Graded identities of matrix algebras†

Daniela do Nascimento Rodrigues*

Introduction

In this poster, we will present a result of S. Vasilovsky [1] exhibiting a finite base for the ideal of graded identities for $M_n(K)$ endowed with a specific grading by the group $\mathbb{Z}_n$, when K is a field of characteristic zero.

Main Theorem. *All graded polynomial identities of the $\mathbb{Z}_n$-graded algebra $M_n(K)$ follow from*

$$x_1x_2 - x_2x_1 = 0, \deg(x_1) = \deg(x_2) = \overline{0}, \tag{1}$$

$$x_1xx_2 - x_2xx_1 = 0, \deg(x_1) = \deg(x_2) = -\deg(x). \tag{2}$$

Definitions and Preliminary Results

Let E_{ij} be the unit matrix, $1 \leq i, j \leq n$. These matrix form a basis of $M_n(K)$ as a vector space. For $t \in \mathbb{Z}$, let $\bar{t}$ denote the residue class in $\mathbb{Z}_n$ that contains t. For each $\alpha \in \mathbb{Z}_n$, let $M_n^{(\alpha)}$ be the subspace of $M_n(K)$ spanned by all matrix units E_{ij} such that $\deg(E_{ij}) = \overline{j-i} = \alpha$. It follows that $M_n^{(\bar{0})}$ consists of diagonal matrices and, for $0 < t \leq n-1$, $M_n^{(\bar{t})}$ consists of the matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & a_{1,t+1} & \cdots & & \cdots & 0 \\ \vdots & & \vdots & \vdots & a_{2,t+2} & & & \vdots \\ \vdots & & \vdots & \vdots & & & \ddots & \vdots \\ 0\cdots & 0 & 0 & \cdots & & \cdots & a_{n-t,n} & \\ a_{n-t+1,1} & \cdots & 0 & 0 & & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & a_{n,t} & 0 & & \cdots & \cdots & 0 \end{pmatrix}$$

Then $M_n(K)$ is a direct sum of the subspaces $M_n^{(\alpha)}$'s:

$$M_n(K) = \bigoplus_{\alpha \in \mathbb{Z}_n} M_n^{(\alpha)}. \tag{3}$$

The decomposition (3) defines a $\mathbb{Z}_n$-grading of the algebra $M_n(K)$. Now, let $X = \cup_{\alpha \in \mathbb{Z}_n} X^{(\alpha)}$, where $X^{(\alpha)} \cap X^{(\beta)} = \emptyset$ if $\alpha \neq \beta$ and consider $K\langle X \rangle$ the free associative algebra freely generated by the set X. The monomials

$$\{x_{i_1} \cdots x_{i_k} : k \in \{1, 2, \dots\}, x_{i_1}, \dots, x_{i_k} \in X\}$$

form a basis of $K\langle X \rangle$ as vector space. An indeterminate $x \in X$ is said to be *of homogenous degree* α, written $\deg(x) = \alpha$, if $x \in X^{(\alpha)}$. The homogenous degree of a monomial $n = x_{i_1}x_{i_2} \cdots x_{i_k}$ is defined by $\deg(n) = \sum_{j=1}^{k} \deg(x_{i_j})$. We can write

$$K\langle X \rangle = \bigoplus_{\alpha \in \mathbb{Z}_n} K\langle X \rangle^{(\alpha)},$$

where $K\langle X \rangle^{(\alpha)}$ designates the subspace of $K\langle X \rangle$ spanned by all the monomials of homogeneous degree α. Clearly $K\langle X \rangle$ is a $\mathbb{Z}_n$-graded algebra and their elements are called *graded polynomials*. A graded polynomial $f \in K\langle X \rangle$ is said to be a *graded polynomial identity* of the M_n if $f(A_1, \dots, A_k) = 0$ for all $A_1, \dots, A_k \in M_n$ such that $A_s \in M_n^{(\deg(x_s))}$, $s \in \{1, \dots, k\}$. The set $T_n(M_n)$ of all graded identities of M_n is a T_n-ideal of $K\langle X \rangle$, i.e., an ideal of $K\langle X \rangle$ that is invariant under any endomorphism φ of $K\langle X \rangle$ such that $\varphi(K\langle X \rangle^{(\alpha)}) \subseteq K\langle X \rangle^{(\alpha)}$ for all $\alpha \in \mathbb{Z}_n$. A graded polynomial f is said to follow from a family of graded polynomial identities $\Upsilon = \{g_\lambda : \lambda \in \Lambda\}$, if f lies in the smallest T_n-ideal containing the family Υ. It is easy to see that M_n satisfies (1), since any two diagonal matrices commute. The verification of (2) is a straightforward computation. Now, let I_n be the T_n-ideal generated by the graded identities (1) and (2). If k is a positive integer, denote by S_k the set of all permutations of the set $\{1, \dots, k\}$. For $x_1, \dots, x_k \in X$ and $\sigma \in S_k$, let

