The Seifert graph of generalized knots and links diagrams

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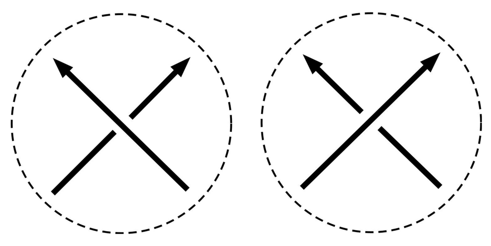
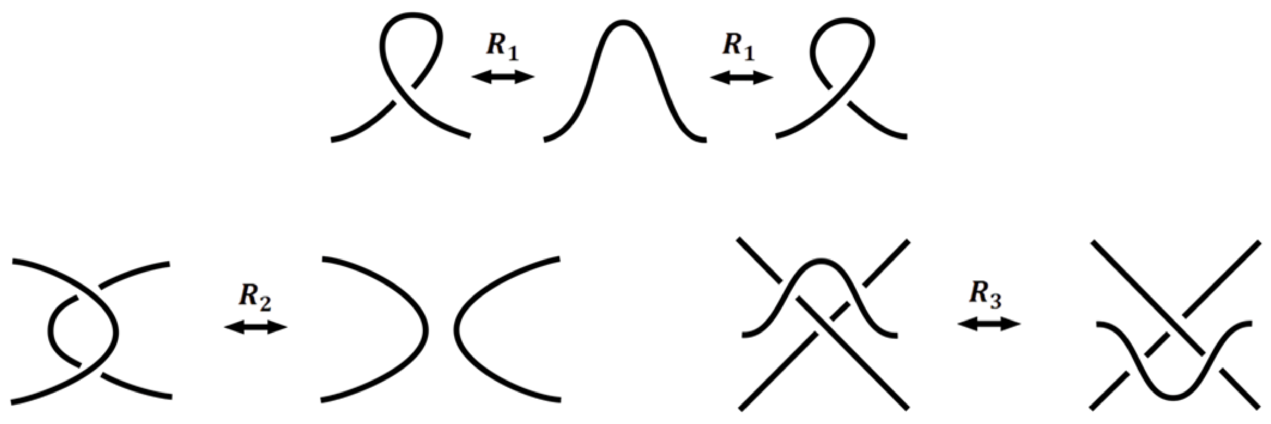
Knots and links

Knot Theory as we know first gained prominence as a physicist's erroneous concept of a model for the atom. It was kept alive by the efforts of a few diligent physicists until the 20th century when mathematicians took up the challenge. Today, mathematical theories regarding knots are being applied to fields of physics, biology and chemistry.

We define a knot as the image of the circle $\mathbb{S}^1$ in $\mathbb{R}^3$ under an embedding. A link is a collection of knots which do not intersect, in particular a knot is a link with one component. In this work we call these geometric objects of classical links. Knots and links are presented usually by regular projections on the plane $\mathbb{R}^2$ which are called their *diagrams*. In this case, we considered the following (real) crossings



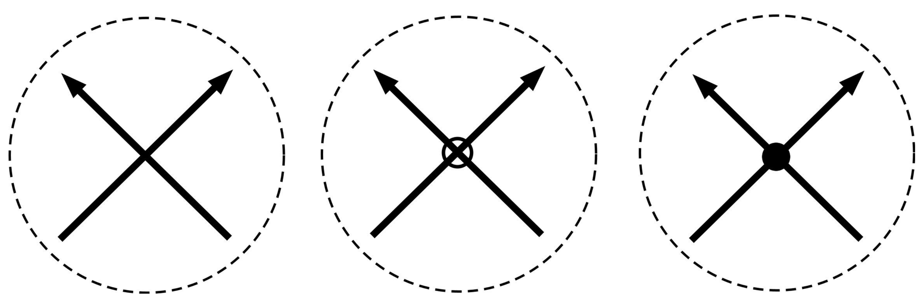
Two links are *equivalent* if one can be obtained the other by a finite sequence of *Reidemeister moves*, represented in the image below.



