Stability of a class of iterative methods for solving nonlinear vectorial systems

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We propose the follow parametric class of iterative schemes for approximating the solutions of nonlinear systems:

$$\begin{aligned} y^{(i)} &= z^{(i)} - \alpha\, F'(z^{(i)})^{-1} F(z^{(i)}), \\ z^{(i+1)} &= z^{(i)} - \Big(\beta\, F'(y^{(i)})^{-1} F'(z^{(i)})\, F'(y^{(i)})^{-1} + \\ &\quad + \gamma\, F'(y^{(i)})^{-1} + \mu\, F'(z^{(i)})^{-1} + \\ &\quad + \delta\, F'(z^{(i)})^{-1} F'(y^{(i)})\, F'(z^{(i)})^{-1}\Big)\, F(z^{(i)}) \end{aligned} \qquad (1)$$

where $\alpha, \beta, \gamma, \mu, \delta \in \mathbb{R}$ should be chosen in order to obtain the four-order of convergence, established by next theorem.

Theorem 1. *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable at each point of an open neighborhood D of $\bar{z} \in \mathbb{R}^n$, that is a solution of the system $F(x) = 0$ and the initial estimation $z^{(0)}$ is close enough to $\bar{z}$. Let us suppose that $F'(x)$ is continuous and nonsingular in $\bar{z}$. Then, sequence $\{z^{(i)}\}_{i\geq 0}$ obtained from expression (1) converges to $\bar{z}$, with order 4, when $\alpha = \frac{2}{3}$, $\beta = \frac{3}{8} - \frac{\gamma}{3}$, $\mu = \frac{5}{8} - \gamma$ and $\delta = \frac{\gamma}{3}$, with parameter γ.*

Stability analysis

By using real multidimensional dynamics tools we determine the elements of these family that posses better performance, in terms of their convergence on the initial estimations used.

We denote by $Op^4(x, \gamma)$ this parametric family applied to n-variable polynomial system $p_i(x) = x_i^2 - 1 = 0$.

x^* is a fixed point if $Op^4(x^*, \gamma) = x^*$, and it is strange fixed point when it is not a root of $p(x) = 0$.

The stability of the fixed points x^* depends on the eigenvalues λ_i of the Jacobian matrix of $Op^4(x^*, \gamma)$. It is attracting if all $|\lambda_j| < 1$, repelling if all $|\lambda_j| > 1$, and saddle if at least one $|\lambda_{j_0}| > 1$.

Theorem 2. *The rational function $Op^4(x, \gamma)$ has 2^n superattracting fixed points whose components are roots of $p(x)$. This operator also has real strange fixed points whose components are combinations of the real roots of polynomial $q(t) = t^6(8\gamma - 423) + t^4(-24\gamma - 180) + t^2(24\gamma - 45) - 8\gamma$ depending on γ, denoted by $q_i(\gamma)$ and the roots of $p(x)$:*

- *If $\gamma < 0$ or $\gamma > \frac{423}{8}$ the roots $q_i(\gamma)$, $i = 1, 2$, are real. Moreover, their eigenvalues of the Jacobian matrix are greater than one (in absolute value). So, the strange fixed point are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) are equal to 1 or -1, it will be a saddle fixed point.*
- *If $0 < \gamma < \frac{423}{8}$, then the roots of polynomial $q(t)$ are complex and there not exist any real strange fixed point.*

We obtain the critical points of $Op^4(x, \gamma)$ i. e. the values of x that make all the eigenvalues of its Jacobian matrix null. When this critical point is not a solution of $p(x) = 0$, are free critical point. In order to analyze other attracting behavior, as attracting periodic orbits or even strange attractors, we can analyze the orbits of the free critical points, if they exist.

Theorem 3. *The components of the free critical points of $Op^4(x, \gamma)$ are the real roots $z_i(\gamma) \neq 0$ of the polynomial $z(s) = s^4(16\gamma + 306) + s^2(152\gamma + 45) + 24\gamma$ for some γ :*

- *If $\gamma \leq -\frac{153}{8}$, $\gamma = -\frac{177}{64}$ or $\gamma \geq 0$, then there not exist free critical points, i. e, the critical points are the roots of $p(x)$.*
- *If $-\frac{153}{8} < \gamma < -\frac{177}{64}$ or $-\frac{177}{64} < \gamma < 0$, then the free critical points are combinations of $z_i(\gamma)$, $i = 1, 2$ or ± 1 (but not all ± 1).*

In the particular case $n = 2$, we can see in Figure 1 that for $\gamma = 10$ there exists only one connected component of each basin of attraction, but $\gamma = -20$ have infinite connected components.