$$m_\sigma = m_\sigma(x_1, \dots . x_k) = x_{\sigma(1)} \cdots x_{\sigma(k)}.$$

The multilinear monomial in $x_1, \dots, x_k$ corresponding to the identity permutation will be denoted by

$$m = m(x_1, \dots, x_k) = x_1 \cdots x_k.$$

Clearly, $\deg(m) = \deg(m_\sigma) = \deg(x_1) + \cdots + \deg(x_k)$. It follows that every multilinear graded polynomial $f(x_1, \dots, x_k)$ can be expressed as

$$f = \sum_{\sigma \in S_k} a_\sigma m_\sigma, \text{ where } a_\sigma \in K.$$

By a *standard substitution* we will understand a substitution S of the form $x_s = E_{i_s j_s}, s \in \{1, \dots, k\}$, where $\overline{j_s - i_s} = \deg(x_s)$. It is easy to see that, if a multilinear graded polynomial f is such that $f(S) = 0$ for every standard substitution S, then f is a graded identity of M_n.

Remark 1. *Observe that, when a substitution S is made, the value of a monomial m_σ differs from zero only if*

$$j_{\sigma(s-1)} = i_{\sigma(s)}, s \in \{2, \dots, k\},$$

in which case $m_\sigma(S) = E_{i_{\sigma(1)} j_{\sigma(k)}}$.

Lemma 2. *If for a permutation $\sigma \in S_k$, there is a standard substitution S such that*

$$0 \neq m_\sigma(S) = m(S),$$

then

$$m_\sigma(x_1, \dots, x_k) \equiv x_1 \cdot n(x_2, \dots, x_k)(\textit{mod } I_n),$$

for some monomial $n(x_2, \dots, x_k) = x_{l_2} \cdots x_{l_k}$.

Lemma 3. *If for two permutations $\sigma, \tau \in S_k$, there exists a standard substitution S such that*

$$m_\sigma(S) = m_\tau(S) \neq 0,$$

then

$$m_\sigma(x_1, \dots, x_k) \equiv m_\tau(x_1, \dots, x_k)(\textit{mod } I_n).$$

Proof of Main Theorem

Proof. Since the characteristic of the field K is zero, we only need to prove that any multilinear graded polynomial identity f of M_n lies in I_n. Let r be the least non-negative interger such that f can be expressed, modulo I_n, as a linear ombination of r multilinear monomials

$$f \equiv \sum_{q=1}^{r} a_{\sigma_q} m_{\sigma_q} (\text{mod } I_n), 0 \neq a_{\sigma_q} \in K, \sigma_q \in S_k.$$

We will show that $r = 0$. Suppose, on the contrary, $r > 0$. By (1), we can find a standard substitution S such that $m_{\sigma_1}(S) \neq 0$. Since

$$m_{\sigma_q}(S) \in \{E_{ij} : i, j \in \{1, \dots, n\}\} \cup \{0\}, q \in \{1, \dots, r\},$$

$$a_{\sigma_1} m_{\sigma_1}(S) = \sum_{q=2}^{r} (-a_{\sigma_q}) m_{\sigma_q}(S),$$

it follows that there is a least one integer $p \in \{2, \dots, n\}$ such that $m_{\sigma_p}(S) = m_{\sigma_1}(S)$. Then, by Lemma (3), $m_{\sigma_p} \equiv m_{\sigma_1} (\text{mod } I_n)$, so that

$$f \equiv \sum_{q=1}^{r} a_{\sigma_q} m_{\sigma_q} \equiv (a_{\sigma_1} + a_{\sigma_p}) m_{\sigma_1} + \sum_{q=}^{p-2} a_{\sigma_q} m_{\sigma_q} + \sum_{q=p+1}^{r} a_{\sigma_q} m_{\sigma_q} (\text{mod } I_n),$$

i.e., f can be expressed, modulo I_n, as a linear combination of no more than $r - 1$ multilinear monomials, which contradicts our choice of r. Thus $f \equiv 0 (\text{mod } I_n)$. □

References

[1] Sergei Yu. Vasilovsky. *$\mathbb{Z}_n$-Graded Polynomial Identities of the Full Matrix Algebra of Order n. Proceedings of the American Mathematical Society, Vol 127, No 12 (Dec., 1999), pp.3517-3524*

† Supported by grant #2022/13058-6, São Paulo Research Foundation (FAPESP).

*Instituto de Ciência e Tecnologia, Universidade Federal de São Paulo
E-mail: dnrodrigues@unifesp.br