Generalizations of knots and links

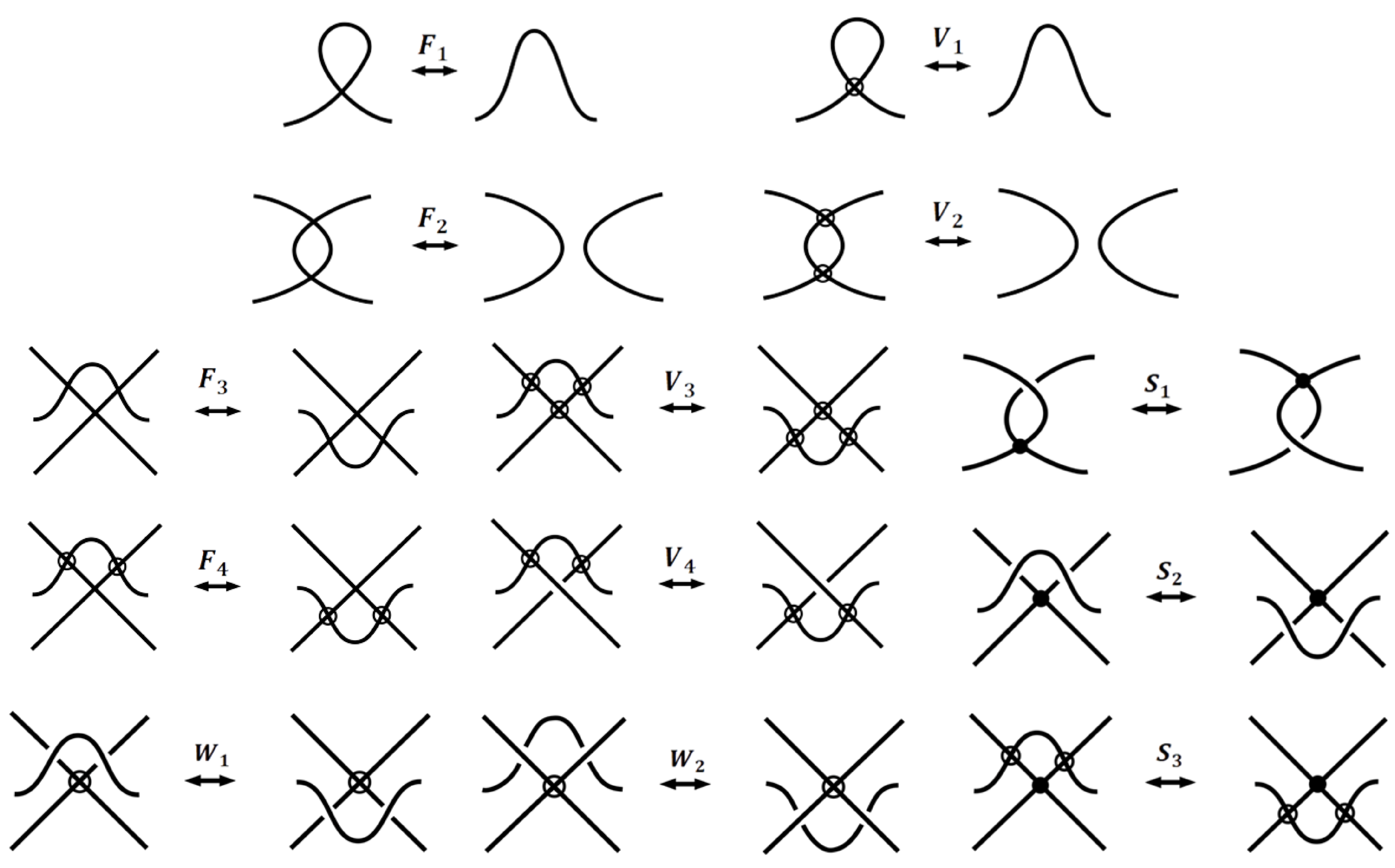
From the end of the 20th century was introduced some generalizations of the notion of a classical link. As we seen, every classical knot or link diagram can be regarded as an immersion of circles in the plane with extra structure at the double points.

This extra structure is usually indicated by the over and under crossing conventions that give instructions for constructing an embedding of the link in three-dimensional space from the diagram. But, we can consider this extra structure as other types of crossings. The other crossings that we considered here are the flat, virtual and singular crossings, as in the image below.



More precisely, a *virtual diagram* is a diagram with real and virtual crossings. A *flat virtual diagram* is a diagram with virtual and flat crossings. *Welded* and *Unrestricted virtual diagrams* are diagrams with real and virtual crossings, differing by a forbidden move, as we shall see later. A *singular diagram* is a diagram with real and singular crossings and a *virtual singular diagram* is a diagram with real, virtual and singular crossings.

