Asymptotics for Integrals in Random Matrix Models

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Abstract

The eigenvalues of random matrices model a wide range of topics, such as Big Data processes and quantum chaos. In turn, the multiplicative statistics of eigenvalues of random matrices are obtained through a complex analysis technique known as Riemann-Hilbert Problems. And to deal with these problems we usually need to find the asymptotic expansions of some complicated integrals. In this work, we obtained the asymptotic expansion for a family of integrals of interest for which classical techniques, such as Laplace and steepest-descent, do not work.

Introduction

The works with matrices whose entries are random variables started in the 1920s with Wishart. Along the time such objects showed that despite their random nature, random matrix ensembles could not be more regular. The Figure 1 exemplifies some of the universality laws in Random Matrix Theory. On the left side one has the *Semicircular Law* and on the right side, the Tracy-Widom distribution.

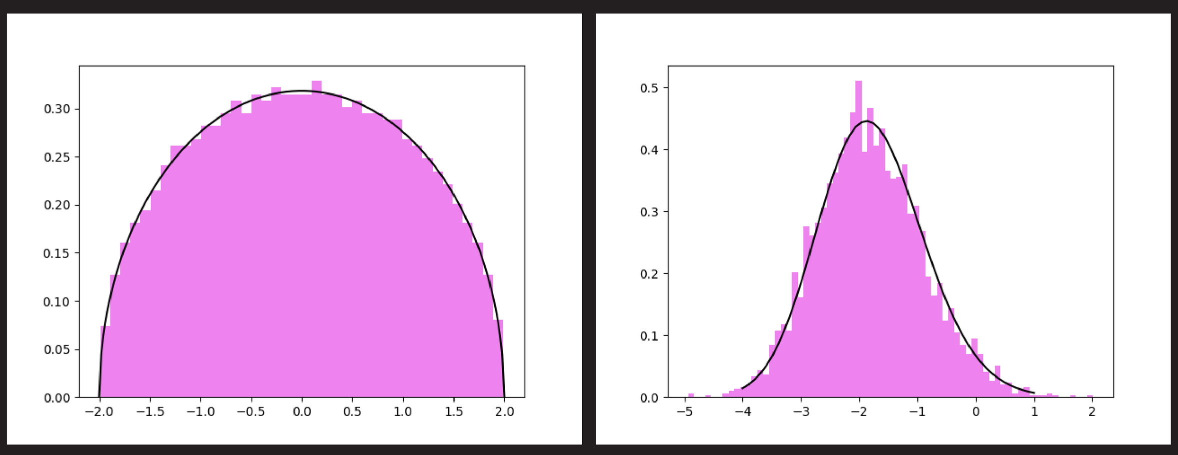

Figure 1: On the left: Empirical distribution of the eigenvalues for a 2000×2000 matrix. On the right: Empirical distribution for the largest eigenvalue of 3000 matrices.

As new applications of RMT were found,the interest in the study of new universality laws increased. Random matrices eigenvalues model particles quantum chaos, log gas, big data, and so on, justifying the study of the eigenvalues behaviour.

Theoretical background

In order to obtain an asymptotic expansion for the eigenvalues in a random matrix ensemble for the large dimension regime, we often apply Riemann Hilbert Problems (RHP) techniques.

Crash course in RHP

Defining the Cauchy transform as

$$Cf(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}\,\mathrm{d}s \quad z \in \gamma,$$

the lateral limits for $C(z)$ gives us the Plemelj formula,

$$C_+f(s) - C_-f(s) = f(s).$$

On the other side, this formula suggests that $g(z) = Cf(z)$ solves the linear RHP

$$g_+(s) - g_-(s) = f(s) \qquad (1)$$

$$g(z) = o(1), \;\; z \to \infty. \qquad (2)$$

This is the basis of the RHP Theory. For more complicated problems, the trick is to apply several invertible transformations until we reach a problem ψ that can be approximate by a simple problem like the one in (1)-(2), in the following sense:

$$\psi(z) = g(z)\left(1 + \sum_s \frac{g_s}{n^s}\right).$$

It gives us the n-asymptotic expansion of the initial RHP.

Initial problem

We want an asymptotic expansion for the multiplicative statistics of eigenvalues ruled by a kernel of orthogonal polynomials with a certain weigh. In order to obtain it we work on the asymptotic expansion for the associated RHP, namely the problem of finding a matrix-valued function $Y(z)$ such that

1. Y is analytic in $\mathbb{C}\setminus\mathbb{R}$.
2. The function has continuous boundary values on $\mathbb{R}$ related by
$$Y_+(x) = Y_-(x)[I + \omega_n(x)E_{12}],$$
where $\omega_n(z) = (1 + \mathrm{e}^{-s-n^{2/3}Q(z)})^{-1}\,\mathrm{e}^{nV(z)}$.
3. As $z \to \infty$,
$$Y(z) = (I + O(z^{-1}))z^{n\sigma_3}.$$

The case $s > s_0$ for some fixed constant $s_0 \in \mathbb{R}$ is done in [1]. The first step for expanding such result for the case $s \to -\infty$ is to find an asymptotic expansion for integrals of the type

$$F(t) = \int_0^a g(w)\ln(1 + \mathrm{e}^{-t(f(w)-\epsilon)})\,\mathrm{d}w. \qquad (3)$$

Such asymptotic is not obtained through the classic methods such as Laplace and Steepest-descent.

Result

The main result so far is shown in Theorem 1.

Theorem 1. *Let f be a locally injective crescent function that attains its only minimum in $[0, a]$ at $w = 0$. For some $0 < \epsilon < f(a)$ fixed consider the integral in (3). If, for $\hat{g}(u) := g(f^{-1}(u))/f'[f^{-1}(u)]$,*

1. *for some $k_1, k_2 > -1$*
$$\hat{g}(u) \sim \begin{cases} (u - f(a))^{k_1} & \text{as } u \to f(a), \\ u^{k_2} & \text{as } u \to 0. \end{cases}$$
2. *As $w \to \epsilon$,*
$$\hat{g}(u) \sim \sum_{n=0}^{\infty} b_n(u-\epsilon)^{\beta_n}.$$

Then,

$$F(t) = t\int_0^\epsilon \hat{g}(\epsilon - u)u\,\mathrm{d}u + I(t), \quad \text{where}$$

$$I(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\zeta(z+1)(1-2^{-z})\Gamma(z)\times \left[\int_0^{f(a)} |u-\epsilon|^{-z}\hat{g}(u)\,\mathrm{d}u\right]\mathrm{d}z,$$

for $0 < c < 1$. Moreover, for any $n > 0$ fixed and $\beta_n + 1 < d < \beta_n + 2$,

$$I(t) = \sum_{s=0}^{n} 2b_s t^{-\beta_s-1}\zeta(\beta_s+2)(1-2^{-\beta_s-1})\Gamma(\beta_s+1) + O(t^{-d}),$$

where $\hat{g}_{n+1}(u) = \hat{g}(u) - \sum_{s=0}^{n} b_s(u-\epsilon)^{\beta_s}$.

Conclusion

There is still a lot to be done concerning to the aimed RHP. However, the results found so far in the asymptotic for integrals will play a crucial role not only in the current problem, but can also be useful in another contexts.

References

[1] Promit Ghosal and Guilherme L. F. Silva. Universality for multiplicative statistics of Hermitian random matrices and the integro-differential Painlevé II equation, *Springer Berlin Heidelberg*, 2022.

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