Nielsen-Borsuk-Ulam number for maps between tori

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Abstract

The results presented here are published in the reference [8].

We compute the Nielsen-Borsuk-Ulam number for any selfmap of $n-$torus, $\mathbb{T}^n$, as well as any free involution τ in $\mathbb{T}^n$, with $n \leqslant 3$. Finally, we conclude that the tori, $\mathbb{T}^1$, $\mathbb{T}^2$ and $\mathbb{T}^3$, are Wecken spaces in Nielsen-Borsuk-Ulam theory. Such a number is a lower bound for the minimal number of pair of points such that $f(x) = f(\tau(x))$ in a given homotopy class of maps.

Introduction

In the literature one can find many different generalizations of the classical Borsuk-Ulam Theorem for maps from the sphere S^n in the Euclidean space $\mathbb{R}^n$. One possible generalization can be the following: given two topological spaces X and Y and a free involution τ on X we can ask if the triple $(X, \tau; Y)$ has the Borsuk-Ulam Property, i. e., if for any continuous map $f : X \to Y$ there exists a point $x \in X$ such that $f(x) = f(\tau(x))$. In [3] this approach was used to study Borsuk-Ulam Property for surfaces with maps on $\mathbb{R}^2$, and it indicated that the answer may depend on the involution, i. e., the same surface can have this property in respect to an involution τ_1, but not for another involution τ_2.

More recently (in [5]) the Borsuk-Ulam Property was stated not for a triple (X, τ, Y) but for each homotopy class of selfmaps of surfaces with Euler characteristic zero. It must be noted that for maps on $\mathbb{R}^n$ there is only one such class, however, this is not generally the case.

From this perspective, while investigating for which homotopy class of maps $f : X \to Y$ it is true that for any f' in such class, there exists a point $x \in X$ such that $f'(x) = f'(\tau(x))$, the studies [1, 2] have taken on a different approach. Using ideas from Nielsen fixed point theory, Nielsen-Borsuk-Ulam classes and a Nilsen-Borsuk-Ulam number were defined, for a homotopy class of maps between triangulated, orientable, closed manifolds. Such invariant is a lower bound, in the homotopy class, for the number of pairs of points satisfying $f(x) = f(\tau(x))$. In the present study we compute the Nielsen-Borsuk-Ulam number for selfmaps of tori until dimension 3.

Interestingly, the results presented here show that, for $n \leqslant 3$, the triples $(\mathbb{T}^n, \tau, \mathbb{T}^n)$ do not have the Borsuk-Ulam Property, for any involution τ, and also, that tori are Wecken spaces in Nielsen-Borsuk-Ulam theory. In each case, we present maps that realize Nielsen-Borsuk-Ulam number.

In the proofs of Theorems 7 and 8 a similar reasoning was adopted, given a map f that represents a homotopy class, we show that there is a small perturbation of f in this same class, usually called f', such that f' realizes the Nielsen-Borsuk-Ulam number. We will always see the n-torus as $\dfrac{\mathbb{R}^n}{\mathbb{Z}^n}$.

Nielsen-Borsuk-Ulam theory

For practical reasons, we will reproduce in this section some definitions and propositions from [1] and [2].

Denoting by $Coin(f, f \circ \tau)$ the coincidence set of the pair $(f, f \circ \tau)$, [1, Theorem 2.1] shows, in the context of simplicial complexes, that we can suppose $Coin(f, f \circ \tau)$ finite. Moreover [1, Theorem 3.5] shows that if two homotopic maps, f and g, are such that $Coin(f, f \circ \tau)$ and $Coin(g, g \circ \tau)$ are both finite, then there exists a homotopy between them with such set finite in each level.

Definition 1. *[2, 2.1] Let $(X, \tau; Y)$ be a triple where X and Y are finite n-dimensional complexes, τ is a free simplicial involution on X for any map $f : X \to Y$ with $Coin(f, f \circ \tau) = \{x_1, \tau(x_1), \ldots, x_m, \tau(x_m)\}$, we define the Borsuk-Ulam coincidence set for the pair (f, τ), as the set of pairs:*

$$BUCoin(f; \tau) = \{(x_1, \tau(x_1)); \ldots ; (x_m, \tau(x_m))\}$$

and we say that two pairs $(x_i, \tau(x_i)), (x_j, \tau(x_j))$ are in the same Borsuk-Ulam coincidence class if there exists a path γ from a point in $\{x_i, \tau(x_i)\}$ to a point in $\{x_j, \tau(x_j)\}$ such that $f \circ \gamma$ is homotopic to $f \circ \tau \circ \gamma$ with fixed endpoints.