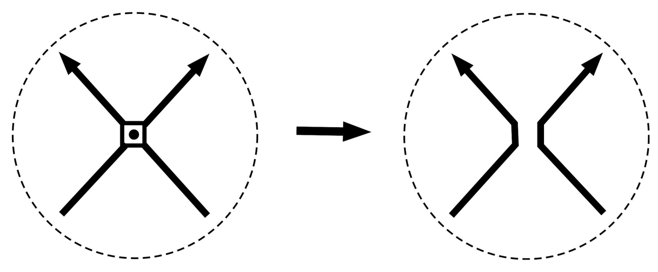
Considering the other crossings, we have that these generalizations of links are equivalents via Reidemeister-type moves.



Moreover, we have also other generalizations of (virtual) knots and links, the so-called doodles and virtual doodles, that are the image of circles in $\mathbb{R}^2$ under an "embedding".

The Seifert graph of generalized knots and links diagrams

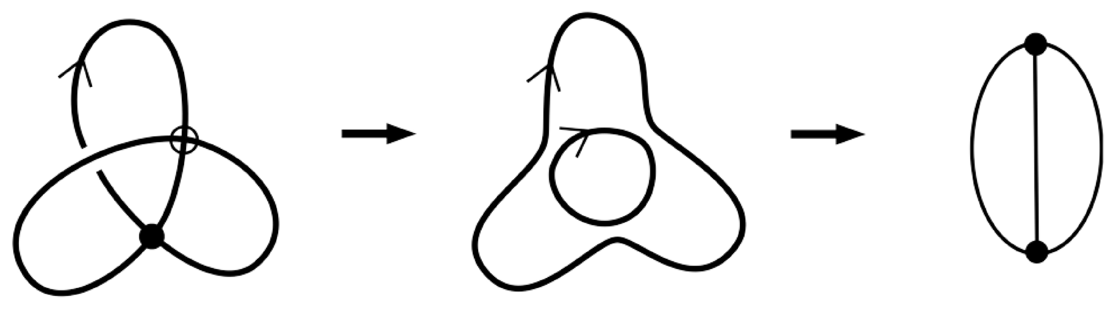
Generalized Seifert circles are circles obtained by smoothing all crossings of a generalized link diagram.



Let $S(D)$ be the number of Seifert circles of D and $c(D)$ the number of crossings in D. The *Seifert graph* $\Gamma(D)$ is a graph with $S(D)$ vertices $v_1, v_2, \ldots, v_{S(D)}$ and $c(D)$ edges $e_1, e_2, \ldots, e_{c(D)}$. Each vertex corresponds to a Seifert circle and each edge correspond to a crossing.

Two distinct vertices v_i and v_j are connected by e_k if the two Seifert circles S_i and S_j (corresponding to v_i and v_j, respectively) are joined by the crossing c_k (corresponding to e_k).

Example 1. For the virtual singular knot below, we have two Seifert circles. Hence, your Seifert graph has two vertices and three edges.



The main result

A graph G is said to be *bipartite* if any cycle has an even length and is said to be *planar* if G is a graph embedded in $\mathbb{R}^2$.

Remarks 2. Note that the Seifert graph of generalized links and knots diagrams is planar by construction.

Let D be a non-classical oriented diagram, we denote by $cl(D)$ an oriented diagram, such that any non-classical crossing in D is transformed into a classical crossing.

Remarks 3. The choice of this transformation is not unique.

Lemma 4. *The Seifert graphs $\Gamma(D)$ and $\Gamma(cl(D))$ are equals.*

Proof. $\Gamma(cl(D))$ is well defined. □

Proposition 5. *Let D be a non-classical oriented diagram, then we have that the Seifert graph $\Gamma(D)$ is bipartite.*

Proof. By previous lemma, $\Gamma(D) = \Gamma(cl(D))$ and we know that the Seifert graph $\Gamma(cl(D))$ is bipartite (see [1]). □

Therefore, we can conclude the following result.

Theorem 6. *The Seifert graph of generalized knots and links diagrams is planar and bipartite.*

References

[1] Caprau, Carmen; de La Pena, Andrew; McGahan, Sarah. *Virtual Singular Links and Braids*. Manuscripta Mathematica, **151**, 147 - 175, 2016.

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Acknowledgments

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