Minimally immersed Klein bottles in $T^1\mathbb{S}^2$

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Abstract

For the area-minimizing vector fields on antipodally punctured unit 2-sphere with even Poincaré indexes, we prove that the topological closure of their image coincides with the image of minimally immersed Klein bottles in $T^1\mathbb{S}^2$.

1 Introduction

Let M be a closed oriented Riemannian manifold and V a unit vector field on M. Consider the unit tangent bundle T^1M equipped with the Sasaki metric. ***The volume of a unit vector field*** V is defined as

$$\mathrm{vol}(V) = \int_M \sqrt{\det(\mathrm{I} + (\nabla \mathrm{V})(\nabla \mathrm{V})^*)}\nu_M,$$

where $\mathbf{I}$ is the identity, and ∇V is considered as an endomorphism of the tangent space with adjoint operator $(\nabla V)^*$.

Theorem 1 (Gluck and Ziller)**.** *The unit vector fields of minimum volume on $\mathbb{S}^3$ are precisely the Hopf vector fields and no others.*

Intuitively speaking, *one hopes that the visually best organized unit vector fields on M are rewarded with minimum possible volume.*

The Gil-Medrano and Llinhares-Fuster's result states that critical points are submanifolds which are minimal immersions:

Theorem 2 (Gil-Medrano and Llinhares-Fuster)**.** *An element $V \in \mathcal{X}^1(M)$ is a critical point of the volume functional restricted to $\mathcal{X}^1(M)$ if and only if $V : M \to (T^1M, g^{Sas})$ is a minimal immersion, where g^{Sas} is the Sasaki mectric.*

I now report the most recent result about area-minimizing unit vector fields on $\mathbb{S}^2\setminus\{\pm p\}$. It says that the volume of a unit vector field V is bounded below by the length of an ellipse naturally associated to it:

Theorem 3 (Brito, ——, Gonçalves and Nicoli, [2])**.** *Let V be a unit vector field defined on $\mathbb{S}^2\setminus\{N, S\}$. If $k = \max\{I_V(N), I_V(S)\}$, $k \neq 0$, $k \neq 2$, then*

$$\mathrm{vol}(V) \geq \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ is the length of the ellipse $\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1$ and $I_V(p)$ stands for the Poincaré index of V around p.

2 Area-minimizing unit vector fields on $\mathbb{S}^2\setminus\{N, S\}$

Let $\mathbb{S}^2\setminus\{N, S\}$ be the Euclidean sphere in which two antipodal points N and S are removed. Denote by g the usual metric of $\mathbb{S}^2$ induced from $\mathbb{R}^3$, and by ∇ the Levi-Civita connection associated to g. Consider the oriented orthonormal frame $\{e_1, e_2\}$ on $\mathbb{S}^2\setminus\{N, S\}$ such that e_1 is tangent to the parallels and e_2 to the meridians. Let k be an integer number and define the ***angle function*** as

$$\begin{aligned}\theta_k : \mathbb{S}^2\setminus\{N, S\} &\longrightarrow \mathbb{R}\\ p &\longmapsto \theta_k(p) = (k-1)t + \frac{\pi}{2},\end{aligned}$$

where $t \in [0, 2\pi)$ is the *longitude* coordinate of $p = (x, y, z)$ in $\mathbb{S}^2\setminus\{N, S\}$. Note that if $\{e_1, e_2\}$ is the oriented orthonormal frame aforementioned,

$$d\theta_k(p)(e_1) = \frac{k-1}{\sqrt{x^2+y^2}} \quad \text{and} \quad d\theta_k(p)(e_2) = 0.$$

Definition 4. *For $k \in \mathbb{Z}$, define the unit vector field $V_{k,2-k}$ as*

$$V_{k,2-k}(p) = \cos(\theta_k(p))\, e_1(p) + \sin(\theta_k(p))\, e_2(p),$$

where $p \in \mathbb{S}^2\setminus\{N, S\}$, θ_k is the angle function and $\{e_1, e_2\}$ is the oriented orthonormal frame on $\mathbb{S}^2\setminus\{N, S\}$ aforementioned.

Corollary 5. *For $k \in \mathbb{Z}\setminus\{0, 2\}$, the unit vector field $V_{k,2-k}$ on $\mathbb{S}^2\setminus\{N, S\}$ is area-minimizing if*

$$\mathrm{vol}(V_{k,2-k}) = \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ is the length of the ellipse $\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1$.