Proposition 2. *[2, 2.4] A Borsuk-Ulam coincidence class C is single if, and only if, it is composed of just one usual coincidence class of the pair $(f, f \circ \tau)$. Moreover, if C is a finite Borsuk-Ulam coincidence class of the pair (f, τ) which is not single (called double), then we can change the labels of the elements of C in a way that:*

- $C = \{(x_1, \tau(x_1)), \ldots, (x_k, \tau(x_k))\}$;
- $C = C_1 \cup C_2$ *where C_1 and C_2 are usual coincidence classes of the pair $(f, f \circ \tau)$;*
- $C_1 = \{x_1, \ldots, x_k\}$ *and* $C_2 = \{\tau(x_1), \ldots, \tau(x_k)\}$.

We have:

$$ind(f, f \circ \tau; c) = \begin{cases} (-1)^n ind(f, f \circ \tau; \tau(c)) \text{ if} \\ \quad \tau \text{ preserves orientation,} \\ (-1)^{n-1} ind(f, f \circ \tau; \tau(c)) \text{ if} \\ \quad \tau \text{ reverses orientation.} \end{cases}$$

where $ind(f, f \circ \tau; c)$ is the usual local index for coincidence and n is the dimension of the manifold.

Definition 3. *Let X and Y be closed orientable triangulable n-manifolds, τ a free involution on X and $f : X \to Y$ a continuous map such that $BUCoin(f, \tau)$ is finite. If $C = \{(x_1, \tau(x_1)), \ldots, (x_k, \tau(x_k))\}$ is a Borsuk-Ulam coincidence class of the pair (f, τ), we define the pseudo-index of C, denoted $|ind|(f, \tau; C)$, by:*

$$\begin{cases} \sum_{x_i \in C} ind(f, f \circ \tau; x_i) \bmod 2 & \textit{(if } C \textit{ is single) and} \\ & (\tau \textit{ reverses orientation and } n \textit{ is even; or } \tau \textit{ preserves orientation and } n \textit{ is odd.)} \\ \dfrac{ind(f, f \circ \tau; C)}{2} & \textit{(if } C \textit{ is single) and} \\ & (\tau \textit{ preserves orientation and } n \textit{ is even; or } \tau \textit{ reverses orientation and } n \textit{ is odd.)} \\ |ind(f, f \circ \tau; C_1)| & \textit{(if } C \textit{ is double, } C = C_1 \cup C_2) \textit{ and} \\ & (\tau \textit{ reverses orientation and } n \textit{ is even; or } \tau \textit{ preserves orientation and } n \textit{ is odd.)} \\ ind(f, f \circ \tau; C_1) & \textit{(if } C \textit{ is double, } C = C_1 \cup C_2) \textit{ and} \\ & (\tau \textit{ preserves orientation and } n \textit{ is even; or } \tau \textit{ reverses orientation and } n \textit{ is odd.)} \end{cases}$$

where C_1 and C_2 are disjoint usual coincidence classes of the pair $(f, f \circ \tau)$.

We call a Borsuk-Ulam coincidence class C essential if $|ind|(f, \tau; C) \neq 0$ and we define $NBU(f, \tau)$, the Nielsen-Borsuk-Ulam number of the pair (f, τ), as the number of essential Borsuk-Ulam coincidences classes. The definitions above are exactly what we need in order to prove:

Proposition 4. *[2, 2.7] If f' is a map homotopic to f then f' has at least $NBU(f, \tau)$ pairs of Borsuk-Ulam coincidence points.*

Nielsen-Borsuk-Ulam number in $\mathbb{T}^2$

In [4, Proposition 30, Proposition 32] we can find a classification for free involutions on $\mathbb{T}^2$, i.e, the authors proved that there are two free involutions in the torus $\mathbb{T}^2$, up to equivalence:

$$(x, y) \mapsto \left(x + \frac{1}{2}, y\right), \qquad (x, y) \mapsto \left(-x, y + \frac{1}{2}\right).$$

