Subgroups of $\mathbf{Diff}(\mathbb{C}, 0)$

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Abstract

We study the group of formal diffeomorphism of $(\mathbb{C}, 0)$ with the usual composition. Then we present the subgroup of holomorphic diffeomorphism of $(\mathbb{C}, 0)$ and analyze the problem of finitude of this subgroups. Finally we present a caracterization of periodic elements of $\text{Diff}(\mathbb{C}, 0)$.

Introduction

The qualitative study of holomorphic vector fields originated essentially with the works of Henri Poincaré (1854-1912). In addition to the qualitative analysis of vector fields, Poincaré developed, in parallel, the study of maps of $\mathbb{C}^n$ in $\mathbb{C}^n$ and its corresponding invariants. This parallel development of the study of analytic maps and vector fields was later continued and deepened by various mathematicians, including Dulac, Siegel, Brjuno, Arnold, Chen, Sternberg, among others.

Objectives

Characterize the periodic elements of the group $\text{Diff}(\mathbb{C}, 0)$.

Results

Consider the following subset of the ring of formal power series.

$$\widehat{\text{Diff}}(\mathbb{C}, 0) = \left\{ \hat{f}(z) := \sum_{j\in\mathbb{N}} a_j z^j \in \mathbb{C}[[z]] \;\middle|\; \begin{array}{l} a_0 = 0, \\ a_1 \neq 0 \end{array} \right\}$$

which is a non-abelian group with product given by the **composition**. An element of $\widehat{\text{Diff}}(\mathbb{C}, 0)$ is called **formal germ of diffeomorphism**.

If $\hat{f} \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ and $\hat{f}(z) = \sum_{j\in\mathbb{N}} a_j z^j$ converges uniformly on some open disc of positive radius, we write $\hat{f} \in \text{Diff}(\mathbb{C}, 0)$. An element of $\text{Diff}(\mathbb{C}, 0)$ is called **germ of diffeomorphism**. Note that

- $\text{Diff}(\mathbb{C}, 0)$ is a subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$.
- The application $\begin{array}{rcl} \mathfrak{T}: \quad (\widehat{\text{Diff}}(\mathbb{C}, 0), \circ) & \longrightarrow & (\mathbb{C}^*, .) \\ \hat{f}(z) := \sum_{j\in\mathbb{N}} a_j z^j & \longrightarrow & \mathfrak{T}(\hat{f}) = a_1 \end{array}$ is a homomorphism of groups.
- If $\hat{g} \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ then $\frac{1}{\mathfrak{T}(\hat{g})}\hat{g}(z) \in \text{Ker}(\mathfrak{T}) = \left\{ \hat{f}(z) := \sum_{j\in\mathbb{N}} a_j z^j \in \widehat{\text{Diff}}(\mathbb{C}, 0) : a_1 = 1 \right\}$.
- For each $\hat{f} \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ define $\ell_{\hat{f}}(z) := \mathfrak{T}(\hat{f})z$, and $\ell_{\hat{f}} \in \text{Diff}(\mathbb{C}, 0)$.

Proposition 1. If G is a **finite** subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$ then

(a) $\{\ell_{\hat{g}} \in \widehat{\text{Diff}}(\mathbb{C}, 0) : g \in G\}$ is a cyclic subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$.

(b) G and $\left\langle z \longmapsto e^{\frac{2\pi i}{|\mathfrak{T}(G)|}} z \right\rangle$ are conjugates (i.e., there is an element $h \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ such that $Gh = h\left\langle z \longmapsto e^{\frac{2\pi i}{|\mathfrak{T}(G)|}} z \right\rangle$).

Corollary 2. Every **finite** subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$ is cyclic.

Note that for each $\hat{f} \in \mathbb{C}[[z]]$ not null we have that $H(\hat{f}) := \{\hat{h} \in \widehat{\text{Diff}}(\mathbb{C}, 0) : \hat{f} \circ \hat{h} = \hat{f}\}$ is a subgroup of $\widehat{\text{Diff}}(\mathbb{C}, 0)$, which is called **invariance group of** $\hat{f}$.

Proposition 3. Let $\hat{f} \in \mathbb{C}[[z]]$ of order ν then $H(\hat{f})$ and $\left\langle z \longmapsto e^{\frac{2\pi i}{\nu}} z \right\rangle$ are conjugates. In particular, $H(\hat{f})$ is finite.

Corollary 4. If $\hat{f} \in \text{Diff}(\mathbb{C}, 0)$ is of order ν then $H(\hat{f}) \subseteq \text{Diff}(\mathbb{C}, 0)$.

