The Hardy parabolic problem with initial data in uniformly local Lebesgue spaces

Brandon Carhuas De La Torre & Miguel Loayza

Universidade Federal de Pernambuco

brandon.carhuas@ufpe.br & miguel.loayza@ufpe.br

UNIVERSIDADE FEDERAL DE PERNAMBUCO

Introduction

The local existence of non-negative solutions of the singular nonlinear parabolic problem

$$\begin{cases} u_t - \Delta u = |\cdot|^{-\gamma} f(u) & \text{in } \mathbb{R}^N \times (0,T), \\ u(0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

with $f \in C([0,\infty))$ non-decreasing, $0 < \gamma < \min\{2, N\}$, $u_0 \in L^r(\mathbb{R}^N)$, $u_0 \geq 0$ has been studied by Miguel et al. [3]. They showed the existence of a critical value

$$p_\gamma^\star = 1 + \frac{(2-\gamma)r}{N} \tag{2}$$

such that for $r > 1$:

- If $\gamma < N/r$, $\limsup_{t\to 0} t^{-(1+\epsilon-\gamma r/N)} f(t) < \infty$ and $\limsup_{t\to\infty} t^{-p_\gamma^*} f(t) < \infty$, for some $\epsilon \in (0, \gamma r/N)$, or $\gamma > N/r$ and $\limsup_{t\to\infty} t^{-p_\gamma^*} f(t) < \infty$, then problem (1) has a non-negative solution for every $u_0 \in L^r(\mathbb{R}^N)$, with $u_0 \geq 0$.
- If $\gamma < N/r$ and $\limsup_{t\to 0} t^{-(1-\gamma r/N)} f(t) = \infty$ or $\limsup_{t\to\infty} t^{-p_\gamma^*} f(t) = \infty$, then there exists $u_0 \in L^r(\mathbb{R}^N)$ with $u_0 \geq 0$ such that problem (1) does not have a non-negative solution.

A similar situation occurs in the case $r = 1$, substituting $\limsup_{t\to\infty} t^{-p_\gamma^*} f(t)$ by $\int_1^\infty G_\epsilon(\sigma)\sigma^{-p_\gamma^*} d\sigma$ where $G_\epsilon(t) = \sup_{0<\sigma\leq t} f(\sigma)/t^{1-\gamma/N+\epsilon}$ for $\epsilon = 0$ or $\epsilon > 0$.

Objective

Our objective is to improve the results given in [3].

For $1 \leq r \leq \infty$, the uniformly local Lebesgue space $L^r_{ul,\rho}(\mathbb{R}^N)$ is defined by

$$L^r_{ul,\rho}(\mathbb{R}^N) = \left\{ u \in L^1_{loc}(\mathbb{R}^N); \|u\|_{L^r_{ul,\rho}(\mathbb{R}^N)} < \infty \right\}, \text{ where}$$

$$\|u\|_{L^r_{ul,\rho}(\mathbb{R}^N)} := \begin{cases} \sup_{y\in\mathbb{R}^N} \left(\int_{B_\rho(y)} |u(x)|^r dx \right)^{1/r} & \text{if } 1 \leq r < \infty, \\ \operatorname{ess\,sup}_{y\in\mathbb{R}^N} \|u\|_{L^\infty(B_\rho(y))} & \text{if } \quad r = \infty, \end{cases}$$

and $B_\rho(y) \subset \mathbb{R}^N$ denotes the open ball centered at y with radius $\rho > 0$. It is clear that $L^\infty_{ul,\rho}(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$. We denote by $\mathcal{L}^r_{ul,\rho}(\mathbb{R}^N)$ the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ in the space $L^r_{ul,\rho}(\mathbb{R}^N)$, that is, $\mathcal{L}^r_{ul,\rho}(\mathbb{R}^N) := \overline{BUC(\mathbb{R}^N)}^{\|\cdot\|_{L^r_{ul,\rho}(\mathbb{R}^N)}}$.

To reduce notation, we write $L^r_{ul}(\mathbb{R}^N)$ and $\mathcal{L}^r_{ul}(\mathbb{R}^N)$ if $\rho = 1$.

First Result

The notion of solution used in the work is the following.

Definition 1. *Let $\gamma > 0$, $u_0 \in L^r_{ul}(\mathbb{R}^N)$, $1 \leq r < \infty$ and $f \in C(\mathbb{R})$. We say that $u \in L^\infty((0,T), L^r_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T), L^\infty(\mathbb{R}^N))$, for some $T > 0$, is a solution of the problem (1) if it verifies*

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)|\cdot|^{-\gamma} f(u(\sigma))\, d\sigma$$

a.e. in $\mathbb{R}^N \times (0,T)$, where $\{S(t)\}_{t\geq 0}$ denotes the heat semigroup.

