Random Functions based on Normal Numbers

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Abstract

The idea of this poster is to use the digits of a normal number of type $\sum_{k=1}^{\infty} \frac{1}{b^{c^k} c^k}$ [1] as random variables to build a random function.

Introduction

In the year 1930 Kolmogorov and others elaborated the basis of the theory of random functions. Random function is an attempt to represent the evolution in time(or other evolution parameter) of a random process. One possibility to achieve this goal is to define a random function by an analytical formula, containing parameters which are random variables. For example, one can consider polynomials (ordinary or trigonometric) with random coefficients

$$\xi(t) = \sum_{k=1}^{\infty} (\xi_k cos(\omega_k t) + \nu_k sin(\omega_k t)) \qquad (1)$$

The concept of normal number was introduced by Borel in 1909. A real number α is *g-normal* $(g \geq 2)$ if every preassigned sequence of digits of length $k \geq 1$ occurs with the expected limiting frequency g^{-k} in the base g expansion of α.

Bailey and Crandall [2] proved that the number

$$\alpha_{b,c} = \sum_{k=1}^{\infty} \frac{1}{b^{c^k} c^k} \qquad (2)$$

is *b-normal* whenever b, c are coprime and $b, c > 1$

Objectives

1. Determine what is the frequency distribution of the first 500000 digits of *10-normal* number

$$\alpha_{10,3} = \sum_{k=1}^{\infty} \frac{1}{10^{3^k} 3^k} \qquad (3)$$

2. Create a random function as defined in (1) , where the coefficients are the digits of the *10-normal* number $\alpha_{10,3}$ and compare the results with coefficients generated by pseudo-random number generators using the Linear Congruence algorithm.

Results

The frequency distribution for the ten digits (0,1,2,..,9) of normal number $\alpha_{10,3}$ is shown in the table below

Digit	Frequency	Relative freq. (%)
0	49970	9.994
1	49993	9.999
2	49987	9.997
3	50020	10.004
4	50117	10.023
5	49941	9.988
6	49954	9.991
7	49978	9.995
8	50032	10.006
9	50008	10.002

Table 1: Frequency distribution of the first 500000 digits of $\alpha_{10,3}$

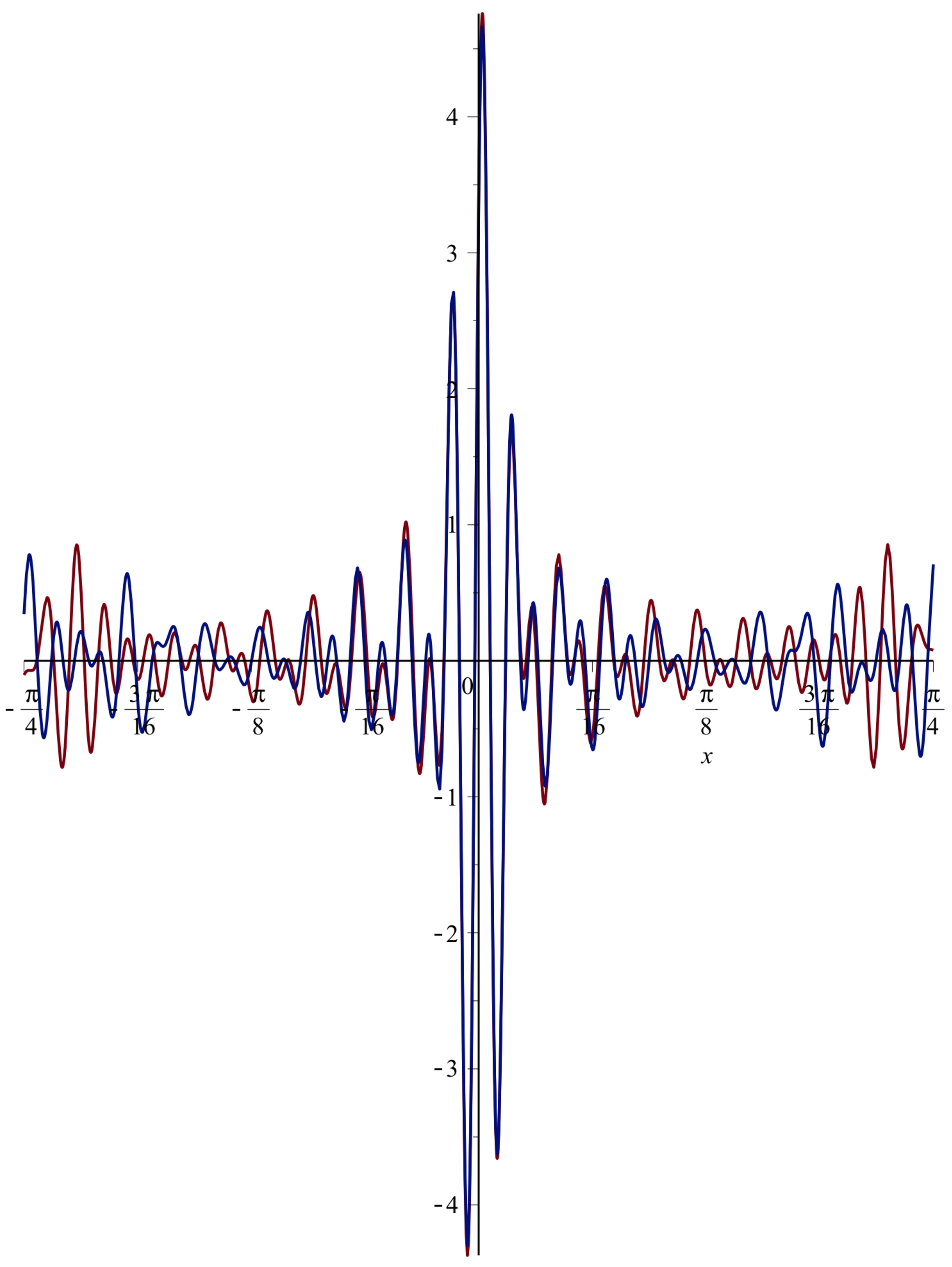

Figure 1: The curves were generated by a random function as defined in (1) with the following simplifications: $\xi_k = \nu_k$, $\omega_k = k$ and $k = 1..75$. On the red line the coefficients are the digits of $\alpha_{10,3}$. On the blue line the coefficients are generated by pseudo-random number generators using the Linear Congruence algorithm

Conclusion

The results show that the qualitative aspect of the random function remains the same for coefficients generated by normal numbers and pseudo-random number generators.

References

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