Corollary 5. Let $\hat{f} \in \mathbb{C}[[z]]$ of order ν. If $H(\hat{f}) \subseteq \text{Diff}(\mathbb{C}, 0)$ there exists $\hat{l} \in \widehat{\text{Diff}}(\mathbb{C}, 0)$ such that $\hat{l} \circ \hat{f}$ is analytic.

Lemma 6. Let H be a subgroup of $\text{Diff}(\mathbb{C}, 0)$ such that every element of $h \in H$ is periodic. Then H is abelian.

Proof. Indeed, let $h, g \in H$ we have that $[h, g] \in \text{Diff}(\mathbb{C}, 0)$, then $\mathfrak{T}(h^{-1} \circ g^{-1} \circ h \circ g) = 1$. Therefore $[h, g](z) = (h^{-1} \circ g^{-1} \circ h \circ g)(z) = z + a_\nu z^\nu + \cdots$. As $[h, g] \in H$ then there exists $n \in \mathbb{N}$ such that $[h, g]^{\circ n} = \text{id}$ so $a_\nu = 0$. Thus $[h, g](z) = z$ and $\langle h^{-1} \circ g^{-1} \circ h \circ g : h, g \in H \rangle = \{\text{id}\}$. From group theory, then H is an abelian subgroup of $\text{Diff}(\mathbb{C}, 0)$. □

Example 7. The cyclic group of mth roots of unity in $\mathbb{C}$ will be denoted by $U_m = \{z \in \mathbb{C} : z^m = 1\} = \langle e^{\frac{2\pi i}{m}} \rangle$ then the set of all roots of unity is the set

$$U = \bigcup_{m=1}^{\infty} U_m = \bigcup_{m=1}^{\infty} \langle e^{\frac{2\pi i}{m}} \rangle$$

which is also an abelian group. Consider the abelian subgroup

$$H = \left\langle z \longmapsto e^{\frac{2\pi i}{m}} z : m \geq 1 \right\rangle$$

of $\text{Diff}(\mathbb{C}, 0)$. Note that every element of H is periodic, but H is not finite.

Lemma 8. Let H be a finitely generated subgroup of $\text{Diff}(\mathbb{C}, 0)$ such that every element of $h \in H$ is periodic. Then H is finite.

Definition 9. Consider $h \in \text{Diff}(\mathbb{C}, 0)$ and U be an open neighborhood of 0 on which h is defined, holomorphic and injective. Let $V \subseteq U$ and $z \in V$ then

- the **V-orbit of a point** z is the set

$$\begin{aligned} O_V(z) = \{h^{\circ p}(z) : h(z), \cdots, h^{\circ p}(z) \in V\} \cup \\ \cup \left\{h^{\circ -q}(z) : h^{-1}(z), \cdots, h^{\circ -q}(z) \in V\right\} \cup \{z\}, \end{aligned}$$

- the **number of iterations of x in V** is

$$\begin{aligned} \mu_V(z) = \sup\{p > 0 : h^{\circ p}(z) \in O_V(z)\} \\ + \sup\{p > 0 : h^{\circ(-p)}(z) \in O_V(z)\} + 1. \end{aligned}$$

We need the following ingredients to achieve our goal

Lemma 10. (J. Lewowicz.) Let K be a connected compact neighborhood of $0 \in \mathbb{R}^n$ and $h : K \to \mathbb{R}^n$ an embedding satisfying $h(0) = 0$. There then exists a point x of the boundary ∂K of K whose number of iterations in K is infinite.

Theorem 11. (Sura-Bura) Every compact component A of a locally compact Hausdorff space X has a neighborhood base in X consisting of open compact subsets of X.

Finally our main goal.

Proposition 12. Suppose that $h \in \text{Diff}(\mathbb{C}, 0)$ is not periodic, then there exists a fundamental system of neighborhoods $\mathcal{U}$ of $0 \in \mathbb{C}$ such that for each $U \in \mathcal{U}$ we have that

(a) the set $\{x \in U : O_U(x) \text{ is infinite}\}$ is uncountable, and

(b) $0 \in \overline{\{x \in U : O_U(x) \text{ is infinite}\}}$.

Corollary 13. An element $f \in \text{Diff}(\mathbb{C}, 0)$ has finite order (i.e., $f^{\circ n} = \text{id}$ for some $n \in \mathbb{N}$) if, and only if, for some neighborhood V of 0 all the orbits of f in V are finite.

Conclusion

There exists infinite subgroups of $\text{Diff}(\mathbb{C}, 0)$ in which all of its elements are periodics. Also, Corollary 13 proved the caracterization of periodic elements of $\text{Diff}(\mathbb{C}, 0)$.

References

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