# Conley Index for GS Super repeller


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## 1 Introduction

Conley in 1978, in an innovative way, introduced an index for vector fields that aggregates, in addition to numerical information, topological information related to homotopy. This index called the Conley index [1]. In 1982, Gutierrez e Sotomayor extended some results on structural stability of vector fields tangents to $2$-dimensional manifolds for manifolds with singularities, which today we call GS manifolds. Moreover, they considered a class of flows over these manifolds, called GS flows [2]. In 2020, de Rezende and Montúfar applied Conley's theory for GS flows and computed the Conley indices for the corresponding singularities [4]. Recently Lima, Raminelli and de Rezende, when studying the dynamical homotopical cancellation theory for GS flows, naturally found generalizations of GS singularities [3] .

### What is the Conley index?

Let $\varphi_t$ be a continuous flow defined over a topological space $M$. A compact subset $N \subset M$ is an **isolating neighborhood** if

$$Inv(N) := \{x \in N | \ \varphi_t(x) \in N, \forall t \in \mathbb{R}\} \subset int(N).$$

$S$ is an **isolated invariant set** if $S = Inv(N)$, for some isolating neighborhood $N$. Let $S$ be an isolated invariant set, a pair of topological spaces $(N, L)$, where $L$ is a closed subspace of $N$, is an **index pair** for $S$ if:

(i) $S = Inv(\overline{N \setminus L})$;

(ii) $L$ is **positively invariant** in $N$;

(iii) $L$ is an **exit set** of the flow for $N$;

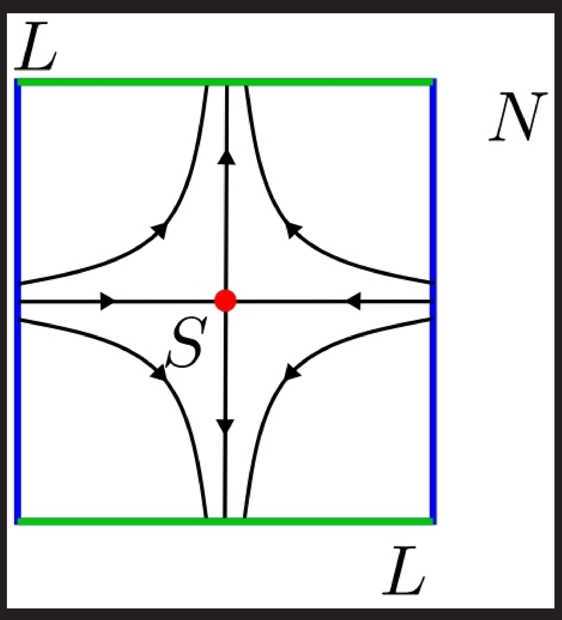


**Figure 1:** Example of index-pair

The **homotopy Conley index** of an isolated invariant set $S$ is defined as

$$h(S) := [N/L, *],$$

where $(N, L)$ is an index pair for $S$.

### Why these singular manifolds?

They are singular manifolds that naturally arose in [3] when they were studying the dynamical cancellation of GS singularities. These flows were defined by Gutierrez and Sotomayor in [2] which, in turn, are defined in singular manifolds, called GS manifolds. These manifolds mix regular parts and singularities of cone, cross-cap, double crossing and triple crossing types.

