Infinite-Dimensional Evolution Algebras generated by Gibbs Measures

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Abstract

Evolution algebras play a significant role in the study of self-reproduction of alleles in non-Mendelian genetics, making them relevant in various mathematical fields, including graph theory, stochastic processes, and mathematical physics. These algebras possess a unique structure, characterized by a null product between distinct elements of the canonical basis, resulting in commutativity and generally non-associativity.

In our research, we focus on evolution algebras with a canonical basis consisting of pairs of configurations of finite spins on countable sets. The product coefficients are determined by Gibbs measures associated with these configurations. Since they represent a particular case of genetic algebras, where the product represents reproduction, we establish a meaningful connection by associating the square of a pair with its offspring. By translating the thermodynamic formalism into the framework of evolution algebras, we unveil intriguing algebraic properties.

Introduction

We define $\Omega = S^{\mathbb{L}}$ as the set of configurations consisting of a countable set of sites, $\mathbb{L}$, and the associated spin space, S, which we suppose to be finite. Let $\mathcal{L} \subseteq 2^{\mathbb{L}}$ be the set of finite subsets of $\mathbb{L}$. An interacting potential $\Phi = (\Phi_A)_{A\in\mathcal{L}}$ is a family of functions that will determine the local Hamiltonians H^{Φ}_{Λ} for each $\Lambda \in \mathcal{L}$ and the local specification γ^{Φ}_{Λ}. Two potentials Φ and Ψ are said equivalent ($\Phi \sim \Psi$) when for any $\Lambda \in \mathcal{L}$ the Hamiltonian $H^{\Phi-\Psi}_{\Lambda}$ is measurable on the "outside" of Λ.

We put $\mathscr{G}(\Phi)$ to be the set of all Gibbs measures determined by the local specification γ^{Φ}_{Λ}.

Fix a partition of $\mathbb{L}$, $\mathscr{C} \subseteq 2^{\mathbb{L}}$, and call each $\Delta \in \mathscr{C}$ a cluster. Given $L \in 2^{\mathbb{L}}$ and $\sigma \in \Omega$, σ_L is the restriction of σ to L. The discrepancy set $\mathcal{D}_{\sigma\eta}$ consists of elements $x \in \mathbb{L}$ such that $\sigma(x) \neq \eta(x)$ and the offspring of σ and η is the set $\Omega_{\sigma\eta}$ whose elements coincide with σ or η in every cluster $\Delta \in \mathscr{C}$.

We set $\mathfrak{B}_{\Omega} := \{e_{(\sigma,\eta)}\}_{(\sigma,\eta)\in\Omega^2}$ to be a basis and let $e_{\sigma\eta}$ stand for $e_{(\sigma,\eta)}$

The $\mathscr{C}$-evolution Gibbs algebra generated by $\mu \in \mathscr{G}(\Phi)$ on Ω is the free module $\mathcal{E}(\mathscr{C}, \mu, \Phi, \Omega) = \langle \mathfrak{B}_{\Omega^2} \rangle$ with product given by ($\mathbb{R}$ or $\mathbb{C}$) bilinear extension of

$$e_{\sigma\eta} \cdot e_{\sigma'\eta'} = \begin{cases} \sum_{(\zeta,\xi)\in\Omega^2_{\sigma\eta}} \mathfrak{c}_{\sigma\eta,\zeta\xi}\, e_{\zeta\xi}, & \text{if } \sigma = \sigma' \text{ and } \eta = \eta'; \\ 0, & \text{otherwise.} \end{cases}$$

where

$$\mathfrak{c}_{\sigma\eta,\zeta\xi} = \frac{\mu(\zeta \mid \sigma_{(\mathfrak{D}_{\sigma\eta})^c})\mu(\xi \mid \sigma_{(\mathfrak{D}_{\sigma\eta})^c})}{\mu^{\otimes 2}(\Omega^2_{\sigma\eta} \mid \sigma_{(\mathfrak{D}_{\sigma\eta})^c})} = c_{\sigma\eta,\zeta} c_{\sigma\eta,\xi}.$$

