Solutions of elliptic systems without variational structure via fixed point in cones

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Abstract

In this work, we follow Cosner [1] to study two existence results of positive solutions for elliptic systems without variational structure via fixed point in cones, which allows us to even deal with superlinear systems. In the first existence result, we will consider the region where the solution is defined as a bounded domain with smooth boundary and the operator is uniformly elliptic in its divergent form with smooth coefficients. In the second result, we add the hypothesis of convexity and consider the Laplacian operator. In both results, we state some assumptions about the nonlinearity term, including some growth conditions. To guarantee the existence results, our main tool is a Fixed Point Theorem at Cones by Amann [2].

Keywords: Ordered Banach Space, Fixed Point Index in Cones, Leray–Schauder Topological Degree, Elliptic Systems.

Introduction

The main goal of this work is to study a class of elliptic systems as a application of the Fixed Point Index theory, based mainly on Amann [2] and Cosner [1]. Particularly, we are interested in the existence of non-negative and non-trivial solution for the following elliptic system of m equations

$$\begin{cases} L_\mu u_\mu = f_\mu(\vec{u}) & \text{in } \Omega, \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

assuming that:

- $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain with smooth boundary;
- $L_\mu \cdot := -\sum_{i,j=1}^{N} \partial_i(a_{\mu_{ij}}(x)\partial_j \cdot) + \sum_{i=1}^{N} b_{\mu_i}(x)\partial_i \cdot + c_\mu(x)\cdot$, $\forall \mu \in \{1, \cdots, m\}$;
- $a_{\mu_{ij}}$, b_{μ_i}, c_μ smooth, $c_\mu(x) \geq 0$, $\forall x \in \Omega$, $(\forall i, j \text{ e } \forall \mu)$.

General and growth conditions about $\vec{f} = (f_1, \cdots, f_m)$:

$(H0)$ $\vec{f} : \mathbb{R}^m \to \mathbb{R}^m$ smooth;

$(H1)$ $x_\mu \geq 0, \ \forall \mu \implies f_\mu(x) \geq 0, \ \forall \mu$;

$(H2)$ $\lim_{x_\mu \to \infty} \dfrac{f_\mu(x)}{x_\mu} > \lambda_1^{\mu^*}$ uniformly in $x_\nu \geq 0$ for $\nu \neq \mu$;

$(H3)$ $\inf_\mu \left\{ a_\mu^0 \lambda_1 - b_\mu^0 \sqrt{\lambda_1} \right\} > 0$;

$(H4)$ There exists $\gamma < \gamma_0$ such that $\left\langle x, \vec{f}(x) \right\rangle \leq \gamma |x|^2$, when $|x| < \gamma_0$;

$(H5)$ $\dfrac{\partial f_\mu(x)}{\partial x_\nu} \geq 0$, for $\mu \neq \nu$ and $x_\mu \geq 0$;

$(HC1)$ $\lim_{|x| \to \infty} \dfrac{|\vec{f}(x)|}{|x|^\beta} = 0$, for $\beta = \dfrac{N+1}{N-1}$;

$(HC2)$ $\lim_{|x| \to \infty} \dfrac{|\vec{f}(x)|}{|x|^\beta} = 0$ for some $\beta < \dfrac{N}{N-2}$ if $N \geq 3$ and for any β if $N = 2$.

Where $b_\mu^0 := \sup_\Omega \left[\sum_{i=1}^{n} (b_{\mu_i}) \right]^{1/2}$, $c_\mu^0 := \inf_\Omega c_\mu$, and $\gamma_0 := \inf_\mu \left\{ a_\mu^0 \lambda_1 - b_\mu^0 \sqrt{\lambda_1} + c_\mu^0 \right\}$.

Main Result

Theorem 0. (Fixed Point Theorem) $\vec{F} : \overline{P_\rho} \to P$ will have a fixed point $\vec{u}$, with $0 < r \leq \|\vec{u}\|_{(C(\overline{\Omega}))^m} < R < \infty$, provided

$$\vec{F}\vec{u} \neq s\vec{u}, \quad s \geq 1, \text{ para } \|\vec{u}\|_{(C(\overline{\Omega}))^m} = r, \tag{2}$$

$$\vec{F}\vec{u} \neq \vec{u} - t\psi_*, \quad t \geq 0, \text{ para } \|\vec{u}\|_{(C(\overline{\Omega}))^m} = R, \tag{3}$$

where $\psi_* > 0$ is some function in the positive cone $P = \left(P_{C(\overline{\Omega})}\right)^m$.