### Results

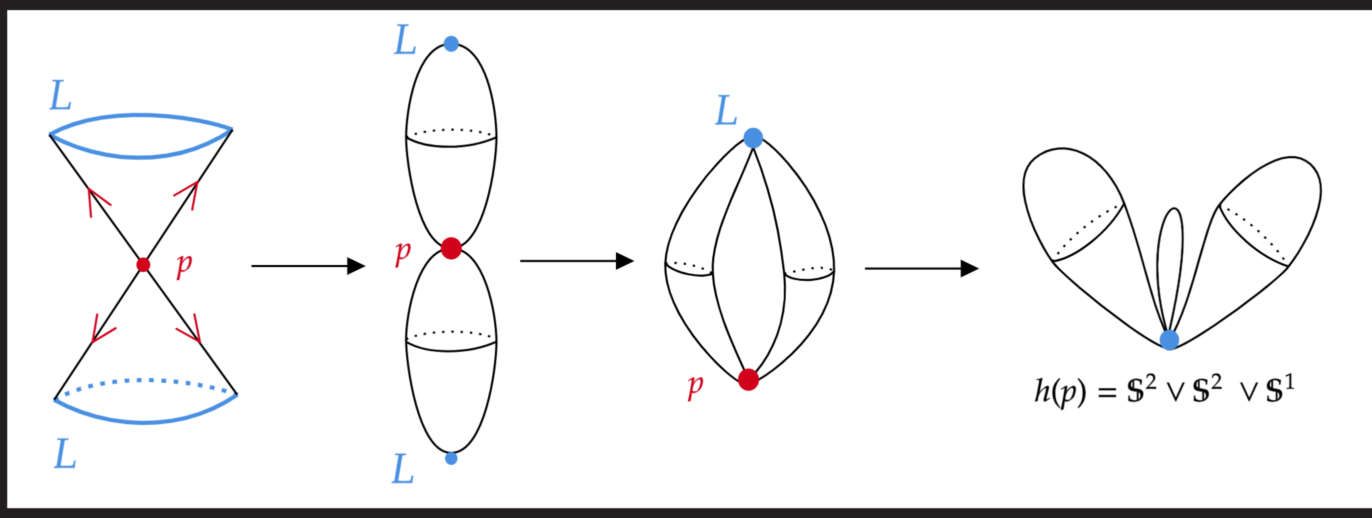


**Figure 2:** Case $n = 2$

**Proposition 1:** Let $p$ be a super repeller of $n$-sheet cone type, where $n \geq 2$. Then, the Conley index of the repeller singularity $p$ is given by:

$$h(p) = \bigvee_{i=1}^{n} \mathbb{S}^2 \vee \bigvee_{j=1}^{n-1} \mathbb{S}^1.$$

**Proposition 2:** Let $p$ be a super repeller of $n$-sheet cross-cap type, where $n \geq 2$. Then, the Conley index of the repeller singularity $p$ is given by:

$$h(p) = \bigvee_{i=1}^{n} \mathbb{S}^2.$$

**Proposition 3:** Let $p$ be a super repeller of $n$-sheet double crossing type, where $n \geq 2$. Then, the Conley index of the repeller singularity $p$ is given by:

$$h(p) = \bigvee_{i=1}^{2n-1} \mathbb{S}^2.$$

**Proposition 4:** Let $p$ be a super repeller of $n$-sheet triple crossing type , where $n = 2k + 1 \geq 3$. Then, the Conley index of the repeller singularity $p$ is given by:

$$h(p) = \bigvee_{i=1}^{6n+1} \mathbb{S}^2.$$

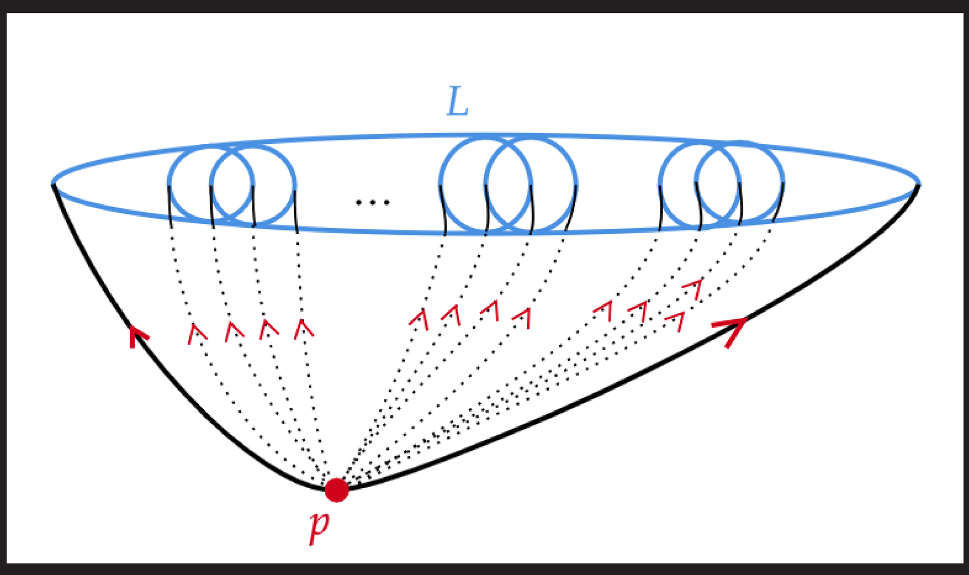


**Figure 3:** $(2(k+1)+1)$-sheet triple crossing

**Proof:** The base is proved in [4]. Now suppose the result holds for a certain $k \geq 1$. We'll prove that it holds for $k+1$. Indeed, consider a super repeller of $(2(k+1)+1)$-sheet triple crossing type, as in Figure 3. This new pair of cones contributes with six more $\mathbb{S}^2$. Thus,

$$h(p) = \left(\bigvee^{6} \mathbb{S}^2\right) \vee \left(\bigvee^{6k+1} \mathbb{S}^2\right) = \bigvee^{6k+7} \mathbb{S}^2 = \bigvee^{6(k+1)+1} \mathbb{S}^2.$$

Therefore, the result holds for all $n \in \mathbb{N}$. □

### Conclusion

Although the Conley index is well defined, it is not always easy to compute. Since the Conley index adds important information to the dynamics of the flow, it is important to build a large collection of examples, so that the application of this tool will be easier.

### References


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