Consider the set of configurations with finite discrepancy of η, $\mathtt{E}^{\eta}$, and define the fertile ideal $\mathtt{F}^{\eta} = \langle \mathtt{E}^{\eta} \rangle$ and $\mathcal{F}_{\Omega} := \{\mathtt{F}^{\eta} : \eta \in \Omega\}$. Consider now $\tilde{\sigma} : \mathcal{F}_{\Omega} \to \Omega$ to be a choice that fixes $\tilde{\sigma}(\mathtt{F}) \in \Omega$ such that $\mathtt{F} = \mathtt{F}^{\tilde{\sigma}(\mathtt{F})}$. Set $\tilde{\Omega} := \{\tilde{\sigma}(\mathtt{F}) : \mathtt{F} \in \mathcal{F}_{\Omega}\}$. Fix $\tilde{\Omega} \subseteq \Omega$ as the set that chooses a unique representative of each fertile ideal. Namely, for all $\eta \in \Omega$ there exists an unique $\sigma \in \tilde{\Omega}$ s.t. $\mathtt{F}^{\sigma} = \mathtt{F}^{\eta}$.

Main results

Some of our results are for $\mathcal{E}(\mathscr{C}, \mu, \Phi, \Omega)$, but since an infinity of elements of the basis lie in the kernel of the map $x \mapsto x^2$, it is convenient to avoid them by considering the quotient or simply generating the algebra from their complementary. That is, set $\mathfrak{N} := \{e_{\sigma\eta} \in \mathfrak{B}_{\Omega^2} : \mathfrak{D}_{\sigma\eta} \notin \mathcal{L}\}$ and define $\mathcal{E}_M = \mathcal{E}_M(\mathscr{C}, \mu, \Phi, \Omega)$ as the subalgebra of $\mathcal{E}(\mathscr{C}, \mu, \Phi, \Omega)$ such that $\mathcal{E}_M := \langle \mathfrak{B}_{\Omega^2} \setminus \mathfrak{N} \rangle$. We call $\mathcal{E}_M$ the Markov $\mathscr{C}$-*evolution Gibbs algebra* generated by the $\mu \in \mathscr{G}(\Phi)$ on Ω. Define $\mathtt{F}_{\sigma\eta} = \langle \mathfrak{B}_{\mathtt{E}^{\sigma}\times\mathtt{E}^{\eta}} \rangle$.

Theorem (*Decomposition of $\mathcal{E}_M$ into a direct sum of ideals*) Let $\mathcal{E}_M := \mathcal{E}_M(\mathscr{C}, \mu, \Phi, \Omega)$ be a Markov $\mathscr{C}$-evolution Gibbs algebra generated by $\mu \in \mathscr{G}(\Phi)$ on Ω. Then $\mathcal{E}_M$ is indeed Markov such that

$$\mathcal{E}_M = \bigoplus_{\sigma\in\tilde{\Omega}} \mathtt{F}_{\sigma\sigma},$$

where each $\mathtt{F}_{\sigma\sigma} \in \mathcal{F}_{\Omega^2}$ is a ideal with countable basis $\mathfrak{B}_{(\mathtt{E}^{\sigma})^2}$. Moreover, if Φ has finite range; then, for all $\sigma, \eta \in \Omega$, $\mathtt{F}_{\sigma\sigma}$ and $\mathtt{F}_{\eta\eta}$ are isomorphic.

