Dirichlet Laplacian in quantum waveguides

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Abstract

Let $-\Delta_D^{\Gamma_\theta}$ be the Dirichlet Laplacian in a rectangular V-shaped waveguide Γ_θ, with θ a parameter of opening. The goal is to prove that the discrete spectrum of $-\Delta_D^{\Gamma_\theta}$ is non-empty and that there are a finite number of discrete eigenvalues. This number depends on θ and tends to infinity as $\theta \to \pi/2$, as it was done in [1].

Geometry of the waveguide

Let $\theta \in (0, \pi/2)$. Denote by Γ_θ the V-shaped rectangular waveguide associated with θ, and by $S = (0,\pi)^2$ the oblique cross-section in Γ_θ, see Figure 1. More precisely,

$$\Gamma_\theta := \Big\{ (x_1, x_2, x_3) \in \mathbb{R} \times (0,\pi) \times \mathbb{R} : \left(|x_1| - \frac{\pi}{\sin\theta} \right) \tan\theta < x_3 < |x_1| \tan\theta \Big\}. \tag{1}$$

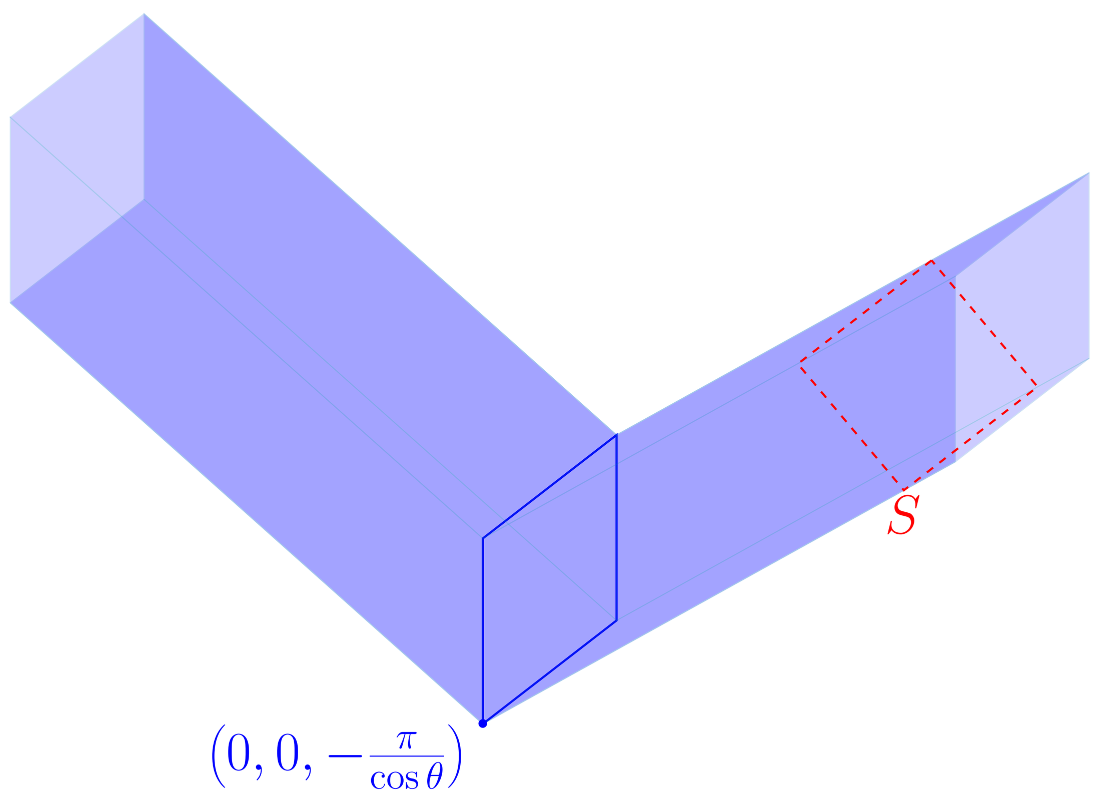

Figure 1: Broken rectangular waveguide Γ_θ, with $\theta = 2\pi/9$.