Theorem 5 ([7, 5])**.** *Let τ_1 be the free involution above. So every class of homotopy $\beta \in [\mathbb{T}^2, \mathbb{T}^2]$ does not have the Borsuk-Ulam property with respect to τ_1.*

Theorem 6 ([7, 5])**.** *Let τ_2 be the aforementioned involution and $\beta \in [\mathbb{T}^2, \mathbb{T}^2]$ a homotopy class. Then β has the Borsuk-Ulam property with respect to τ_2 if, and only if, homomorphism $\beta_\# : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by*

$$\beta_\# = \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}$$

where $(m_1, n_1) \neq (0, 0)$, m_2 and n_2 are even.

Those results will help us with the following:

Theorem 7. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a map. Then,*

$$\mathrm{NBU}(f, \tau_1) = 0$$

and

$$\mathrm{NBU}(f, \tau_2) = \begin{cases} 2 \textit{ if } f_\# = \begin{pmatrix} p & 2k \\ q & 2l \end{pmatrix}; \\ 0 \textit{ otherwise,} \end{cases}$$

with $p, q, k, l \in \mathbb{Z}$, $(p, q) \neq (0, 0)$ and $f_\#$ being the homomorphism induced from f in the fundamental group. Moreover, for each map $f : \mathbb{T}^2 \to \mathbb{T}^2$ there exists f' homotopic to f which realizes the $\mathrm{NBU}(f, \tau_i)$, i.e., the torus $\mathbb{T}^2$ is a Wecken type space in Nielsen-Borsuk-Ulam theory.

Nielsen-Borsuk-Ulam number in $\mathbb{T}^3$

In [6] we can find classifications for free involutions on $\mathbb{T}^3$. Up to equivalence there are four such involutions. These are described in [6] by:

$$h_1(x, y, z) = \left(x, y, z + \frac{1}{2}\right)$$
$$h_2(x, y, z) = \left(-x, -y, z + \frac{1}{2}\right)$$
$$h_3(x, y, z) = \left(x, -y, z + \frac{1}{2}\right)$$
$$h_4(x, y, z) = \left(x + y, -y, z + \frac{1}{2}\right).$$

Theorem 8. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a map such that $f_\# : \pi_1(\mathbb{T}^3) \to \pi_1(\mathbb{T}^3)$ is represented by the matrix $f_\# = \begin{pmatrix} a & b & c \\ r & s & t \\ u & v & w \end{pmatrix}$. Then,*

$$\mathrm{NBU}(f, h_1) = 0, \quad \mathrm{NBU}(f, h_3) = 0, \quad \mathrm{NBU}(f, h_4) = 0 \quad \textit{and}$$

$$\mathrm{NBU}(f, h_2) = \begin{cases} 4 \textit{ if } c, t, w \textit{ are even}, (a, r, u) \neq (0, 0, 0), \\ \quad (b, s, v) \neq (0, 0, 0) \textit{ and } (p, q) \neq (0, 0) \\ \quad \textit{or} \\ \quad \textit{if } c, t, w \textit{ are even}, (a, r, u) \neq (0, 0, 0), \\ \quad (b, s, v) \neq (0, 0, 0), (p, q) = (0, 0) \textit{ and } u = 0 \\ 0 \textit{ otherwise,} \end{cases}$$

with $p = \det \begin{pmatrix} r & s \\ u & v \end{pmatrix}$ and $q = \det \begin{pmatrix} a & b \\ u & v \end{pmatrix}$.

Remark 9. *The fact that the torus $\mathbb{T}^3$ is a Wecken space in the Nielsen-Borsuk-Ulam theory has already been demonstrated, see [2, Theorem 3.5].*

Nielsen-Borsuk-Ulam number in $\mathbb{T}^n$

For the $n-$torus, with $n > 3$, there is no classification of free involutions in the literature. That way, the study of the Nielsen-Borsuk-Ulam number in this particular space cannot be made in general. What we can do is consider a free involution τ and calculate the Borsuk-Ulam number $\mathrm{NBU}(f, \tau)$ for any map $f : \mathbb{T}^n \to \mathbb{T}^n$.

