The well-posedness in Besov-Herz spaces for the inhomogeneous incompressible Euler equations

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Abstract

In this work we study the inhomogeneous incompressible Euler equations in $\mathbb{R}^n$ with $n \geq 3$. We obtain well-posedness and blow-up results in a new framework for inhomogeneous fluids, covering particularly critical cases of the regularity. Our results provide a larger initial data class for the existence of a well-defined flow.

Introduction

We consider the density-dependent incompressible Euler equations

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho\,(\partial_t u + u \cdot \nabla u) + \nabla \pi = \rho f, \\ \text{div } u = 0, \\ (\rho, u)_{|t=0} = (\rho_0, u_0), \end{cases} \qquad (\text{IE}_\rho)$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and $n \geq 3$, ρ is the density, the field $u = (u_1, \ldots, u_n)$ is the velocity of the fluid, π is the scalar pressure and f denotes a given external force. Moreover, we assume that there are three constants $\underline{\rho}, \overline{\rho}, \widetilde{\rho} > 0$ such that

$$0 < \underline{\rho} \leq \rho_0(x) \leq \overline{\rho} < \infty \qquad (1)$$

and $\rho_0(x) \to \widetilde{\rho}$ when $|x| \to \infty$. Making the change of variable $a = 1/\rho - 1$, for $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ we can rewrite (IE_ρ) as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u + (1+a)\nabla \pi = f, \\ \text{div } u = 0, \\ (a, u)_{|t=0} = (a_0, u_0). \end{cases} \qquad (\text{IE}_a)$$

Functional setting

Definition 1. For $1 \leq p, q, r \leq \infty$ and $\alpha, s \in \mathbb{R}$, we define the nonhomogeneous Besov-Herz space $BK^{\alpha,s}_{p,q,r}(\mathbb{R}^n)$ as follows

$$BK^{\alpha,s}_{p,q,r} := \{u \in \mathcal{S}' : \Delta_j u \in K^{\alpha}_{p,q},\ \forall\ j \geq -1\}$$

and $\|u\|_{BK^{\alpha,s}_{p,q,r}} < \infty$, where the norm is given by

$$\|u\|_{BK^{\alpha,s}_{p,q,r}} := \begin{cases} \left(\sum_{j \geq -1} \left(2^{sj} \|\Delta_j u\|_{K^{\alpha}_{p,q}}\right)^r \right)^{1/r} < \infty, & \text{if } r < \infty, \\ \sup_{j \geq -1} 2^{sj} \|\Delta_j u\|_{K^{\alpha}_{p,q}} < \infty, & \text{if } r = \infty, \end{cases}$$

and $K^{\alpha}_{p,q}(\mathbb{R}^n) := \{u \in L^p_{\text{loc}}(\mathbb{R}^n) : \|u\|_{K^{\alpha}_{p,q}} < \infty\}$ with

$$\|u\|_{K^{\alpha}_{p,q}} := \left(\sum_{k \geq -1} \left(2^{\alpha k} \|u\|_{L^p(A_k)}\right)^q \right)^{1/q} < \infty,$$

where $A_{-1} := \{x \in \mathbb{R}^n : |x| < 2^{-1}\}$ and for each $k \geq 0$ $A_k := \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k\}$.

Our main results

Below we state our main results.

Theorem 1. Consider $1 < p < \infty$, $1 \leq q, r \leq \infty$, $0 \leq \alpha < n(1 - 1/p)$, and $s \geq n/p + 1$. Suppose also that $r = 1$ when $s = n/p + 1$.

(i) Let $a_0 \in BK^{\alpha,s}_{p,q,r}$, $f \in L^1_T(BK^{\alpha,s}_{p,q,r})$ and $u_0 \in BK^{\alpha,s}_{p,q,r}$ with $\text{div}\, u_0 = 0$. There exist $T \in (0, \infty)$ and a small constant $c > 0$ such that if $\|a_0\|_{BK^{\alpha,s}_{p,q,r}} \leq c$, then system ($\text{IE}_a$) admits a unique solution $(a, u, \nabla\pi)$ satisfying

$$a \in C([0,T]; BK^{\alpha,s}_{p,q,r}),\ u \in C([0,T]; BK^{\alpha,s}_{p,q,r}),$$
$$\nabla \pi \in L^1_T(BK^{\alpha,s}_{p,q,r}).$$

(ii) Let $\{(a_{0,k}, u_{0,k})\}_{k \in \mathbb{N}}$ be a bounded sequence of pairs in $BK^{\alpha,s}_{p,q,r}$ such that $a_{0,k} \to a_0$ and $u_{0,k} \to u_0$ in $BK^{\alpha,s-1}_{p,q,r}$ as $k \to \infty$. Consider (a_k, u_k) and (a, u) the solutions obtained in item (i) with the respective initial data $(a_{0,k}, u_{0,k})$ and (a_0, u_0). Then, there exists $T > 0$ such that $\{a_k\}_{k \in \mathbb{N}}$ and $\{u_k\}_{k \in \mathbb{N}}$ are bounded in $L^\infty_T(BK^{\alpha,s}_{p,q,r})$ and $(a_k, u_k) \overset{k \to \infty}{\longrightarrow} (a, u)$ in the space

$$C([0,T]; BK^{\alpha,s-1}_{p,q,r}) \times C([0,T]; BK^{\alpha,s-1}_{p,q,r}).$$

Theorem 2. Consider $1 < p, q < \infty$, $0 \leq \alpha < n(1 - 1/p)$ and $0 < T^* < \infty$ and assume a_0, u_0 and f as in Theorem 1.