Application I

In order to obtain a strictly positive solution, an additional assumption is necessary.

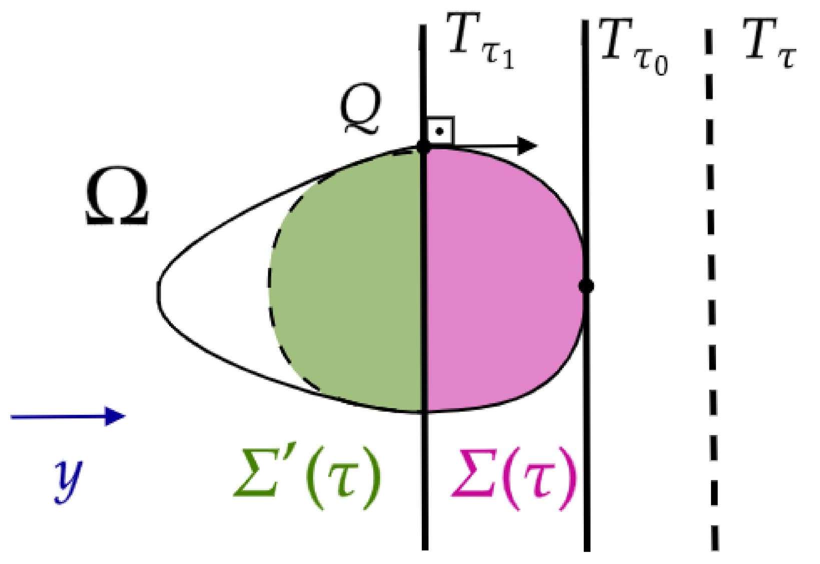
Definition 1. (Quasi-irreducible function) A function $\vec{f} : \mathbb{R}^m \to \mathbb{R}^m$ is quasi-irreducible if $x \in \mathbb{R}^m$ is such that $x_\mu \geq 0$ for all $\mu \in \{1, \cdots, m\}$ and $x_\mu > 0$ for $\mu \in \Gamma \subsetneq \{1, \cdots, m\}$, then $f_\nu(x) > 0$ for some $\nu \notin \Gamma$.

Theorem 1. (First Existence Theorem) Let $N \geq 3$ and suppose that $\vec{f}$ satisfies $(H0)$, $(H1)$, $(H2)$, $(H3)$, $(H4)$ and $(HC1)$. Then, (1) has a nontrivial nonnegative solution. If $\vec{f}$ is quasi-irreducible then each component of the solution is strictly positive in Ω.

Application II

For this application the operator L_μ must be replaced by $-\Delta$.

Definition 2. (Moving Plane Method) Let y be a vector in $\mathbb{R}^n$, and let $T_\tau = \{x \in \mathbb{R}^n : \langle x, y \rangle = \tau\}$. If τ is such that T_τ intersects Ω, define $\Sigma(\tau) := \{x \in \mathbb{R}^n : \langle x, y \rangle > \tau\}$. Define $\Sigma'(\tau)$ to be the reflection of $\Sigma(\tau)$ through T_τ.



Theorem 2. (Second Existence Theorem) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth and convex boundary. Suppose that $\vec{f}$ satisfies $(H0)$, $(H1)$, $(H2)$, $(H4)$, $(H5)$ and $(HC2)$. Then (1) has a nontrivial nonnegative solution. If $\vec{f}$ is quasi-irreducible, then each component of the solution is strictly positive in Ω.

Conclusion

We know that several phenomena in Physics, Biology, Engineering, etc., are modeled by systems of partial differential equations. And in many situations, it is necessary to look for non-negative solutions to these problems. We conclude that, through applications I and II, the Fixed Point Theory in Cones is a powerful tool in obtaining solutions for such systems, especially when there is no variational structure.

References

[1] Cosner, C.. Positive solutions for superlinear elliptic systems without variational structure. Nonlinear Analysis: Theory, Methods e Applications 8 (1984), no. 12, 1427–1436.

[2] Amann, H.. Fixed point theorems and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Review 19 (1977), no. 4, vii–vii.

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