Fixed product maps, alternative division rings and the octonion ring

Douglas de Araujo Smigly

Institute of Mathematics and Statistics
University of São Paulo-Brazil (IME-USP)

dsmigly@ime.usp.br

34° Colóquio Brasileiro de Matemática - IMPA, Rio de Janeiro

Abstract

The goal of this work is the characterization of maps between alternative division rings taking products equal to one fixed element to products equal to another fixed element and study when these maps are either automorphisms or anti-automorphisms. Moreover, we give additional results about these maps on octonion rings.

This is a joint work with Bruno L. M. Ferreira (Federal University of Technology-Paraná) and Elisabete Barreiro (University of Coimbra).

Keywords: Alternative Algebras; automorphisms; alternative division rings; Octonion ring.

Introduction

Alternative rings arose out of the work of Ruth Moufang in the 1930's [1], she turned her attention to the multiplicative structure of an alternative division ring.

In 1949, Hua [4] proved that every bijective semi-automorphism on a division ring is an automorphism or an anti-automorphism. This result was reformulated by Artin in 1957 as [5, Theorem1.15]. In 2005, Chebotar *et. al* [2] extended Artin's result towards a larger class of maps. Recently, Ferreira and Ferreira [3] in 2019 extended Hua's result to a class of alternative rings.

Objetives

In the present work we provide a description of additive maps between alternative division rings taking products equal to one fixed element to products equal to another fixed element. More concretely, we would like to answer the following question:

> **Question** *Let $\mathbb{A}$ be an alternative division ring. Fix two elements c, d in $\mathbb{A}$. Can we find a description of a bijective additive map $\varphi\colon \mathbb{A} \to \mathbb{A}$ such that $\varphi(x)\varphi(y) = d$ whenever $xy = c$?*

We characterize this maps in case of alternative division rings and hence in octonion ring. First, under certain assumptions, we are able to achieve that every bijective additive map is a multiple of a semi-automorphism. Even more, we provide a description for a specific case of ring, namely the Octonion ring.

Main Results

Inspired by the result of Chebotar *et. al*, for achieve this aim we establish a result for the class of alternative division rings that reads as follows:

> **Theorem 1.** *Let $\mathbb{A}$ be an alternative division ring with commutative center $\mathcal{Z}(\mathbb{A})$ and $\varphi : \mathbb{A} \to \mathbb{A}$ a bijective additive map with $\lambda = \varphi(1)$. In this condition, the two assertions*
>
> *1. $\varphi(a^{-1})\varphi(a) = \varphi(b^{-1})\varphi(b)$ for any nonzero $a, b \in \mathbb{A}$;*
>
> *2. $\lambda^3\varphi(ab \cdot c + c \cdot ba) = (\varphi(a)\varphi(b))(\lambda\varphi(c)) + (\varphi(c)\lambda)(\varphi(b)\varphi(a))$ for all $a, b, c \in \mathbb{A}$;*
>
> *are satisfied if and only if $\varphi = \lambda\xi$, where $\lambda = \varphi(1) \in \mathcal{Z}(\mathcal{A})$ and $\xi\colon \mathbb{A} \to \mathbb{A}$ is either an automorphism or an anti-automorphism.*

The main ingredients of the proof are a direct consequence of the well-known Hua's identity for the case of alternative division rings

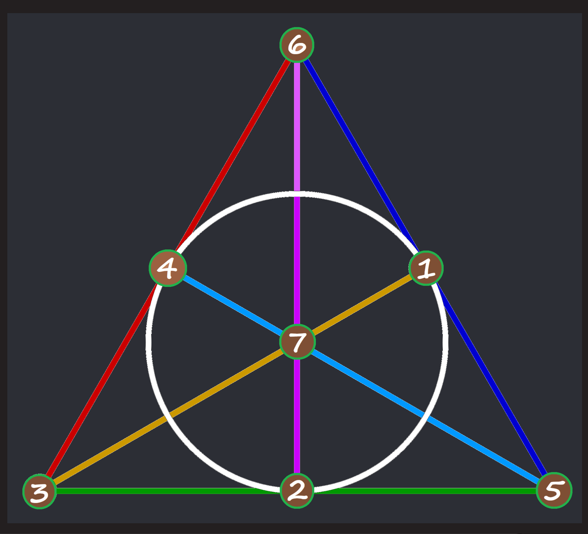
$$\left(a^{-1} - (a - b^{-1})^{-1}\right)^{-1} = a - aba$$

and some appointments about φ and $\mathcal{Z}(\mathcal{A})$, such as $\varphi(1) \in \mathcal{Z}(\mathcal{A})$ and $\varphi(1^{-1})\varphi(1) \in \mathcal{Z}(\mathcal{A})$.

