Existence and regularity of the solution of a semilinear elliptic equation problem with singular nonlinearity

Jadde Thaine dos Santos Oliveira [1] & Luís Henrique de Miranda[1]

Universidade de Brasília[1]

jadde.oliveira@aluno.unb.br
demiranda@unb.br

Abstract

In this work we will show the existence and regularity of solutions of a semilinear elliptic equation problem with singular nonlinearity, following the studies of Lucio Boccardo and Luigi Orsina in [1]. In this specific case, we work with a bounded open set of dimension greater than or equal to two, any non-negative function belonging to some Lebesgue space and a bounded elliptic matrix. To discuss the problem, we use the method of approximation and results such as the Maximum Principle, Schauder's fixed point theorem and other estimates made in [2].

Introduction

In this work, we will show the existence and regularity of solutions to problem (1), following the studies of Lucio Boccardo and Luigi Orsina [1].

$$\begin{cases} -\text{div}(M(x)\nabla u) = \frac{f(x)}{u} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open subset of $\mathbb{R}^N$, with $N \geq 2$, f is a non-negative function belonging to a Lebesgue space and M is a bounded elliptic matrix, i.e. $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi \ \forall \xi \in \mathbb{R}^N$ and $|M(x)| \leq \beta$.

Objectives

Our goals are to guarantee the existence of unical solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ for an approximate problem and to obtain certain properties about it. Furthermore, using the approximation results, the main objective is to obtain answers about the existence and regularity of the original problem.

Results

Approximate problem

Consider the following problem

$$\begin{cases} -\text{div}(M(x)\nabla u_n) = \frac{f_n}{u_n + \frac{1}{n}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where f is a non-negative measurable function, $n \in \mathbb{N}$, $f = min\{f(x), n\}$ and M is a bounded elliptic matrix. To guarantee (1), we need the following results on problem (2):

Lemma 1. Problem (2) has unique non-negative solution $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. In addition, the following properties apply:

i) u_n is increasing with respect to n;

ii) $u_n > 0$ in Ω;

iii) For every $\widehat{\Omega} \subset\subset \Omega$, there is a $K_{\widehat{\Omega}} > 0$, regardless of n, we have $u_n(x) \geq K_{\widehat{\Omega}} > 0$, for every $x \in \Omega$ and for every $n \in \mathbb{N}$.

In order to guarantee that $u_n \in W_0^{1,2}(\Omega)$, Schauder's fixed point method and the Maximum Principle are used, and for $u_n \in L^\infty(\Omega)$ estimates produced in [2] are used. In the second part, item **i)**, it is enough to choose an appropriate test function on the hypothesis of ellipticity and use the weak formulation of the problem. In items **ii)** and **iii)** the Strong Maximum Principle is needed.

Remark 1. The solution given by Lemma 1 is unique.

Since u_n is increasing in n, we define u as a point limit of u_n. Since $u \geq u_n$, then item **iii)** of the previous result is valid for u, that is, $u(x) \geq K_{\widehat{\Omega}} > 0$ for all $x \in \Omega$ and for all $n \in \mathbb{N}$.

Existence and regularity of solution in (1)

After showing that there is a solution to the approximate problem, it is possible to prove the existence and regularity of the solution to the original problem. For this, we guarantee that if u_n is a solution of (2) and $f \in L^1(\Omega)$, then u_n is bounded by $W_0^{1,2}(\Omega)$. Adding this with the weak formulation of problem (2) and the fact that u is the point limit of u_n, we prove the next theorem:

Theorem 1. Let $f \in L^1(\Omega)$ be non-negative and not identically zero. Then there is a solution $u \in W_0^{1,2}(\Omega)$ of (1), in the sense that

$$\int_\Omega M(x)\nabla u \cdot \nabla\phi = \int_\Omega \frac{f\phi}{u} \quad \forall \phi \in C_0^1(\Omega)$$

Since the integrability of u depends on f, we get the next lemma.

Lemma 2. Let $f \in L^m(\Omega)$, with $m \geq 1$. Then, the solution u of (1) given by the previous theorem is such that:

i) If $m > \frac{N}{2}$, then $u \in L^\infty(\Omega)$;

ii) If $1 \leq m < \frac{N}{2}$, then $u \in L^s(\Omega)$, $s = \frac{2Nm}{N-2m}$.

To show item **i)** of Lemma 2, an appropriate test function is chosen to use in the ellipticity hypothesis and concluded with a result found in [2]. As for item **ii)**, when $m = 1$, just use Sobolev immersion, and when $1 < m < \frac{N}{2}$, using, again, an appropriate test function in the hypothesis of ellipticity and $\delta > 1$, we guarantee the estimate:

$$\left(\int_\Omega u_n^{2^*\delta}\right)^{\frac{2}{2^*}} \leq C\|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(2\delta-2)m'}\right)^{\frac{1}{m'}}$$

Being $\frac{2}{2^*} > \frac{1}{m'}$, it remains to choose a δ such that $2^*\delta = (2\delta - 2)m'$ to ensure that u_n is bounded at $L^s(\Omega)$. So $u \in L^s(\Omega)$.

Conclusion

With the results from the approximate problem and using the fact that u_n is bounded by $W_0^{1,2}(\Omega)$, it was possible to establish the existence of a solution to problem (1) in $W_0^{1,2}(\Omega)$ and show that $u \in L^p(\Omega)$, where $p = \infty$ or $p = \frac{2Nm}{N-2m}$, depending on the m where $f \in L^m(\Omega)$. It is noteworthy that u_n being bounded by $W_0^{1,2}(\Omega)$ gives us more regularity than would be expected by the Classical Stampacchia Theory.

References

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