(i) Let $s > n/p + 1$ with $1 \leq r \leq \infty$. Then, the corresponding local solution $(a, u) \in C([0,T]; BK^{\alpha,s}_{p,q,r})$ and $\nabla \pi \in L^1_T(BK^{\alpha,s}_{p,q,r})$, given in Theorem 1 (i), blows up at time $T^* > T$ in $BK^{\alpha,s}_{p,q,r}$, that is, $\limsup_{t \nearrow T^*} \|a(t)\|_{BK^{\alpha,s}_{p,q,r}} = \infty$ or $\limsup_{t \nearrow T^*} \|u(t)\|_{BK^{\alpha,s}_{p,q,r}} = \infty$, if and only if

$$\int_0^{T^*} \|\nabla \times u(t)\|_{\dot{B}^0_{\infty,\infty}} dt = \infty.$$

(ii) Let $s = n/p + 1$ with $r = 1$. Then, the corresponding local solution $(a, u) \in C([0,T]; BK^{\alpha,n/p+1}_{p,q,1})$ and $\nabla \pi \in L^1_T(BK^{\alpha,n/p+1}_{p,q,1})$, given in Theorem 1 (i), blows up in time $T^* > T$ in the space $BK^{\alpha,n/p+1}_{p,q,1}$, that is, $\limsup_{t \nearrow T^*} \|a(t)\|_{BK^{\alpha,n/p+1}_{p,q,1}} = \infty$ or $\limsup_{t \nearrow T^*} \|u(t)\|_{BK^{\alpha,n/p+1}_{p,q,1}} = \infty$, if and only if

$$\int_0^{T^*} \|\nabla \times u(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty.$$

Linear estimates for the proof of Theorems 1 and 2

Proposition 1. Assume that $1 \leq p < \infty$, $1 \leq q, r \leq \infty$, $\alpha \geq 0$, and $s \geq n/p + 1$ with $r = 1$ if $s = n/p + 1$. Consider $a_0 \in BK^{\alpha,s}_{p,q,r}$ and a field $u \in BK^{\alpha,s}_{p,q,r}$ with $\nabla \cdot u = 0$ and $\nabla u \in L^\infty(\mathbb{R}^n)$ for $T > 0$. If $a \in L^\infty_T(BK^{\alpha,s}_{p,q,r})$ is a solution of the transport equation

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0 \\ a(\cdot, 0) = a_0 \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \qquad (0.1)$$

then, for some constant $C > 0$ we have the estimate

$$\|a\|_{L^\infty_T(BK^{\alpha,s}_{p,q,r})} \leq C \exp\left(CT\|u\|_{L^\infty_T(BK^{\alpha,s}_{p,q,r})}\right) \|a_0\|_{BK^{\alpha,s}_{p,q,r}}.$$

Proposition 2. Let $1 < p < \infty$, $1 \leq q, r \leq \infty$, $0 \leq \alpha < n(1 - 1/p)$ and $s \geq n/p + 1$ with $r = 1$ if $s = n/p + 1$. Consider $u_0 \in BK^{\alpha,s}_{p,q,r}$, a divergence-free vector field $v \in L^\infty_T(BK^{\alpha,s}_{p,q,r})$, $f \in L^1_T(BK^{\alpha,s}_{p,q,r})$ and $a \in L^\infty_T(BK^{\alpha,s}_{p,q,r})$ for $T > 0$. Suppose also that

$$(u, \nabla \pi) \in L^\infty_T(BK^{\alpha,s}_{p,q,r}) \times L^1_T(BK^{\alpha,s}_{p,q,r})$$

solves the linearized Euler equations

$$\begin{cases} \partial_t u + v \cdot \nabla u + (1+a)\nabla \pi = f \\ \text{div } u = 0 \end{cases}, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+,$$

with the initial data $u(\cdot, 0) = u_0$. Then, there holds

$$\begin{aligned} &\|u\|_{L^\infty_T(BK^{\alpha,s}_{p,q,r})} + \|\nabla \pi\|_{L^1_T(BK^{\alpha,s}_{p,q,r})} \\ &\leq C \exp\left(C \int_0^T \|v(\tau)\|_{BK^{\alpha,s}_{p,q,r}}\, d\tau \right) \\ &\quad \times \left(\|u_0\|_{BK^{\alpha,s}_{p,q,r}} + \|f\|_{L^1_T(BK^{\alpha,s}_{p,q,r})} + \|a\|_{L^\infty_T(BK^{\alpha,s}_{p,q,r})} \|\nabla \pi\|_{L^1_T(BK^{\alpha,s}_{p,q,r})} \right). \end{aligned}$$

Idea of the proofs of Theorems 1 and 2

The proof of Theorem 1 is based on an approximate linear scheme and its uniform boundedness via the aforementioned linear estimates, and then showing the convergence of the approximate solutions. For the continuous dependence w.r.t. the initial data, we also need to prove some contraction-type estimates. While the proof of Theorem 2 is based on logarithm-type inequalities as well as arguments come from the proofs of the linear estimates.

References

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