1 Essential spectrum

Denote by $-\Delta_D^{\Gamma_\theta}$ the Dirichlet Laplacian in Γ_θ, i.e., the self-adjoint operator associated with the quadratic form

$$Q_D^{\Gamma_\theta}(\psi) = \int_{\Gamma_\theta} |\nabla\psi|^2 \mathrm{dx}, \quad \mathrm{dom}\ Q_D^{\Gamma_\theta} = H_0^1(\Gamma_\theta); \tag{2}$$

$\mathrm{x} = (x_1, x_2, x_3)$ denotes a point in Γ_θ, and $\nabla\psi$ denotes the gradient of ψ. Given the two-dimensional Dirichlet Laplacian operator in S, $-\Delta_D^S$, its first eigenvalue is

$$\lambda_1(-\Delta_D^S) = 2.$$

Note that, outside a compact set, Γ_θ is the union of two isometrically affine straight tubes to $(0,\infty) \times (0,\pi)^2$. Thus, we have

Proposition 1. For each $\theta \in (0, \pi/2)$, the essential spectrum of operator $-\Delta_D^{\Gamma_\theta}$ coincides with $[2, \infty)$.

2 Existence of the discrete spectrum

Let A be the self-adjoint operator associated with Q. The Rayleigh quotients of A can be defined as

$$\lambda_j(A) = \inf_{\substack{G \subset \mathrm{dom}\, Q \\ \dim G = j}} \sup_{\substack{\psi \in G \\ \psi \neq 0}} \frac{Q(\psi)}{\|\psi\|_H^2}. \tag{3}$$

Let $\mu = \inf \sigma_{ess}(A)$. The sequence $\{\lambda_j(A)\}_{j\in\mathbb{N}}$ is non-decreasing and satisfies (i) If $\lambda_j(A) < \mu$, then it is a discrete eigenvalue of A; (ii) If $\lambda_j(A) \geq \mu$, then $\lambda_j(A) = \mu$; (iii) The j-th eigenvalue of A less than μ (if it exists) coincides with $\lambda_j(A)$. We use the notation $\mathcal{N}(A, \lambda)$ (or $\mathcal{N}(Q, \lambda)$) to indicate the maximal index j such the j-th Rayleigh quotient of A is less than λ.

According to (3) and Proposition 1, it is enough to show that there exists a non null function $\psi \in \mathrm{dom}\ \mathcal{Q}_D^{\Gamma_\theta} \setminus \{0\}$ so that $\mathcal{Q}_D^{\Gamma_\theta}(\psi) - \|\psi\|_{L^2(\Gamma_\theta)} < 0$. In fact, one has the following result

Proposition 2. For each $\theta \in (0, \pi/2)$, $\lambda_1(-\Delta_D^{\Gamma_\theta}) < 2$.

3 Finite number of discrete eigenvalues

The V-shaped effect in Γ_θ induces the question on the number of discrete eigenvalues. Denote by $\mathcal{N}(-\Delta_D^{\Gamma_\theta}, \lambda)$ the maximal index j so that $\lambda_j(-\Delta_D^{\Gamma_\theta})$ is $< \lambda$. To answer the question above, we have the following result,

Proposition 3. For each $\theta \in (0, \pi/2)$, $\mathcal{N}(-\Delta_{\Gamma_\theta}^D, 2)$ is finite.

4 Number of eigenvalues for θ close to 0

Inspired by [2], it is possible to find an interval of variation for θ so that the discrete spectrum of operator $-\Delta_D^{\Gamma_\theta}$ has a single discrete eigenvalue, as it is enunciated in the following result

Proposition 4. For each $\theta \in (0, \arctan(3/2)]$, the operator $-\Delta_D^{\Gamma_\theta}$ has exactly one discrete eigenvalue.

5 Accumulation of eigenvalues for $\theta \to \pi/2$

Given a parallelepiped $\mathcal{P}_\theta$ contained in Γ_θ, as shown in the Figure 2, by the monotonicity of the Dirichlet spectrum, one has

$$\lambda_j(-\Delta_D^{\mathcal{P}_\theta}) \geq \lambda_j(-\Delta_D^{\Gamma_\theta}), \quad \forall\, j \geq 1. \tag{4}$$

