On area estimates for stable H-hypersurfaces and applications

Elaine Sampaio

Universidade Federal de Alagoas - Instituto de Matemática

elaine.carlos@im.ufal.br

Abstract

We show some area estimates for a stable constant mean curvature (CMC) hypersurface immersed in a Riemannian manifold which has either scalar or sectional curvature bounded from below. Our results can be read as natural extensions to CMC context of the results to stable minimal surfaces due to Munteanu, Sung and Wang [2]. As applications we derive upper estimates for the bottom spectrum of these hypersurfaces.

Introduction

A minimal hypersurface Σ is said stable if it minimizes the area functional $\mathcal{A}(t)$ up the second order for all compactly supported variations. The concept of CMC stability has been extensively investigated in the last decades since it was introduced by Barbosa and do Carmo [1]. CMC hypersurfaces with mean curvature H, or simply H-hypersurfaces, are critical points of the functional

$$\mathcal{J}(t) = \mathcal{A}(t) - 2H\mathcal{V}(t),$$

where $\mathcal{A}(t)$ and $\mathcal{V}(t)$ denote the area and the volume enclosed of the variation between 0 and t. An H-hypersurface is said to be *strongly stable* if, for compactly supported variations,

$$\frac{d^2}{dt^2}\mathcal{J}(t)\bigg|_{t=0} \geq 0.$$

This is a natural generalization of the stability of a minimal surface ($H = 0$). Its is possible to show that if Σ is a strongly stable H-surface in M, then

$$\int_\Sigma (|h|^2 + \mathrm{Ric}(\nu,\nu))f^2 \leq \int_\Sigma |\nabla f|^2, \quad \forall f \in C_0^\infty(\Sigma).$$

Results

Inspired by [2], we show some area estimates for stable H-surfaces. Our first result can be stated as

Theorem 1. *Let $B_p(R)$ be a geodesic ball in a stable H-surface Σ, with $|H| < 1$, in a three dimensional manifold M. Assume that $B_p(R)$ does not intersect the boundary of Σ.*

1. *If the scalar curvature S of M satisfies $S \geq -6$, then*
$$A(R) \leq C_1 e^{2\sqrt{1-H^2}R}$$
for some absolute constant $C_1 > 0$.
2. *If the sectional curvature K of M satisfies $K \geq -1$, then*
$$A(R) \leq C_1 e^{\frac{4\sqrt{1-H^2}}{\sqrt{7}}R}$$
for some absolute constant $C_1 > 0$.

When $M = \mathbb{H}^2 \times \mathbb{R}$, we can improve the constant in Theorem 1. More precisely, if $H^2 < \dfrac{1}{2}$, then

$$A(R) \leq Ce^{aR}, \quad \text{where } a = \frac{4\sqrt{\frac{1}{2} - H^2}}{\sqrt{7}}.$$

Since the geometry of minimal surfaces in the euclidean space $\mathbb{R}^3$ is very similar to the geometry of 1-surfaces in $\mathbb{H}^3$, we have:

Theorem 2. *Let Σ be a stable 1-surface in $\mathbb{H}^3$. Then there exists a universal constant R_0 such that for any geodesic ball $B_p(R)$ with no intersection with the boundary of Σ, it holds*

$$L(r) \leq 2\pi r\left(1 + \frac{10}{\ln R}\right); \quad A(r) \leq \pi r^2\left(1 + \frac{10}{\ln R}\right),$$

for all $r \leq \sqrt{R}$ and $R \geq R_0$. Here $L(r)$ denotes the length of geodesic circle $\partial B_p(r)$ and $A(r)$ denotes the area of $B_p(r)$.

As a direct application, we have a new proof of the following well known result proved by da Silveira [4].

Corollary 3. *Let Σ be a complete noncompact stable 1-surface in $\mathbb{H}^3$, then*

$$A(r) \leq \pi r^2$$

for all $r > 0$. In particular, Σ is a horosphere.

Application: Bottom spectrum estimates

The bottom spectrum of a complete manifold Σ can be defined as the best constant of the Poincaré inequality

$$\lambda_0(\Sigma)\int_\Sigma \phi^2 \leq \int_\Sigma |\nabla\phi|^2,$$

where $\phi \in C_0^\infty(\Sigma)$. According to Li and Wang [5], we have

$$\lambda_0(\Sigma) \leq \frac{1}{4}\left(\liminf_{R\to\infty}\frac{\ln A(R)}{R}\right)^2. \tag{0.1}$$

We can give an upper estimate of the bottom spectrum. More precisely, we have

Corollary 4. *Let Σ be a stable complete H-surface, with $|H| < 1$, in a three dimensional Riemannian manifold M:*

a) *If the scalar curvature S of M satisfies $S \geq -6$, then*
$$\lambda_0(\Sigma) \leq 1 - H^2.$$

b) *If the sectional curvature K of M satisfies $K \geq -1$, then*
$$\lambda_0(\Sigma) \leq \frac{4 - 4H^2}{7}.$$

Higher dimension

In higher dimensions, our approach does not work to get volume estimates for geodesic balls. However, following the technique in [2], we were able to show the following bottom spectrum estimate:

Theorem 5. *Let Σ be a complete stable H-hypersurface in a $(n+1)$-manifold M with $n = 3$ or $n = 4$. If the sectional curvature K of M satisfies $K \geq -1$, then*

a) *For $n = 3$, if $|H| < \dfrac{\sqrt{10}}{3}$, we have that*
$$\lambda_0(\Sigma) \leq \frac{10 - 9H^2}{4};$$

b) *For $n = 4$, if $|H| < \dfrac{\sqrt{7}}{2}$, then*
$$\lambda_0(\Sigma) \leq 24 - 12H^2.$$

References

[1] J. L. Barbosa, M. do Carmo. *Stability of hypersurfaces with constant mean curvature.* Math. Z. 185, 339-353. 1984

[2] O. Munteanu, C. J. A. Sung, J. Wang. *Area and spectrum estimates for stable minimal surfaces.* The Journal of Geometric Analysis (2023), 33(2), 1-34.

[3] M.Ranieri, E. Sampaio, F. Vitório. A note on area estimates for stable H-hypersurfaces and applications. arXiv preprint arXiv:2305.03155 (2023).

[4] A. da Silveira. *Stability of complete noncompact surfaces with constant mean curvature.* Math. Ann. 277, (1987), 629-638.

[5] P. Li, J. Wang. *Complete manifolds with positive spectrum.* J. Differential Geom. 58 (2001), 501-534.