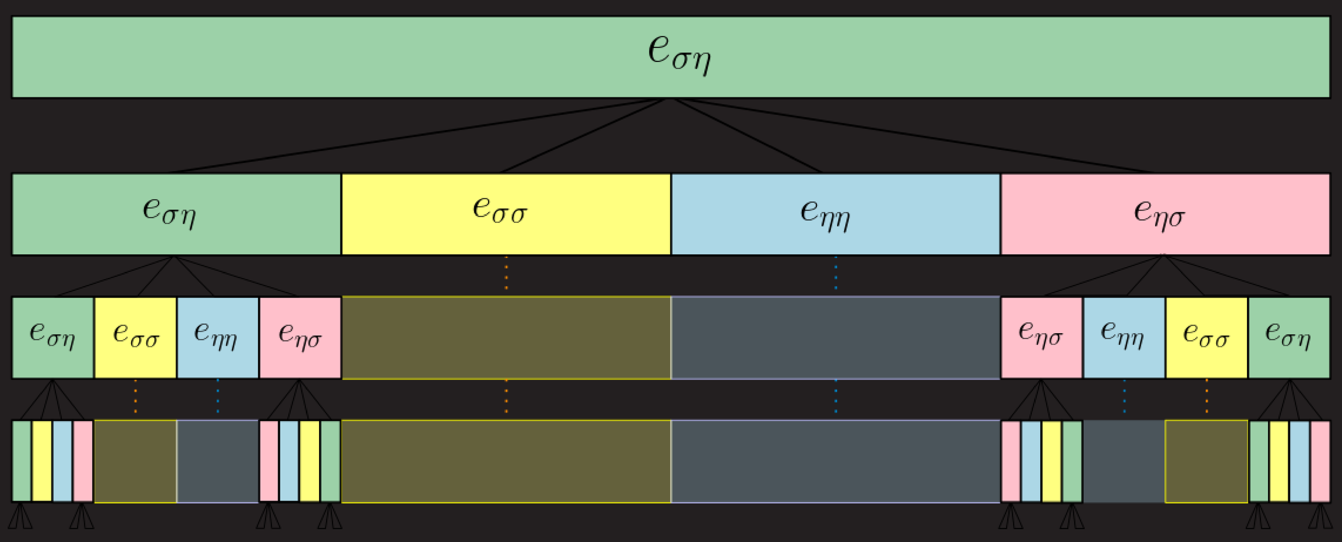

Figure 1: For $\sigma \neq \eta$ the genealogical tree of $e_{\sigma\eta}$ exhibits a self-similar structure of the gene flow.

For Markov $\mathscr{C}$-evolution algebras, we say $\mathcal{E}_m$ is τ-*isomorphic to* $\mathcal{E}'_M$ when the linear map such that $\phi(e_{\sigma\eta}) = e'_{\tau\sigma\tau\eta}$ determines an isomorphism of algebras.

Theorem (*τ-isomorphism*) For reasonable τ, let $\mathcal{E}_M = \mathcal{E}_M(\mathscr{C}, \mu, \Phi, \Omega)$ and $\mathcal{E}'_M = \mathcal{E}_M(\tau(\mathscr{C}), \mu', \Psi, \Omega)$ be two evolution Gibbs algebras. If $\Phi \sim \tau^{-1}(\Psi)$, then the algebra $\mathcal{E}_M$ is τ-isomorphic to $\mathcal{E}'_M$. Moreover, the converse holds when $\mathscr{C} = \mathscr{C}_{\odot}$ is the set of atomic clusters.

Theorem (*Stability under phase transition*) Let $\mu, \mu' \in \mathscr{G}(\Phi)$ be Gibbs measures on Ω. Then the algebra $\mathcal{E}_M(\mathscr{C}, \mu, \Phi, \Omega)$ is isomorphic to $\mathcal{E}_M(\mathscr{C}, \mu', \Phi, \Omega)$.

Theorem (*Evolution algebras generated by products of Gibbs measures*) Let $\{\mathbb{L}_i\}_{i=1}^{n}$ be a sequence of countable sets such that, for each $i \in \{1, \dots, n\}$, $\mathscr{C}_i$ is a partition of $\mathbb{L}_i$ associated with a Gibbs measure $\mu_i \in \mathscr{G}(\Phi^i)$ on $\Omega_i = S^{\mathbb{L}_i}$ with S a fixed finite set of spins.

Consider the evolution Gibbs algebras $\mathcal{E}_{M,i} := \mathcal{E}_M(\mathscr{C}_i, \mu_i, \Phi^i, \Omega_i)$ for all $i \in \{1, \dots, n\}$, and

$$\mathcal{E}_M = \mathcal{E}_M\left(\bigsqcup_{i=1}^{n} \mathscr{C}_i, \bigotimes_{i=1}^{n} \mu_i, \bigoplus_{i=1}^{n} \Phi^i, \prod_{i=1}^{n} \Omega_i\right).$$

Then $\mathcal{E}$ is isomorphic to the tensor algebra $\bigotimes_{i=1}^{n} \mathcal{E}_{M,i}$ equipped with the ordinary product.

Open questions

- Is it possible to modify the algebras preserving part of their properties to identify the phase transition phenomenon?
- Techniques from functional analysis could be interesting to study more properties of the algebras. How do the algebras change when consider a Schauder basis for the fertile ideals?
- How to define similar algebras when S infinite?

References

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