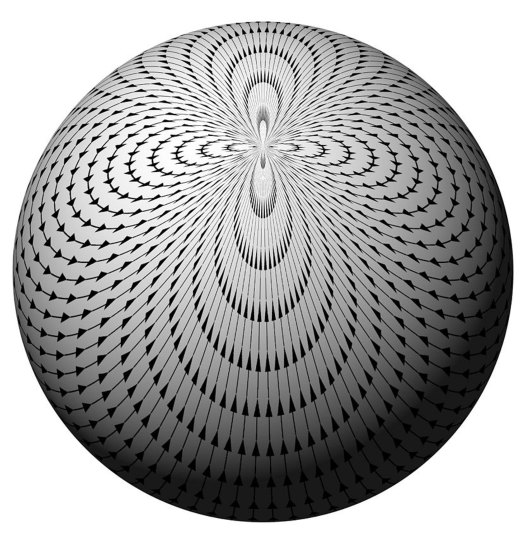

Figure 1: *Vector field on $\mathbb{S}^2$ with only one singular point.*

Proposition 3. *For each integer d with $|d| \geq 2$, there exists a 2-valued map $\phi_d = \{f_0, f_1\}$ such that deg$(\phi_d) = |d|$ and ϕ_d has $N(\phi_d) = 2$ fixed points.*

Proof. We start this proof considering the tangent bundle of $\mathbb{S}^2$, $p : T\mathbb{S}^2 \longrightarrow \mathbb{S}^2$, with fibre $T_x \approx \mathbb{R}^2$ and the normal bundle of $\mathbb{S}^2$, $q : N\mathbb{S}^2 \longrightarrow \mathbb{S}^2$, with fibre $N_x \approx \mathbb{R}$ (Figure 2 illustrates these bundles).

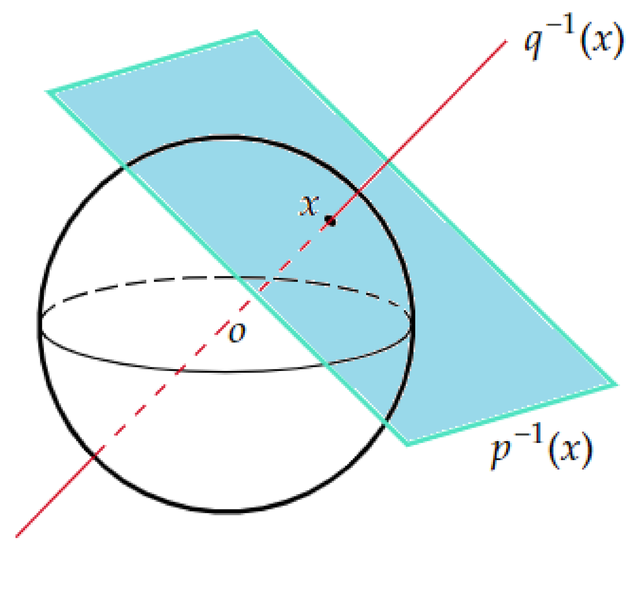

Figure 2: *Normal and tangent bundles at $x \in \mathbb{S}^2$.*

Note that there is a homeomorphism $h : N\mathbb{S}^2 \oplus T\mathbb{S}^2 \longrightarrow \mathbb{S}^2 \times \mathbb{R}^3$, given that for all $x \in \mathbb{S}^2$, every $y \in \mathbb{R}^3$ is uniquely written as a sum $rx + w$, where $< w, x >= 0$ and $r \in \mathbb{R}$. Furthermore, the Withney sum $N\mathbb{S}^2 \oplus T\mathbb{S}^2$ is a trivial bundle over $\mathbb{S}^2$ with fibre $N_x \times T_x \approx \mathbb{R}^3$. Therefore, we can consider the trivial sphere bundle $\bar{p} : S(N\mathbb{S}^2 \oplus T\mathbb{S}^2) \to \mathbb{S}^2$ and the homeomorphism $\bar{h} : S(N\mathbb{S}^2 \oplus T\mathbb{S}^2) \to \mathbb{S}^2 \times \mathbb{S}^2$. Given a section $s : \mathbb{S}^2 \longrightarrow S(N\mathbb{S}^2 \oplus T\mathbb{S}^2)$ it can be associated with a map $f_s : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$, since $\bar{h} \circ s(x) \in \{x\} \times \mathbb{S}^2$. Denote each element $((x, rx), (x, w)) \in S(N\mathbb{S}^2 \oplus T\mathbb{S}^2)$ by $(x, (r, w))$. Note that the constant section $x \mapsto (x, (1, \mathbf{0}))$, where $\mathbf{0} \in T_x$ is the zero vector, is associated with the identity map $Id_{\mathbb{S}^2}$, since $\bar{h}(x, (1, \mathbf{0})) = (x, 1 \cdot x + \mathbf{0}) = (x, x)$. Each element $x \in \mathbb{S}^2$ can be written uniquely as $x = (t, u)$, where $-1 \leq t \leq 1$, $u = (1-t^2)^{1/2}v$ and $v \in S(\mathbb{R}^2) = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}$. According to this notation, we define the northern and southern hemispheres of $\mathbb{S}^2$, respectively, by

$$S_+ = \{(t, u) \in \mathbb{S}^2 \mid t \geq 0\} \text{ and } S_- = \{(t, u) \in \mathbb{S}^2 \mid t \leq 0\}.$$

To define the sections s_0 and s_1 such that $f_{s_0} = f_0$ and $f_{s_1} = f_1$, as in the statement, we still need to choose orthogonal trivializations $\theta_+ : p^{-1}(S_+) \longrightarrow S_+ \times \mathbb{R}^2$ and $\theta_- : p^{-1}(S_-) \longrightarrow S_- \times \mathbb{R}^2$ such that when we consider $\mathbb{R}^2$ as the subspace of $\mathbb{R}^3$ generated by $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ we have that θ_+ preserves the base at the north pole $(1, \mathbf{0})$ and θ_- preserves the base at the south pole $(-1, \mathbf{0})$ (Figure 3 illustrates an example of pair (θ_+, θ_-)).

