Curvature-normalized Yamabe flow and bounded geometry

Bruno Caldeira

Departamento de Matemática - UFSCar

brunocarlotti@estudante.ufscar.br

Instituto de Matemática Pura e Aplicada

Abstract

This work presents results regarding the global existence of a curvature-normalized Yamabe flow on the manifolds with bounded geometry, which can be found in [2]. The results presented in [2] were obtained in collaboration with Luiz Hartmann (UFSCar) and Boris Vertman (Universität Oldenburg).

Introduction

Hamilton's suggestion for approaching the Yamabe problem, see [3], may be interpreted, up to rescaling, as searching for a a family $\{g(t)\}_{t\geq 0}$ of Riemannian metrics on a closed manifold (M^m, g_0), $m \geq 3$, satisfying

$$\partial_t g(t) = -\operatorname{scal}(g(t)) \cdot g(t), \quad g(0) = g_0. \tag{1}$$

The Eq.(1) is known as the *Yamabe flow equation* and it is currently well-understood in compact manifolds. Nowadays, it has been studied for noncompact and/or singular manifolds. A natural direction to head is to consider (M, g_0) as a *manifolds with bounded geometry*. Simple examples of such class include compact manifolds, Euclidean spaces, and finite products of manifolds with bounded geometries, among a few others.

One interesting property for such class is that , every $p \in M$ admits a neighborhood of uniform radii δ such that, in normal coordinates, there are quasi-isometries

$$\Psi_p : B^{T_pM}(0;\delta) \to B^M(p;\delta) \tag{2}$$

between the Euclidean space and the manifold itself.

Goals

1. Prove local existence of the Yamabe flow on manifolds with bounded geometry.
2. Normalized the Yamabe flow via its curvature.
3. Obtain global existence of the curvature normalized Yamabe flow on manifolds with bounded geometry assuming negative initial scalar curvature.

Results

First, for $\eta = (m-2)/4$, write (w.l.o.g.)

$$[g_0] = \{u^{1/\eta} g \mid u > 0\}. \tag{3}$$

Thus, by considering $g(t) = u(t)^{1/\eta} g_0$, one can rewrite (1), omitting the time variable t, as

$$\begin{aligned} &\partial_t u = (m-1)u^{-1/\eta}\Delta_{g_0} u - \eta \operatorname{scal}(g_0) u^{1-1/\eta}, \\ &u|_{t=0} = 1. \end{aligned} \tag{4}$$

Regularity-wise, set $M_T = M \times [0,T]$, $\alpha \in (0,1)$ and let us consider $u \in C^{k,\alpha}(M_T)$, which is a Banach space, for $k \geq 2$. Furthermore, write $u = 1 + v$ to get, up to rescaling,

$$\begin{aligned} &(\partial_t - \Delta_{g_0})v = (F_1 + F_2)v, \\ &v|_{t=0} = 1, \end{aligned} \tag{5}$$

where F_1 is a second-order operator and F_2 is zeroth order geometry-based operator. By localizing the norm $\|\cdot\|_{k,\alpha}$ and extending the quasi-isometry Ψ_p to $B^M(p;\delta)_T$, one can import regularity results in [4] from the Euclidean spaces to (M, g_0). Hence, the parametrix $\mathbf{Q}$ for

$$(\partial_t - a\Delta_{g_0})v = f, \quad v|_{t=0} = 0, \tag{6}$$

with $a \in C^{k,\alpha}(M_T)$ bounded from below away from zero and $f \in C^{k,\alpha}(M)$, maps

$$\mathbf{Q} : C^{k,\alpha}(M_T) \to \left(C^{k+2,\alpha} \cap t\, C^{k,\alpha}\right)(M_T) \tag{7}$$

continuously. Moreover, employing the Omori-Yau maximum principle as presented in [1], we get

Theorem 1. *If* $\operatorname{scal}(g_0) \in C^{k,\alpha}(M)$ *then there exists* $u \in C^{k+2,\alpha}(M_T)$ *solution of* (4) *for* T *sufficiently small. Furthermore, the YF on* (M, g_0) *of bounded geometry is unique (for as long as it exists).*

Now, we introduce the curvature-normalized YF of interest. Set

$$\rho(t) = \sup \operatorname{scal}(g(t)).$$

From Theorem 1 it follows that ρ is well-defined and such choice is justified by the fact that $\operatorname{vol}(M, g(t))$ may be infinite, thus precluding the usual normalizing term. This choice was originally introduced by Suarez-Serrato and Tapie in [5]. Thus, we are interested in the following flow equation, now named CYF$^+$:

$$\partial_t g(t) = (\rho(t) - \operatorname{scal}(g(t))) \cdot g(t). \tag{8}$$

For such flow, one can check that it exists if and only if the original one exists. Moreover, by analyzing the evolution of $\operatorname{scal}(g(t))$ along CYF$^+$, given by

$$\begin{aligned} \partial_t \operatorname{scal}(g(t)) =& (m-1)\Delta_{g(t)} \operatorname{scal}(g(t)) \\ &+ \operatorname{scal}(g(t))(\operatorname{scal}(g(t)) - \rho(t)), \end{aligned} \tag{9}$$

as well as obtaining uniform bounds for the conformal factor, it is possible to prove the following:

Theorem 2. *Assume* $\operatorname{scal}(g_0) \in C^{k,\alpha}(M)$ *is negative and bounded from above away from zero, with* $k \geq 4$. *Then CYF*$^+$ *exists for all positive time with conformal factor* $u \in C^{k,\alpha}(M \times [0,\infty))$.

Conclusion

- The Yamabe flow exists on manifolds with bounded geometry exists for a sufficiently small time.
- For as long as it exists, the Yamabe flow is unique.
- Assuming $\operatorname{scal}(g_0)$ negative and bounded from above away from zero, CYF$^+$ exists for all $T > 0$.

References

[1] Alías, L. J., Mastrolia, P., Rigoli, M.; *Maximum principles and geometric applications*, Springer Monographs in Mathematics. Springer, Cham, 2016. xvii+570 pp. ISBN: 978-3-319-24335-1; 978-3-319-24337-5 MR3445380

[2] Caldeira, B., Hartmann, L., Vertman, B.; *Normalized Yamabe flow on manifolds with bounded geometry*, Ann Glob Anal Geom 63, 23 (2023).

[3] Hamilton, R.; *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. MR0954419

[4] Krylov, N. V.; *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Mathematics, 96. American Mathematical Society, Providence, RI, 2008. xviii+357 pp. ISBN: 978-0-8218-4684-1 MR2435520

[5] Suárez-Serrato, P., Tapie, S.; *Conformal entropy rigidity through Yamabe flows*, Math. Ann. 353 (2012), no. 2, 333–357. MR2915539

Aknowledgements

This study was financed by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.