Consider the following free involutions in $\mathbb{T}^n$:

$$\tau_1(x_1, x_2, \ldots, x_{n-1}, x_n) = \left(x_1, x_2, \ldots, x_{n-1}, x_n + \frac{1}{2}\right)$$
$$\tau_3(x_1, x_2, \ldots, x_{n-1}, x_n) = \left(x_1, x_2, \ldots, x_{n-2}, -x_{n-1}, x_n + \frac{1}{2}\right)$$
$$\tau_4(x_1, x_2, \ldots, x_{n-1}, x_n) = \left(x_1 + x_2, -x_2, x_3, \ldots, x_{n-1}, x_n + \frac{1}{2}\right).$$

Observe that the mentioned involutions are generalizations to the $n-$torus of the involutions h_1, h_3 and h_4 of $3-$torus. Applying the same method used previously in $\mathbb{T}^3$ to these involutions, we can demonstrate that the Nielsen-Borsuk-Ulam number is zero for those involutions, i.e., for any map $f : \mathbb{T}^n \to \mathbb{T}^n$ we have

$$\mathrm{NBU}(f, \tau_1) = 0, \qquad \mathrm{NBU}(f, \tau_3) = 0 \quad \text{and} \quad \mathrm{NBU}(f, \tau_4) = 0.$$

For the free involution

$$\tau_2(x_1, x_2, \ldots, x_{n-1}, x_n) = \left(-x_1, -x_2, \ldots, -x_{n-1}, x_n + \frac{1}{2}\right)$$

in $\mathbb{T}^n$, we have $\mathrm{NBU}(f, \tau_2) \neq 0$ for some map $f : \mathbb{T}^n \to \mathbb{T}^n$. Indeed, let $g : \mathbb{T}^n \to \mathbb{T}^n$ be a map such that

$$g_\# = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 2b_1 \\ 0 & 1 & 0 & \cdots & 0 & 2b_2 \\ 0 & 0 & 1 & \cdots & 0 & 2b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 2b_n \end{pmatrix}$$

where $b_i \in \mathbb{Z}$, i.e.,

$$g(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, 2b_1x_1 + 2b_2x_2 + \ldots + 2b_nx_n).$$

Let $g' : \mathbb{T}^n \to \mathbb{T}^n$ be defined by

$$g'(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, 2b_1x_1 + 2b_2x_2 + \ldots + 2b_nx_n + \epsilon(x_n)),$$

where $\epsilon : \mathbb{T}^1 \to \mathbb{T}^1$ is given by $\epsilon(x) = \frac{1}{n_0}\,\mathrm{sen}(2\pi x)$, with $n_0 \in \mathbb{N}$ conveniently chosen. Note that g' is homotopic to g and

$$g'(x_1, \ldots, x_n) = (g' \circ \tau_2)(x_1, \ldots, x_n) \Leftrightarrow \begin{cases} x_1 = 0, \frac{1}{2} \\ x_2 = 0, \frac{1}{2} \\ \vdots \\ x_n = 0, \frac{1}{2} \end{cases}.$$

Then, we have that the cardinality of the coincidence set of pair $(g', g' \circ \tau_2)$ is equal to 2^n, $\#\,\mathrm{Coin}(g', g' \circ \tau_2) = 2^n$, and the cardinality of the Borsuk-Ulam coincidence set of pair (g', τ_2) is 2^{n-1}, $\#\,\mathrm{BUCoin}(g', \tau_2) = 2^{n-1}$. Therefore, there exists 2^{n-1} essential Borsuk-Ulam coincidence classes. Thus, we can conclude that $\mathrm{NBU}(g, \tau_2) = 2^{n-1}$.

The results obtained here for the Nielsen-Borsuk-Ulam number in low dimension $n-$torus, $n = 1, 2, 3$, and the example of the map g in $\mathbb{T}^n$ above, induces the formulation of the following conjecture:

Conjecture 10. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a map and τ a free involution in $\mathbb{T}^n$. Then*

$$\mathrm{NBU}(f, \tau) = \begin{cases} 2^{n-1} \textit{ or} \\ 0. \end{cases}$$

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