Optimal Spherical Codes, Hopf Fibration and Quaternion Algebras for Constructing 4D Modulations

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Abstract

High-order modulations have been highlighting through the use of some formats that use the light polarization to transmit data and that can be optimized by using operations of rotation and translation of the symbols of the constellations defined by polytopes. The aim of this presentation is construct four-dimensional (4D) modulations applied in coherent optical communications systems, using concepts and results of optimal spherical codes, discrete Hopf fibration and quaternion algebras.

Initial Concepts

A *Sphere* S^{n-1} can be defined in an Euclidean $n-$space $\mathbb{R}^n$ by

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 = r^2\}, \quad (1)$$

where r is the sphere radius. When $r = 1$ then S^{n-1} is an *Unit Sphere*. A *Spherical Code* is a finite set of M points on this sphere. Given a dimension n and an integer number $M > 0$, a spherical code with M points such that the minimum distance between two points in the code is the largest possible is called *Optimal Spherical Codes*.

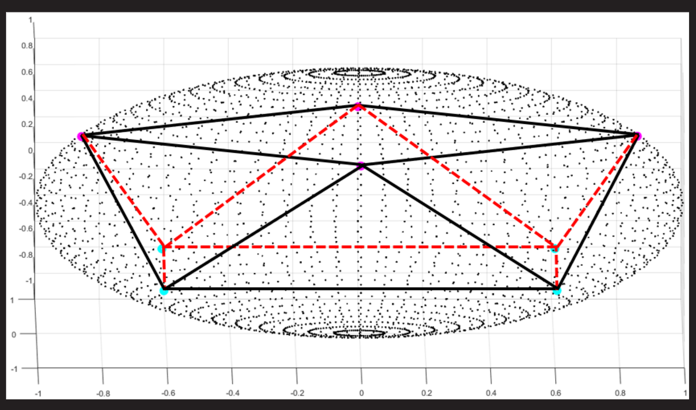

Figure 1: Optimal Spherical Codes with $M = 8$ points in S^2.

The *Stokes Parameters* are a set of values $\{S_0, S_1, S_2, S_3\}$ that describe the polarization state of electromagnetic radiation. They can be written in S^2 as

$$S^2 = \{(S_1, S_2, S_3) \in \mathbb{R}^3 \mid S_1^2 + S_2^2 + S_3^2 = S_0^2\}.$$

The term fibration means the decomposition of a given geometric space into smaller subspaces called *Hopf fibrations*. We are interested here in $(S^1 \times S^2) \to S^3$ which means that the space S^3 is bundled by circles S^1 and a base space S^2. The *Inverse Hopf Map* has the following form

$$\begin{cases} x_1 = cos\left(\frac{\theta}{2}\right) \cdot cos\left(\varphi + \frac{\phi}{2}\right) \\ x_2 = cos\left(\frac{\theta}{2}\right) \cdot sen\left(\varphi + \frac{\phi}{2}\right) \\ x_3 = sen\left(\frac{\theta}{2}\right) \cdot cos\left(\varphi - \frac{\phi}{2}\right) \\ x_4 = sen\left(\frac{\theta}{2}\right) \cdot sen\left(\varphi - \frac{\phi}{2}\right) \end{cases}, \quad (2)$$

where $0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq 2\pi$ e $-\pi \leq \phi \leq \pi$.

A *Quaternion Algebra* $\mathcal{A} = (a, b)_{\mathbb{K}}$ over a field $\mathbb{K}$ is a central simple algebra of dimension 4 with basis $\{1, i, j, k\}$ satisfying $i^2 = a$, $j^2 = b$ and $k = ij = -ji$, where $a, b \in \mathbb{K}/\{0\}$. The standard example of quaternion algebra over real number field is the *Hamilton's Quaternions* $\mathcal{H} = (-1, -1)_{\mathbb{R}}$. There is an isomorphism between $\mathcal{H}$ and $\mathbb{R}^4$, in which

$$x = x_1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot k \;\mapsto\; (x_1, x_2, x_3, x_4). \quad (3)$$

For $x = x_1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot k \in \mathcal{H}$, the Cayley-Dickson construction shows us that x can be written as follows

$$\begin{aligned} x &= x_1 + x_2 \cdot i + x_3 \cdot j + x_4 \cdot k \\ &= (x_1 + x_2 \cdot i) + (x_3 + x_4 \cdot i) \cdot j\,, \\ &= z_1 + z_2 \cdot j \end{aligned} \quad (4)$$

where $z_1 = x_1 + x_2 i \in \mathbb{C}$, $z_2 = x_3 + x_4 i \in \mathbb{C}$ e $j^2 = -1$.

Results

The modulation in 4 dimensions (4D) is given by

$$m\text{PolSK} - n\text{PSK}, \quad (5)$$

where m is the number of points on the sphere S^2, n is the number of samples considered in the torus, PolSK(*Polarization Shift Keying*) is a polarization shift modulation that uses the polarization of light to transmit data and QPSK(*Quadrature Phase Shift Keying*) is a quadrature phase shift modulation.

