Log Calabi-Yau geometry and Cremona maps

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Abstract

In the context of algebraic geometry, decomposition and inertia groups are special subgroups of the Cremona group which preserve a certain subvariety of $\mathbb{P}^n$ as a set and pointwise, respectively. Castelnuovo, and more recently, Blanc, Pan and Vust have proved a bunch of interesting results and have provided some descriptions of them for the case of plane curves. In the particular case where this fixed subvariety is a hypersurface of degree $n+1$, we have the notion of Calabi-Yau pair which allows us to use new tools to deal with the study of these groups and one of them is the so-called volume preserving Sarkisov Program. Using this different approach it is possible to recover a result by Pan and deduce some very nice properties of birational plane maps that preserve a nonsingular cubic. With the same strategy and regarding the 3-dimensional case, we give a description of which weighted blowups are volume preserving (or crepant) for the case of a normal quartic surface having canonical singularities.

Introduction

The study of Calabi-Yau pairs has been an active research area in Algebraic Geometry. In part, this is because they can be seen as distinguished minimal models of the classical Minimal Model Program (MMP) or its log version. Moreover, the interior of a maximal log Calabi-Yau pair is expected to have notable properties predicted from mirror symmetry. One important tool in the study of Calabi-Yau pairs is a relatively new version of the Sarkisov Program for volume preserving maps between Mori fibered Calabi-Yau pairs [CK, Theorem 1.1].

A *Calabi-Yau (CY) pair* is a pair (X, D) with mild singularities consisting of a normal projective variety X and a reduced Weil divisor on X such that $K_X + D \sim 0$. Many results on decomposition and inertia groups can be interpreted as statements about the *birational geometry of the Calabi-Yau pair* $(\mathbb{P}^n, D)$.

1 Decomposition group of a nonsingular plane cubic

Under restrictions on the singularities of a CY pair $(\mathbb{P}^n, D)$, the decomposition group of the hypersurface D, denoted by $\mathrm{Dec}(D)$ or simply $\mathrm{Bir}(\mathbb{P}^n, D)$, coincides with the group of birational self maps of $\mathbb{P}^n$ that preserve a volume form ω satisfying $D + \mathrm{div}(\omega) = 0$, up to scaling. Such maps are naturally called *volume preserving*.

The Sarkisov Program asserts that we can factorize any Cremona map, an element of $\mathrm{Bir}(\mathbb{P}^n)$, into a composition of elementary links between Mori fibered spaces, the so-called *Sarkisov links*:

$$
\begin{array}{ccccccc}
\mathbb{P}^n = X_0 & \dashrightarrow & X_1 & \dashrightarrow \cdots \dashrightarrow & X_{m-1} & \dashrightarrow & X_m = \mathbb{P}^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) = Y_0 & & Y_1 & & Y_{m-1} & & Y_m = \mathrm{Spec}(\mathbb{C})
\end{array}
$$

There also exists a volume preserving version of this result established by Corti & Kaloghiros [CK].

Theorem 1.1 ([Alv2]). *Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. The standard Sarkisov Program applied to an element of $\mathrm{Dec}(C)$ is automatically volume preserving.*

The following result negatively answers the question posed in [BPV]:

Theorem 1.2 ([Alv2]). *The canonical complex of the pair $(\mathbb{P}^2, C)$ does not admit any splitting at C when we write $\mathrm{Aut}(C) = C \rtimes \mathbb{Z}_d$.*

2 Birational geometry of log Calabi-Yau pairs $(\mathbb{P}^3, D)$ of coregularity 2

Given a CY pair (X, D), the coregularity is the most important discrete volume preserving invariant. It is defined to be the dimension of the smallest log canonical center on a dlt modification of (X, D) and denoted by $\mathrm{coreg}(X, D)$.

The case of $\mathrm{coreg}(\mathbb{P}^3, D) = 2$ occurs if and only if D is a normal quartic surface with canonical singularities.

The first step in a Sarkisov decomposition of a Cremona map always consists of a divisorial extraction. In the 3-dimensional case and 0-dimensional center $P \in \mathbb{P}^3$, by [Kaw, Theorem 1.1], in suitable analytic coordinates this divisorial extraction can be described as the weighted blowup of P with weights $(1, a, b)$, where $\mathrm{GCD}(a, b) = 1$. We call such map a *Kawakita blowup* of P.

The following result extends the classification given in [ACM] contemplating all types of surface canonical singularities that can be corresponded with simple-laced Dynkin diagrams of type ADE.

Theorem 2.1 ([Alv1]). *Let $(\mathbb{P}^3, D)$ be a log Calabi-Yau pair of coregularity 2 and $\pi\colon (X, D_X) \to (\mathbb{P}^3, D)$ be a volume preserving toric $(1, a, b)$-weighted blowup of a closed point. Then this point is necessarily a singularity of D and, up to permutation, the only possibilities for the weights, depending on the type of singularities, are listed in the following Table 1.*

type of singularity	volume preserving weights
A_1	(1,1,1)
A_2	(1,1,1), (1,1,2),
A_3	(1,1,1), (1,1,2), (1,1,3)
A_4	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
A_5	(1,1,1), (1,1,2), (1,1,3), (1,2,3)
$A_{\geq 6}$	(1,1,1), (1,1,2), (1,1,3), (1,2,3), (1,2,5)
D_4	(1,1,1), (1,1,2)
$D_{\geq 5}$	(1,1,1), (1,1,2), (1,2,3)
E_6	(1,1,1), (1,1,2), (1,2,3)
E_7	(1,1,1), (1,1,2), (1,2,3)
E_8	(1,1,1), (1,1,2), (1,2,3)

Table 1: Table summarizing volume preserving weights obtained, up to permutation.

Remark 2.2. The colorful weights obtained are not volume preserving for a generic D in the corresponding coarse moduli space of such quartics.

Theorem 2.1 can be regarded as a first step in the explicit classification CY pairs $(\mathbb{P}^3, D)$ of coregularity two, up to volume preserving equivalence. Such classification is already settled for coregularity ≤ 1 by Ducat.

References

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