Instanton bundles on contact Fano manifolds

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Abstract

In their seminal work Atiyah, Drinfeld, Hitchin and Manin established a correspondence between the ASD solutions of the Yang-Mills equations on the four-sphere S^4 and certain holomorphic vector bundles, referred to as *instantons*, on its twistor space $\mathbb{P}^3$. Both twistor geometry and Yang Mills theory can be generalized to a 4n-dimensional *Quaternion Kähler* manifold M; this allows to define instantons on the twistor space Z, a so called *contact Fano* manifold. We study instantons and their moduli in the case $M = G_2/SO(4)$ and $Z = G_2/U(1) \cdot SU(2)$.

Quaternion Kähler and contact Fano manifolds

A Riemannian manifold (M, g) of real dimension $4n$ is *Quaternion Kähler* if $Hol(M) \subset Sp(n) \cdot Sp(1)$. This holds $\iff \exists \mathcal{G} \subset \mathrm{End}(TM)$ a rank 3 bundle satisfying:

- $\mathcal{G}$ is preserved by the connection induced by the Riemannian connection;
- $\mathcal{G}$ is locally spanned by 3 almost complex structures I, J, K such that $I^2 = J^2 = -1$, $IJ = -JI = K$ and such that g is Hermitian with respect to $I,\ J,\ K$.

The sphere subbundle Z of $\mathcal{G}$, consisting of the $aI+bJ+cK$ of unit norm is the *twistor space of* M.

If M *positive QK* (i.e. has positive scalar curvature) $\Rightarrow Z$ is a *contact Fano manifold* namely K_Z is ample and TZ fits in

$$0 \to F \to TZ \xrightarrow{\theta} L \to 0 \qquad (1)$$

with $d\theta_F : \bigwedge^2 F \to L$ nowhere degenerate.

ASD connections and ASD instanton bundles

The bundle $\bigwedge^2 T^*M$ has the following decomposition:

$$\bigwedge^2 T^*M = S^2\mathsf{H} \oplus S^2\mathsf{E} \oplus (S^2\mathsf{H} \oplus S^2\mathsf{E})^{\perp} \qquad (2)$$

for H and E the bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively.

Let us now consider a connection ∇ on a complex vector bundle F on M. We say that ∇ is *anti-self-dual* if its curvature R_∇ belongs to $End(F) \otimes S^2\mathsf{E}$. Pulling back complex vector bundles endowed with ASD connections via $Z \xrightarrow{\pi} M$ we establish the following 1-1 correspondence:

Ward correspondence. *There exists a 1-1 correspondence between complex vector bundles F with ASD connections on M and holomorphic vector bundles $\tilde{F}$ on Z such that:*

- $\tilde{F}|_{\pi^{-1}(x)}$ *is trivial* $\forall\ x \in M$;
- $\exists$ *an anti-holomorphic isomorphism* $\tau : \tilde{F} \xrightarrow{\simeq} \sigma^*\tilde{F}^*$

where σ denotes the real structure on Z induced by the quaternionic structure on M.

The holomorphic bundles $\tilde{F} = \pi^*(F)$ on Z obtained in this way are referred to as an *ASD instanton bundles*.

Instantons on $G_2/U(1) \cdot SU(2)$

We focus our attention on the case $M = G_2/SO(4)$ and $Z = G_2/U(1) \cdot SU(2)$. Mamone and Capria proved the existence of a rank 3 G_2-homogeneous instanton bundle F_0 on Z. Later, Nagatomo proved that, like it is the case for the projective space $\mathbb{P}^3$, F_0 can be represented as the cohomology of a monad of the form:

$$\mathcal{U} \xrightarrow{f} V \otimes \mathcal{O} \xrightarrow{g} \mathcal{U}^* \qquad (3)$$

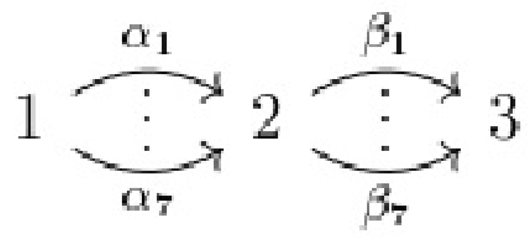
where V is the standard 7-dimensional representation of G_2 and $\mathcal{U}$ is the pullback of the tautological rank 2 bundle on $Gr(2, V)$ via $Z = Gr(2, V) \cap \mathbb{P}(\mathfrak{g}_2) \hookrightarrow Gr(2, V)$.

The family of ASD instantons F with $ch(F) = ch(F_0)$, defined as **1-instantons**, and admitting a monadic representation is actually *complete*.

Theorem (Nagatomo). *Every 1-instanton F on Z is the cohomology of a monad of the form (3).*

Moduli spaces of 1-instantons on $G_2/U(1) \cdot SU(2)$

We can then construct the moduli space of 1-instantons as a *Quiver moduli space* (as done in [3] for instantons on $\mathbb{P}^3$). To each monad (3) we associate a representation of the quiver $\mathcal{Q}$:



(subjected to relations imposed by the condition $g \circ f = 0$). These quiver moduli depend on the choice of a *stability parameter* $\theta = (\alpha, \beta, \gamma) \in \mathbb{Z}^3$ orthogonal to the dimension vector $(1, 7, 1)$ of our representations. The space of stability parameters, which identifies with the (α, γ) plane, has a wall and chamber decomposition: the moduli space is unaltered as soon as the stability parameter varies in a chamber whilst it is subject to a birational transformation when we cross a wall.

Theorem.
- *The moduli space of θ-semistable representations is empty outside the 4-th quadrant of the (α, γ) plane.*
- *In the 4-th quadrant of the (α, γ) plane there exists a unique wall of equation $\alpha = -\gamma$;*
- *Representations of $\mathcal{Q}$ corresponding to ASD-instantons are everywhere stable in the 4-th quadrant of the (α, γ)-plane.*

A more detailed description can be given of the moduli in the two chambers

Theorem. *The moduli spaces in the 4th quadrant of the (α, γ)-plane are both isomorphic to $\mathbb{P}^7$. Moreover:*

- *For $\alpha < -\gamma$ this $\mathbb{P}^7$ is a component of the Gieseker-Maruyama moduli space. This moduli consists of*
  1. *A family of μ-stable vector bundles (containing the ASD instantons) isomorphic to $\mathbb{P}^7 \setminus (\mathbb{P}^6 \cup Q^6)$*
  2. *A family of strictly μ-semistable (but stable) vector bundles isomorphic to $\mathbb{P}^6 \setminus Q^5$;*
  3. *A family of strictly μ-semistable (but stable) sheaves F such that $F^{\vee\vee} \simeq \mathcal{O}^3$ and singular along a $F(0, 1, 2)$, isomorphic to Q_6.*
- *The wall crossing "replaces" the strictly μ-semistable sheaves F with $\mathrm{RH}om(F, \mathcal{O})$.*

Next goals

- How to extend the definition of instanton in order to include all objects appearing in the quiver moduli?
- What about different values of $ch(F)$? Are ASD instantons still cohomologies of monads?
- On $G_2/U(1) \cdot SU(2)$ instantons share several common features with instantons on $\mathbb{P}^3$. Is this still true on other contact Fano manifolds?

References

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[3] Jardim, M, Silva, D. Instanton sheaves and representations of quivers. Proceedings of the Edinburgh Mathematical Society, (2020)