Integer linear optimization inside an n-dimensional hypersphere

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Abstract

In this poster, we present an algorithm to solve the problem of maximizing a linear function with positive integer variables on the ball defined by a hypersphere centered at the origin and with radius r, in a n-dimensional space. To this purpose, we estimate an upper bound for the number of integer points contained in the ball, and we also prove that the ball contains a polytope of integer vertices with the property that the interior of the region bounded by the polytope and the hypersphere does not contain integer points. Using these results, we prove that, once the space dimension is fixed, the algorithm is polynomial in terms of the radius of the hypersphere.

Notations

$\hat{0} \in \mathbb{R}^n$ and $\hat{1} \in \mathbb{R}^n$ denote the vectors with all components equal to 0 and 1, respectively, and $\{e_i\}_{j=1}^n$ denotes the canonical base of $\mathbb{R}^n$. For $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the integer part of r. In addition, we define the following sets

$S_{n,r} = \{x \in \mathbb{R}^n : ||x|| \leq r\}$, where $||,||$ is the Euclidean norm;

$B_r = \{x \in \mathbb{R}^n : |x_i| \leq \lfloor r \rfloor, i \in \{1,..,n\}\}$;

$ch(W)$ is the convex hull of $W \subset \mathbb{R}^n$;

V_W is the vertex set of $ch(W)$ where $W \subset \mathbb{Z}^n$ finite set.

The IOS_r problem.

Let $r \in \mathbb{R}_+^*$ and $c \in \mathbb{R}_+^n$ be given, we consider the problem

$$IOS_r : \max\{\, cx : x \in S_{n,r} \cap \mathbb{Z}_+^n\}.$$

Theorem 1. Let $n \in \mathbb{N}$ and $r \in \mathbb{R}$. If $r \geq 3$ then $|S_{n.r} \cap \mathbb{Z}_+^n| \leq (\lfloor r \rfloor + 1)^n - 1$.

Definition. Let P be a non-empty subset of $\mathbb{R}^n$. We say that P is an integral polytope if there exist $V \subset \mathbb{Z}^n$ such that $P = ch(V)$.

Theorem 2. Let $r \in \mathbb{R}_+^*$. There exists an integral polytope P such that $S_{n,r} \cap \mathbb{Z}_+^n \subset P \subset S_{n,r}$.

Consider a set $L \subset S_r \cap \mathbb{Z}_+^n$ such that

$$|(S_{n,r} \setminus ch(L)) \cap \mathbb{Z}_+^n| = 0.$$

The IOS_r problem can be solved by solving the equivalent problem $P : max\{cx : x \in ch(L) \cap \mathbb{Z}_+^n\}$. The linear programming problem obtained by relaxing the problem P, $P' : max\{cx : x \in ch(L) \cap \mathbb{R}_+^n\}$, reaches its optimal value at some point in $V_{ch(L)}$. So, the problem P' is equivalent to the problem $P'' : max\{cx : x \in V_{ch(L)} \cap \mathbb{Z}_+^n\}$, which means that problems P and P'' are also equivalente and they can be reformulated as as $max\{cx : x \in L\}$.

The $AIOS_r$ Algorithm.

The algorithm AIOSr presented below finds a set L as described above

Let $f_r : S_{n-2,r} \to \mathbb{R}$ and $g_r : S_{n,r} \to \mathbb{R}$ define by

$$f_r(x_1, \ldots, x_{n-2}) = \left\lfloor \sqrt{r^2 - \|(x_1, \ldots, x_{n-2})\|^2} \right\rfloor,$$

$$g_r(x_1, \ldots, x_n) = \left\lfloor \sqrt{r^2 - \|(x_1, \ldots, x_{n-1})\|^2} - x_n \right\rfloor.$$

Algorithm 1 AIOS_r

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Data: n ∈ ℕ, n ≥ 2, r ∈ ℝ+, c ∈ ℝ+^n
Result: a* the optimal solution of the IOSr problem (1)
Initialize: a* = 0
if n = 2 then
    a^0 = (⌊r⌋ + 1, 0)
    for k = 1 to (⌊r⌋ + 1) − ⌊r/√2⌋ do
        a^k = a^{k−1} − e_1 + g_r(a^{k−1} − e_1)e_2
        a* = argmax{ca*, ca^k, câ^k}
    end for
    for i_1 = 0 to f_r(0, 0, ..., 0) do
        ⋮
        for i_{n−3} = 0 to f_r(i_1, ..., i_{n−4}, 0, 0) do
            for i_{n−2} = 0 to f_r(i_1, i_2, ..., i_{n−3}, 0) do
                r̄ = √(r^2 − Σ_{l=1}^{n−2} i_l^2)
                a^0 = (i_1, ..., i_{n−2}, f_r(i_1, i_2, ..., i_{n−3}, 0) + 1, 0)
                for k = 1 to (⌊r̄⌋ + 1) − ⌊r̄/√2⌋ do
                    a^k = a^{k−1} − e_{n−1} + g_r(a^{k−1} − e_{n−1})e_n
                    a* = argmax{ca*, ca^k, câ^k}
                end for
            end for
        end for
        ⋮
    end for
end if
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Theorem 4. The algorithm AOIS_r finds an optimal solution of the OIS_r problem.

Theorem 5. In order to find a solution of the OIS_r problem, the algorithm AOIS_r requires at most $O(r^{n-1})$ iterations and performs at most $O(nr^{n-1})$ basic operations real numbers (the addition, subtraction, multiplication and comparison of two real numbers, and taking the square root and the integer part of a real number).

Conclusions

We have presented a pseudo polynomial algorithm for solving the integer nonlinear programming problem IOS_R.The proposed algorithm, which builds on the reformulation of this problem as a linear programming problem, has an iteration complexity of $O(nr^{n-1})$, which means that the problem IOS_R is polynomial when we fix the dimension n of the space where the problem is defined. As future work, we intend to develop a ”descent-type”algorithm to solve the Knapsack problem using the methodology presented int his work to generate sequences of interior feasible points with decreasing values of the objective function.

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