Theorem 1. *Suppose that $f \in C(\mathbb{R})$ is a nondecreasing function, $0 < \gamma < \min\{2, N\}$, p_γ^* defined by (2), and one of the following conditions hold:*

(i) $u_0 \in L^1_{ul}(\mathbb{R}^N)$ and

$$\int_1^\infty \sigma^{-p_\gamma^*} \tilde{F}(\sigma) d\sigma < \infty, \text{ where } \tilde{F}(t) := \sup_{1\leq|\sigma|\leq t} \frac{f(\sigma)}{\sigma}. \tag{3}$$

(ii) $r > 1$ and

$$\limsup_{|t|\to\infty} |t|^{-p_\gamma^*} |f(t)| < \infty, \text{ if } u_0 \in \mathcal{L}^r_{ul}(\mathbb{R}^N), \tag{4}$$

$$\lim_{|t|\to\infty} |t|^{-p_\gamma^*} |f(t)| = 0, \text{ if } u_0 \in L^r_{ul}(\mathbb{R}^N). \tag{5}$$

Then, problem (1) has a solution u defined on some interval $(0,T)$. Moreover, $t^{N/2r}\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C$ for some $C > 0$ and all $t \in (0,T)$.

Remark For $f(t) = |t|^{p-1}t$, $t \in \mathbb{R}$, $p > 1$, conditions (3) and (5) are verified if $p < p_\gamma^*$, and condition (4) holds if $p \leq p_\gamma^*$. Thus, for $0 < \gamma < \min\{2, N\}$, problem (1) has a local solution for every $u_0 \in L^r_{ul}(\mathbb{R}^N)$, $r \geq 1$ and only for $u_0 \in \mathcal{L}^r_{ul}(\mathbb{R}^N)$ if $p = p_\gamma^*$.

More results

When we consider non-negative initial data we have the following.

Theorem 2. *Let $f : [0,\infty) \to [0,\infty)$ be a continuous and non-decreasing function, and let $0 < \gamma < \min\{2, N\}$. Problem (1) has a local non-negative solution for every $u_0 \in \mathcal{L}^r_{ul}(\mathbb{R}^N)$, $u_0 \geq 0$, $r \geq 1$ if and only if*

$$\begin{cases} \int_1^\infty \sigma^{-[1+(2-\gamma)/N]} F(\sigma) d\sigma < \infty & \text{if } r = 1, \\ \limsup_{t\to\infty} t^{-p_\gamma^*} f(t) < \infty & \text{if } r > 1, \end{cases} \tag{6}$$

where $F(t) = \sup_{1\leq\sigma\leq t} f(\sigma)/\sigma$, $t > 0$.

Assuming that $f \in C(\mathbb{R})$ is locally Lipschitz, non-decreasing it is well defined the non-decreasing functions $\mathcal{G} : [0,\infty) \to [0,\infty)$ given by

$$\mathcal{G}(s) = \sup_{\substack{|u|,|v|\leq s \\ u\neq v}} \frac{f(u) - f(v)}{u - v} \quad s > 0; \quad \mathcal{G}(0) = 0.$$

We also establish unconditional and conditional uniqueness results.

Theorem 3. *Let $0 < \gamma < \min\{2, N\}$ and $f \in C(\mathbb{R})$*

(i) Assume that there exists $M : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$|f(\tau) - f(t)| \leq M(|\tau|, |t|)\, |\tau - t|, \tag{7}$$

for $\tau, t \in \mathbb{R}$ and $\sup_{t\in(0,T)} \|M(|u(t)|, |v(t)|)\|_{L^\alpha_{ul}(\mathbb{R}^N)} < \infty$ for $u, v \in L^\infty((0,T), L^r_{ul}(\mathbb{R}^N))$ with $\alpha > N/(2-\gamma)$ and $1/\alpha + 1/r + \gamma/N < 1$. Then the problem (1) has a unique solution in the class $L^\infty((0,T), L^r_{ul}(\mathbb{R}^N))$.

(ii) Suppose that f is locally Lipschitz. Problem (1) admits a unique solution in the class

$$\{u \in L^\infty((0,T), L^r_{ul}(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T), L^\infty(\mathbb{R}^N)); \sup_{t\in(0,T)} t^{\frac{N}{2r}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_0\}$$

if $\int_0^T \mathcal{G}^q(C_0\sigma^{-N/2r}) d\sigma < \infty$ with $\gamma/N + 1/r < 1$, $1/q + \gamma/2 < 1$.

Conclusion

We obtain local existence results allowing sign-changing solutions. In particular, when we consider non-negative solutions a condition necessary and sufficient is obtained. We also establish a conditional and unconditional uniqueness result.

References

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Acknowledgement

This work was supported by CNPq, Brazil.