3 Minimally immersed Klein bottles in $T^1\mathbb{S}^2$ arising from area-minimizing unit vector fields on $\mathbb{S}^2\setminus\{N, S\}$

For each class of index vector field, by Theorem 2, provide us a minimal surface in $T^1\mathbb{S}^2$. We pose the natural question:

What is the topological type of these minimal surfaces arising from area-minimizing vector fields on $\mathbb{S}^2\setminus\{\pm p\}$?

In 2010, Borrelli and Gil-Medrano proved that the Pontryagin vector fields are area-minimizing on $\mathbb{S}^2\setminus\{p\}$. They obtained that the images of Pontryagin vector fields are homeomorphic to the projective plane:

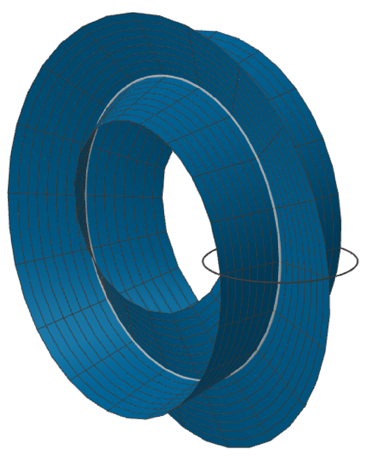
Theorem 6 (Borrelli and Gil-Medrano)**.** *The only minimal surfaces in $T^1\mathbb{S}^2(r)$ homeomorphic to the projective plane arising from vector fields without boundary are Pontryagin cycles.*

In 2022, for an area-minimizing unit vector field $V_{k,2-k}$ on $\mathbb{S}^2\setminus\{\pm p\}$ with $\{I_V(p), I_V(-p)\} = \{k, 2-k\}$, where $k \in 2\mathbb{Z}\setminus\{0, 2\}$ we proved that the topological closure of the image of $V_{k,2-k}(\mathbb{S}^2\setminus\{N, S\})$ is a minimally immersed Klein bottle in $T^1\mathbb{S}^2(1)$.

Theorem 7 (Brito, —— , Gonçalves and Nunes, [1])**.** *Let $V_{k,2-k}$ be an area-minimizing unit vector field on $\mathbb{S}^2\setminus\{N, S\}$. If the Poincaré index around the singularity N (or S) is $k \in 2\mathbb{Z}\setminus\{0, 2\}$, then the topological closure of the image of $V_{k,2-k}(\mathbb{S}^2\setminus\{N, S\})$ is a minimally immersed Klein bottle in $T^1\mathbb{S}^2(1)$.*

Consider the decomposition $\mathbb{S}^2 = \mathbb{S}^2_+ \cup \mathbb{S}^2_-$, where $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are respectively the northern and southern hemisphere.

Lemma 8. *If Poincaré index $k \in 2\mathbb{Z}\setminus\{0, 2\}$ around the singularity N (or S) in $\mathbb{S}^2$, then the topological closure of $V_{k,2-k}(\mathbb{S}^2_+\setminus\{N\})$ (or $V_{k,2-k}(\mathbb{S}^2_-\setminus\{S\})$) in $T^1\mathbb{S}^2$ is the image of an immersed Moebius strips with boundary $V_{k,2-k}(\partial(\mathbb{S}^2_+\setminus\{N\}))$ (or $V_{k,2-k}(\partial(\mathbb{S}^2_-\setminus\{S\}))$).*



An immersed Moebius strip in $\mathbb{R}^4$ given by the topological closure of the image of $V_{4,-2}$

Proof of the Theorem 7. A smooth immersion of the Klein bottle in $T^1\mathbb{S}^2(1)$ is obtained by gluing two images of the Moebius strip along their boundary given by Lemma 8. It follows from Corollary 5 that $V_{k,2-k}$ is area-minimizing vector field in its topological conjugation class. By Theorem 2 the section seen as surface in $T^1\mathbb{S}^2(1)$ is geometrically minimal, i.e., its has zero mean curvature. Therefore, the topological closure of $V_{k,2-k}$ is a minimal surface in $T^1\mathbb{S}^2(1)$. □

References

[1] Fabiano Brito, Jackeline Conrado, Icaro Gonçalves, and Giovanni Nunes. Minimally Immersed Klein Bottles in the Unit Tangent Bundle of the Unit 2-Sphere Arising from Area-Minimizing Unit Vector Fields on $\mathbb{S}^2\setminus\{N, S\}$. *Journal of Geometric Analysis*, 33(5):142, 2023.

[2] Fabiano Brito, Jackeline Conrado, Icaro Gonçalves, and Adriana Nicoli. Area minimizing unit vector fields on antipodally punctured unit 2-sphere. *Comptes Rendus Mathématique*, 359-10:1225–1232, 2021.