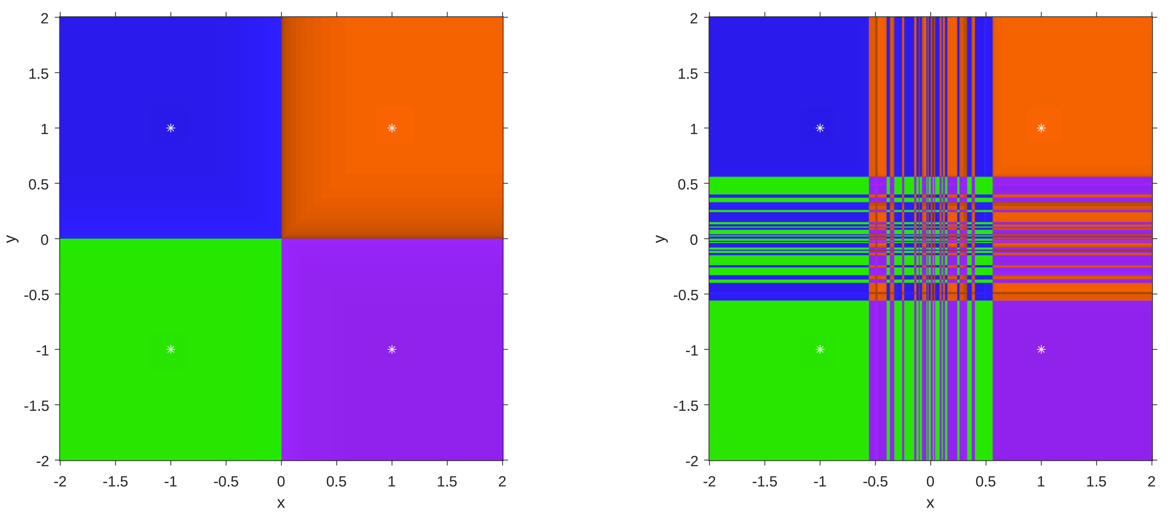

Figura 1: Stable dynamical planes ($\gamma = 10, \gamma = -20$)

Most of the free critical points converges to the roots, only the case $(z_1(\gamma), z_1(\gamma))$ when $-\frac{153}{8} < \gamma < -\frac{177}{64}$, present a black small region around $\gamma = -18.75$ and a narrower one around $\gamma = -16.9$.

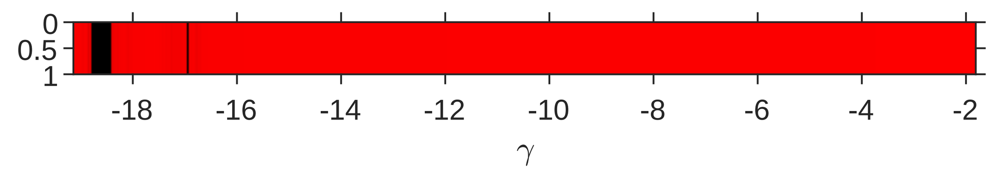

Figura 2: Parameter line for $-\frac{153}{8} < \gamma < -\frac{177}{64}$,

We use Feigenbaum diagrams to analyze the bifurcations, starting with each free critical points and observing the γ behavior after 500 iterations in a mesh of 3000 subintervals.

