A new and unusual perspective for the definition of solid angles

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Abstract

The present work aims to show a deduction behind steradian's definition (angle equals area divided by the square of radius). Therefore, it has educational purposes. The concepts of linear transformations and simple algebra were used to guide the deduction towards the definition through two properties: angle is linear to surface area and can be the same for different sphere sizes. As a result, it was possible to do mathematical modeling and get to the relationship involving the angle, area, and radius. Concluding, this new approach could be useful to inspire professors, students, and math specialists on how to teach and learn solid angles in university lessons.

Introduction

Solid angle is a concept that represents a 3-dimensional angle which goes all throughout a sphere [1]. It is defined as the result of the surface area covered divided by the square of radius, and gets the name of "steradian" [1].

Objectives

The general objective of this work is to show a deduction behind this definition and inspire math specialists to present these deductions to their students, when teaching solid angles. For this, we will use concepts of linear transformations and simple algebra to see the essence of the mentioned definition of steradian.

Analysis of the surface area

It is expected that a given solid angle, α , depends on the surface area formed and on the sphere's radius:

$$\alpha = f(A, r)$$

Where A is the area formed by alpha and r is the radius

To define how it varies with the area, we must adopt two visibles properties of A (see fig. 1):

- The area doubles as alpha doubles
- The solid angle referred to A=0 is equal to zero

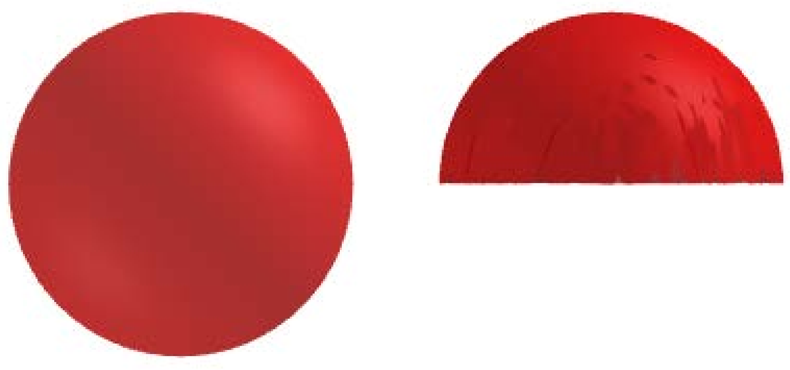

Figure 1: Different solid angles. On the left, the angle covers the full sphere. So the area A reaches its maximum value. On the right, the angle is reduced to its half. So the sphere's area covered is also reduced by half. Source: own autorship (created using Geogebra)

Then, general properties are:

$$\lambda\alpha = f(\lambda A, r)$$
$$0 = f(0, r)$$

These relations indicate that alpha may be interpreted as a linear transformation of A, if we assume r constant. Thus, wherever the constant r is coupled on the general function f, alpha must obey the special relation:

$$\alpha = aA + b,\ a \in \mathbb{R},\ b = 0$$

The function f, then, can only be defined as:

$$\alpha = f(A, r) = g(r) \cdot A$$

where r is now a variable and its variation does not affect the linearity that alpha has with A.

Analysis of the radius

We already know that a solid angle is represented by the relation:

$$\alpha = g(r) \cdot A$$

In this stage, it is necessary to define g. For this, we adopt another very important property of the angle: alpha can be the same for a small sphere with a small given radius and for a bigger sphere with a bigger radius, since we adjust all correct proportions. The analysis of this so relevant aspect, alpha being the same for different sphere sizes, will be the key for defining g. Therefore, we write:

$$\alpha_r = \alpha_{r+\Delta r}$$
$$g(r) \cdot A = g(r + \Delta r) \cdot (A + \Delta A)$$

Considering a complete surface area (full solid angle):

$$g(r) \cdot 4\pi r^2 = g(r + \Delta r) \cdot 4\pi (r + \Delta r)^2$$

$$g(r) \cdot 4\pi r^2 = g(r + \Delta r) \cdot \left[4\pi r^2 + 8\pi r\Delta r + 4\pi(\Delta r)^2\right]$$

$$g(r) \cdot r^2 = g(r + \Delta r) \cdot \left[r^2 + 2r\Delta r + (\Delta r)^2\right]$$

$$\frac{r^2}{r^2 + 2r\Delta r + (\Delta r)^2} = \frac{g(r + \Delta r)}{g(r)}$$

$$\frac{r^2}{(r + \Delta r)^2} = \frac{g(r + \Delta r)}{g(r)}$$

This can be interpreted as a functional equation, whose possible intuitive solution is:

$$g(r) = \frac{c}{r^2},\ c \in \mathbb{R}$$

Therefore:

$$\alpha = g(r) \cdot A$$
$$\alpha = c \cdot \frac{A}{r^2},\ c \in \mathbb{R}$$

Conclusions

The mentioned relation respects both properties of solid angles: alpha is linear with the area and alpha can be the same for different sphere sizes. However, it's desirable to have a relation that corresponds to the convention of several applications, such as electricity, radiation, etc. Thus, we want 1 steradian to be the solid angle of a surface that measures 1 and radius 1. In this case, c comes to be equal to 1.

Reference

[1] NUSSENZVEIG, Moyses. Curso de Física Básica. Vol 3: Eletromagnetismo. 4ª ed. São Paulo: Blucher, 32-33, 2019.