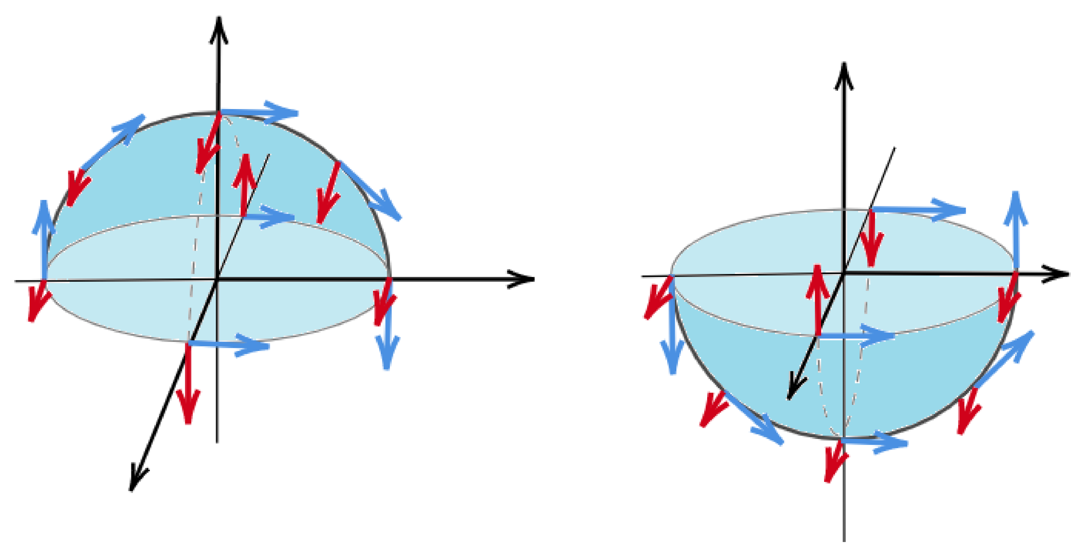

Figure 3: *In each point, the image of the vector e_1 is red and the image of vector e_2 is blue. Extending this continuously we get θ_+ and θ_-.*

Let $\kappa : S(\mathbb{R}^2) \longrightarrow O(\mathbb{R}^2)$ the clutching map on the equator that satisfies $\kappa(v)\theta_+(x)v = \theta_-(x)v$ for $v \in S(\mathbb{R}^2)$, where $O(\mathbb{R}^3)$ the orthogonal transformations of $\mathbb{R}^2$. Let $g : S(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ such that deg$(g) = d + 1$. So we define s_0 and s_1 as follows: for $v \in S(\mathbb{R}^2)$, $0 \geq t \geq 1$ and $x = (t, (1-t^2)^{1/2}v)$ let

$$(1 \oplus \theta_+(x)) \circ p_2 \circ s_0(t, (1-t^2)^{1/2}v) = (t, (1-t^2)^{1/2}g(v))$$

$$(1 \oplus \theta_+(x)) \circ p_2 \circ s_1(t, (1-t^2)^{1/2}v) = (-t, -(1-t^2)^{1/2}g(v))$$

and for $-1 \leq t \leq 0$ let

$$(1 \oplus \theta_-(x)) \circ p_2 \circ s_0(t, (1-t^2)^{1/2}v) = (t, (1-t^2)^{1/2}\kappa(v)g(v))$$

$$(1 \oplus \theta_-(x)) \circ p_2 \circ s_1(t, (1-t^2)^{1/2}v) = (-t, -(1-t^2)^{1/2}\kappa(v)g(v)),$$

where $p_2(x, (r, w)) = (r, w)$. By construction, $s_0(x) \neq s_1(x)$ for all $x \in \mathbb{S}^2$. Furthermore, $s_0(x) = (x, (1, \mathbf{0}))$ if and only if $x = (1, \mathbf{0})$ and $s_1(x) = (x, (-1, \mathbf{0}))$ if and only if $x = (-1, \mathbf{0})$. Thus, the north pole $(1, \mathbf{0})$ is the single fixed point of f_{s_0} and the south pole is the single fixed point of f_{s_1}. In a small enough neighborhood of $(1, 0)$, the trivialization θ_+ slightly changes the images of the base elements e_1 and e_2. Then, in this neighborhood, the map f_{s_0} can be described in suitable coordinates by the self-map of $\mathbb{R}^2$ that takes rv, for $r \geq 0$ and $v \in S(\mathbb{R}^2)$, to $rv + rg(v)$ so that its fixed point is $\mathbf{0} \in \mathbb{R}^2$. Therefore, the index of f_{s_0} is equal to the degree of the map $-g : S(\mathbb{R}^2) \longrightarrow S(\mathbb{R}^2)$, that is, I$(f_{s_0})$ =deg$(g) = d+1$. By calculation, the Lefschetz number of f_{s_0} is L$(f_{s_0}) = 1+$deg(f_{s_0}). Using the Lefschetz-Hopf Theorem we obtain deg$(g) = 1 +$ deg$(f_{s_0}) \Rightarrow$ deg$(f_{s_0}) = d$ and consequently deg$(f_{s_1}) = -d$ by Lemma 2; thus $f_0 = f_{s_0}$ and $f_1 = f_{s_1}$ are the maps we are looking for. □

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