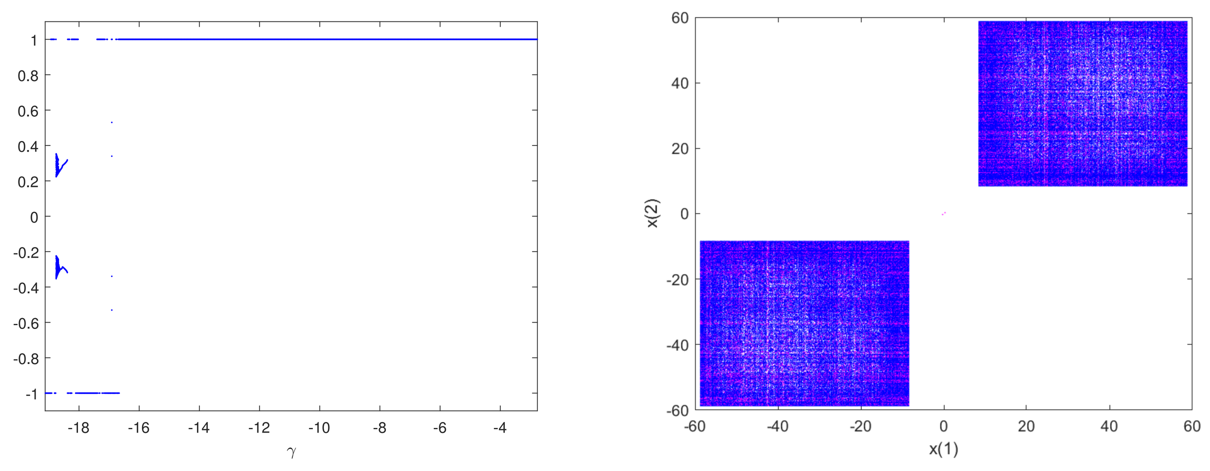

Figura 3: Feigenbaum diagram for $-\frac{153}{8} < \gamma < -\frac{177}{64}$ and strange attractors for $\gamma = -18.75$

Left Figure 3 corresponds to the bifurcation diagrams in the black area for $-\frac{153}{8} < \gamma < -\frac{177}{64}$. In general, we observe convergence to one of the roots, but in a small interval around $\gamma = -18.75$ several period-doubling cascades appear. Right Figure 3 show, with 2500 different initial estimations the (x_1, x_2)-space the orbit of $x^{(0)} = (0.29, 0.29)$ by $Op^4\left((x_1, x_2), \gamma\right)$, for $\gamma = -18.75$. This unstable performance, can be checked by plotting the associated dynamical planes associated of γ in the black regions of the parameter line (Figure 2). For example, the phase space for $\gamma = -18.45$ and $\gamma = -16.91$ in Figure 4. In them, 4-period orbits appear linked by yellow lines. In all cases, more attracting orbits exist, with symmetric coordinates.

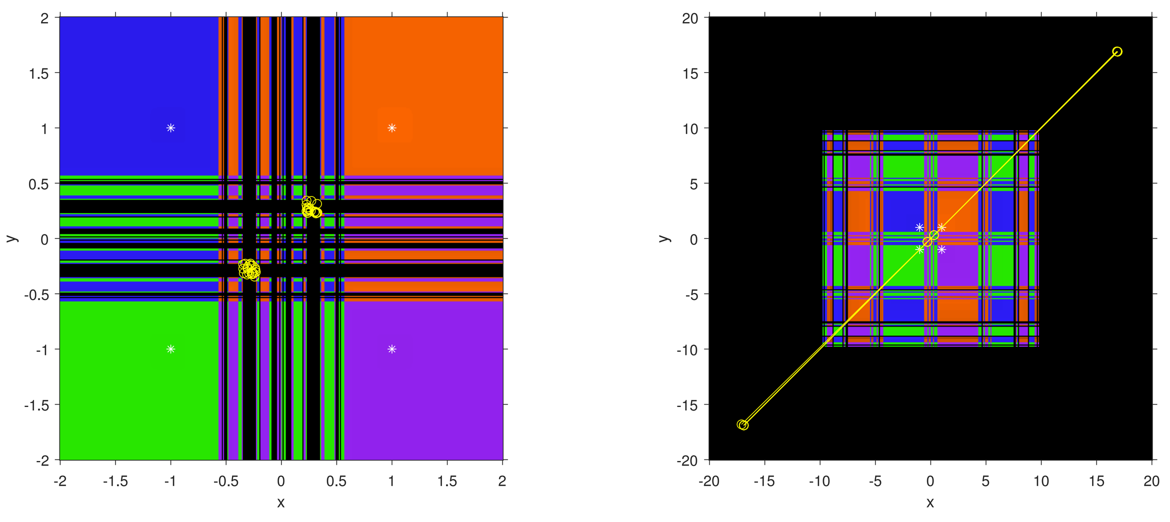

Figura 4: Unstable dynamical planes ($\gamma = -18.75, \gamma = -16.91$)

Conclusion

The main performance of this class of iterative methods on this kind of polynomial systems is very stable. Some numerical tests show the performance of the new methods, confirm the theoretical results and allow to compare the proposed schemes with other known ones.

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