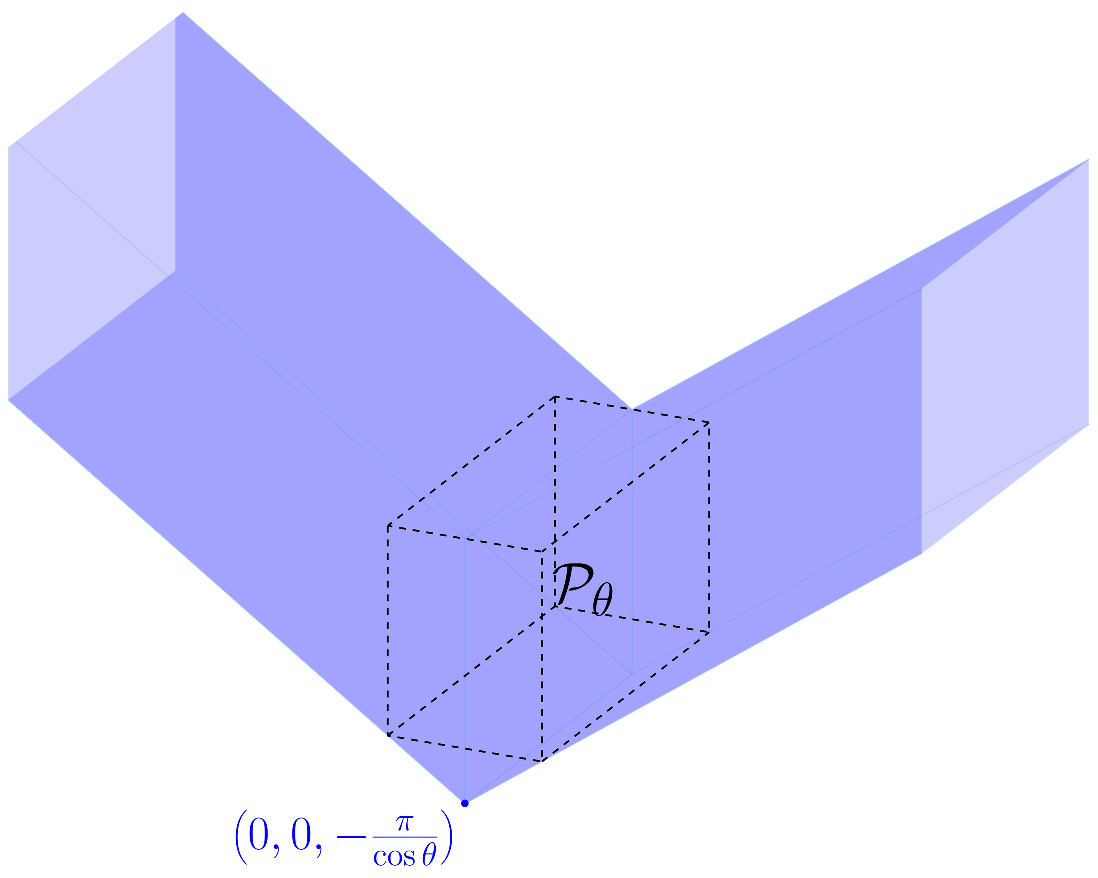

Figure 2: $\mathcal{P}_\theta$ contained in the broken rectangular waveguide Γ_θ, with $\theta = 21\pi/50$.

By (1), choosing $\mathcal{P}_\theta$ in the form of a parallelepiped limited by the lines $x_2 = 0$, $x_2 = \pi$, $x_3 = -\alpha\pi$, $x_3 = 0$, and $x_1 = \pm\tau\pi$, where α and τ satisfy

$$\alpha \in \left(0, \frac{1}{\cos\theta}\right) \quad \text{and} \quad \tau\pi = (-\alpha\cos\theta + 1)\frac{\pi}{\sin\theta},$$

one has the following result

Proposition 5. For $\theta \in (0, \pi/2)$, the number of discrete eigenvalues of $-\Delta_D^{\Gamma_\theta}$ tends to infinity as $\theta \to \pi/2$.

References

[1] M. Dauge, Y. Lafranche, and N. Raymond, *Quantum waveguides whit corners*, ESAIM, Proc. **35**, 14-45 (2012).

[2] S. A. Nazarov, and A. V Shanin, Trapped modes in angular joints of 2D waveguides. *Appl. Anal.* **93**(3), 572-582 (2014).

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