Next, we present the stages of construction of the modulation:

- Obtaining the electromagnetic signal.
- Association of the signal to the Stokes parameters.
- Association of Stokes parameters to optimal spherical codes ($S_0 = 1$).
- Conversion of points (S_1, S_2, S_3) from S^2 to angles θ and ϕ.
- Association of φ angles by n-PSK.
- Use of the angles θ, ϕ and φ in Hopf's inverse mapping to obtain 4D modulation coordinates.
- Application of stereographic projection.

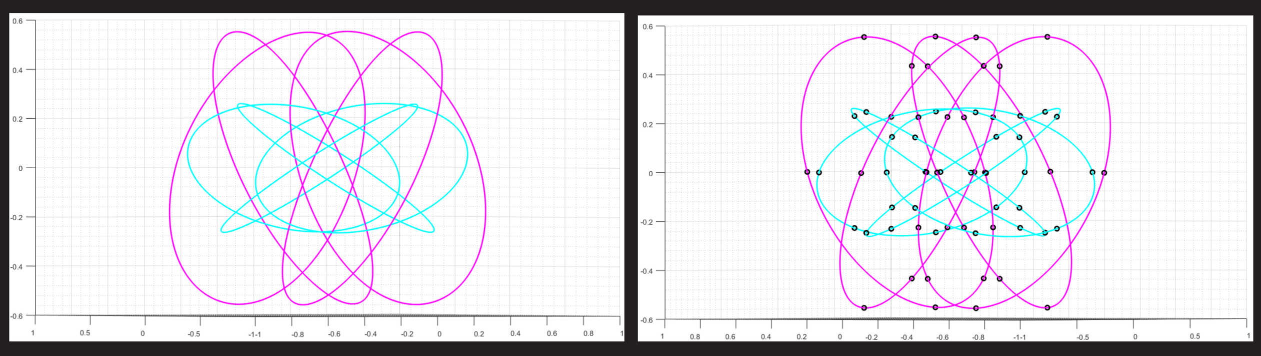

Figure 2: 8PolSK Projection and 8PolSK-8PSK Modulation

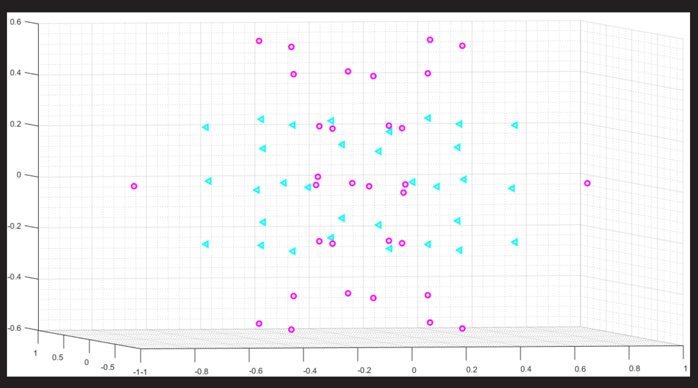

Figure 3: Optimal constellation of signs for 8PolSK-8PSK

We can labeling and get the QAM *(Quadrature Amplitude Modulation)* partitions for the signal constellations.

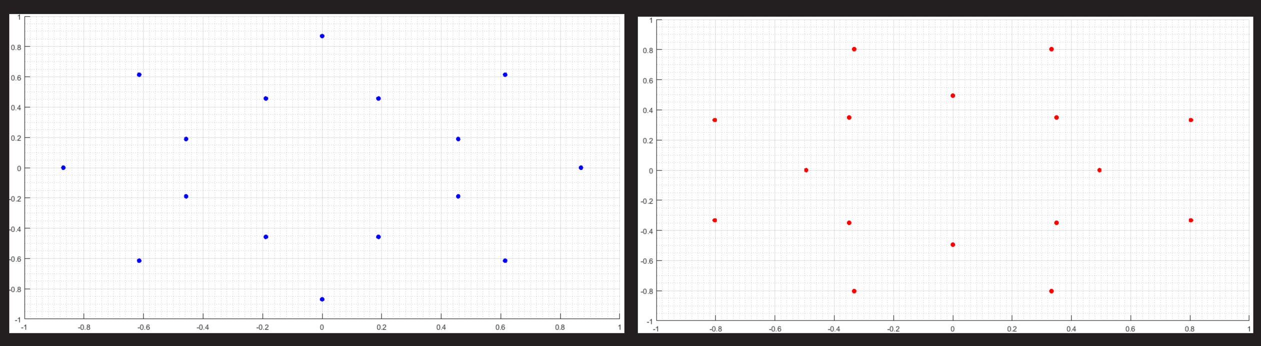

Figure 4: QAM partitions for optimal points $x_1 + x_2 i$ and $x_3 + x_4 i$.

References

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Acknowledgements

This work was supported by FAPESP 2013/25977-7, 2020/03613-7 and 2023/06476-9.