> **Theorem 2.** *Let $\mathbb{A}$ be an alternative division ring. Consider $\varphi\colon \mathbb{A} \to \mathbb{A}$ a bijective additive map such that the following statements hold true:*
>
> ♥ *For all $a, b, c \in \mathbb{A}$, we have that*
>
> $$\varphi(ab{\cdot}c + c{\cdot}ba) = \varphi(a)\varphi(b){\cdot}\varphi(c) + \varphi(c){\cdot}\varphi(b)\varphi(a);$$
>
> ♣ *$\varphi(a^{-1}) = \varphi(a)^{-1}$ for any nonzero element a in $\mathbb{A}$;*
>
> ♦ *$\varphi(1) = 1$.*
>
> *Then φ is either an automorphism or an anti-automorphism.*

For the next results, we will denote the octonion ring on generators $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ by $\mathbb{O}$, and their multiplication is given in Table , which describes the result of multiplying the element in the i-th row by the element in the j-th column:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_1	$-e_1$	e_4	$-e_3$	e_6	$-e_5$	$-e_8$	e_7
e_3	e_3	$-e_4$	$-e_1$	e_2	e_7	e_8	$-e_5$	$-e_6$
e_4	e_4	e_3	$-e_2$	$-e_1$	e_8	$-e_7$	e_6	$-e_5$
e_5	e_5	$-e_6$	$-e_7$	$-e_8$	$-e_1$	e_2	e_3	e_4
e_6	e_6	e_5	$-e_8$	e_7	$-e_2$	$-e_1$	$-e_4$	e_3
e_7	e_7	e_8	e_5	$-e_6$	$-e_3$	e_4	$-e_1$	$-e_2$
e_8	e_8	$-e_7$	e_6	e_5	$-e_4$	$-e_3$	e_2	$-e_1$



> **Corollary 1.** *Let $\mathbb{O}$ be the octonion ring and let $\varphi : \mathbb{O} \to \mathbb{O}$ be a bijective additive map such that $\varphi(a)\varphi(b) = 1_{\mathbb{O}}$ whenever $ab = c$ for c fixed element in $\mathbb{O}$. Then $\varphi = \lambda\xi$, where $\lambda = \varphi(1_{\mathbb{O}})$ and $\xi : \mathbb{O} \to \mathbb{O}$ is a semi-automorphism.*

Example 1. Consider the octonion ring $\mathbb{O}$ and fix $d = e_5 \notin \mathcal{N}(\mathbb{O})^\times$ and $c = e_7$. Define the bijective additive map $\varphi\colon \mathbb{O} \to \mathbb{O}$ given by $\varphi(e_i) = e_{\sigma(i)}$, where σ is the permutation $(1537)(268) \in S_8$, more precisely, φ is defined by the table:

e_i	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
$\varphi(e_i)$	e_5	e_6	e_7	e_4	e_3	e_8	e_1	e_2

Straightforward calculations show that $\varphi(a)\varphi(b) = e_5$ whenever $ab = e_7$. By Corollary 1, we have $\varphi = \lambda\xi = \varphi(e_1)\xi = e_5\xi$, where ξ is a semi-automorphism.

Bibliography

[1] Moufang, R.; Alternativk**ö**rper und der Satz vom vollständigen Vierseit (D_9). *Abh. Math. Sem. Univ. Hamburg* **9** (1933), no. 1, 207–222.

[2] Chebotar, M. A.; Ke, W.-F.; Lee, P.-H.; Shiao, L.-S.; On maps preserving products. *Canad. Math. Bull.* **48** (2005), no. 3, 355–369.

[3] Ferreira, Bruno Leonardo Macedo; Ferreira, Ruth Nascimento; Automorphisms on the alternative division ring, *Rocky Mountain J. Math.* **49** (2019), 73–78.

[4] Hua, Loo-Keng; On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 386–389.

[5] Artin, E.; Geometric algebra. *Interscience Publishers, Inc., New York-London* , 1957. x+214 pp.