Spectral Theory of the Laplacian on Vector Bundles

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Abstract

The spectrum of Laplace Beltrami operator Δ_g on a compact n-dimensional riemannian manifold (M,g) acting on space $C^\infty(M)$ has been a fundamental tool to understand global properties about the topology and geometry of the manifold and much work has been done that be found in literature [1]. The Spectral Theory of Laplacian Operator has intersection with Physics, String Theory and Gauge Theory. In this presentation we study the Connection Laplace Operator acting on smooth sections on a smooth vector bundle E over (M,g), its symmetries and relationship between the Gauge Transformation and linear connection on E. Some results we can find with more details in [2,3].

Introdution

Let $E \to (M,g)$ be a C^∞ vector bundle, with hermitian structure on fibers and rank r. Let us consider the Connection Laplacian operator $L_g^{\widetilde{\nabla}}$ which operates on $\Gamma(E)$, the space of C^∞ section of E. The operator $L_g^{\widetilde{\nabla}}$ is a second order, self-adjoint, elliptic differential operator locally expressed with respect to a locally unitary frama of E and local coordinates of M as

$$L_g^{\widetilde{\nabla}} = -g^{jk}\nabla_j\nabla_k - 2g^{jk}\omega_j\nabla_k - g^{jk}(\nabla_j\omega_k + \omega_j\omega_k)$$

where ∇_j is Levi-Civita connection induced by g, $w = (w_\alpha^\beta)$ is a $r\times r$ matrix of 1-forms

Remark 1. *Connection Laplacian Operator is a positive formally selfadjoint elipptic operator of second order and have compact resolvent and we will denote by $Sp(M,g,E,\widetilde{\nabla})$ the spectrum of E*

Discussion and Results

Continuous sections of a smooth bundle $\times$ G-equivariant functions

Let P be a G-principal bundle over (M,g), where G is a Lie group. Consider V a finite dimensional vector space. Given a linear group representation $\rho : G \to GL(V)$ we can construct an associated bundle $E := P\times_G V$ such that $(p,v) \sim (p\cdot g, \rho(g^{-1})v)$ and, denote by $\Gamma(E)$ the space of smooth sections of E.

Let $C^\infty(P,V)^G$ be the space of all smooth functions $\varphi : P \to V$ satisfing $\varphi(p\cdot g) = \rho(g^{-1})\varphi(p)$. Then, we can construct a isomorphism between $C^\infty(P,V)^G$ and $\Gamma(E)$, for more details see [4]. This results allows us study the Connection Laplacian by means of an associate Laplace Beltrami operator acting on $C^\infty(P,V)^G$. Recall that a connection on P is given by an assignement of equivariant horizontal distribution of P. Since E is an associated vector bundle induced by G-principal bundle P, we have an induced linear connection $\nabla : \mathcal{X}(M)\times\Gamma(E) \to \Gamma(E)$ which associates $\nabla_X s$ to $X^h\cdot\varphi$, where X^h is the horizontal lifting of X to P and $\varphi \in C^\infty(P,V)^G$. Furthermore, the 2-form curvature induced by these connections are equivalent. Then we can rewrite the Connection Laplacian as

$$L_g^{\nabla}s = -\sum(\nabla_{X_i}(\nabla_{X_i}s) - \nabla_{D_{X_i}X_i}(s))$$

where D is a covariant derivative over P associated to the metric already defined, and as $\nabla_X s$ is given by $X^h\cdot\varphi$, it follows that

$$L_g^{\widetilde{\nabla}}s \sim \Delta_h\varphi = -\sum(X_i^h\circ X_i^h - D_{X_i^h}X_i^h)\varphi$$

where, Δ_h is the horizontal Laplacian defined for a riemannian submersion with totally geodesic fibers from P to (M,g). It is important to know that the horizontal Laplacian satisfies the properties

$$\Delta_p = \Delta_v + \Delta_h, \quad [\Delta_v,\Delta_h] = 0$$

where Δ_v is the vertical Laplacian and Δ_P the Laplace Beltrami Operator over P.

Consequently, we have that

$$L_g^{\widetilde{\nabla}}s \sim \Delta_h\varphi = \Delta_p\varphi - (\Delta_G\rho)\varphi$$

where G is a Lie group with bi-invariant metric and ρ is the irreducible representation of G on V. For more details see [5,6]

Gauge Transformation

Definition 0.1. *Let $\psi : E \to E$ be a diffeomorphism which maps each fiber isometrically and linearly onto itself. Given a linear connection $\widetilde{\nabla}$ on E, we can define a connection $\psi^*\widetilde{\nabla} = \psi^{-1}\widetilde{\nabla}\psi$ given by gauge transformation.*

Two connections $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ on E are Gauge equivalent to each other if there is a gauge tranformation ψ on E such that $\widetilde{\nabla}' = \psi^*\widetilde{\nabla}$ and denoted by $\widetilde{\nabla} \sim_{(g)} \widetilde{\nabla}'$

Proposition 1. *Suppose $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ on E. If $\widetilde{\nabla} \sim_{(g)} \widetilde{\nabla}'$ holds, then $Sp(M,g,E,\widetilde{\nabla}) = Sp(M,g,E,\widetilde{\nabla}')$. Let ω and ω' be the connection 1-forms of $\widetilde{\nabla}$ and $\widetilde{\nabla}'$, respectively, with respect to a local section e of E. Then, $\widetilde{\nabla} \sim_{(g)} \widetilde{\nabla}'$ holds iff $(\omega-\omega')/2\pi i$ is an integral 1-form*

Proposition 2 (Konstant)**.** *Assume M is simply connected and assume $\Omega/2\pi i$ is an integral 2-form on M. Then, up to gauge equivalence there is a unique line bundle with $\widetilde{\nabla}$ such that the curvature of $\widetilde{\nabla}$ is equal Ω.*

Theorem 0.1. *Let (S^2,g_0) be a the sphere with canonical metric, and $(E_m,\widetilde{\nabla}_m)$, $m \in \mathbb{Z}$ be a complex line bundle with connection whose curvature form is $\Omega_m = i/2\sin dVol_{g_0}$. Then, the $Sp(S^2,g_0,E_m,\widetilde{\nabla}_m)$ is given by the set*

$$\left\{ l(l+1) - \frac{m^2}{4}; l = \frac{|m|}{2}, \frac{|m|}{2}+1, \frac{|m|}{2}+2, \dots \right\}$$

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