Henstock Kurzweil Integral and Applications

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INTRODUCTION

The concept of integral arises from the attempt to calculate areas and volumes of figures and one of the techniques employed is precisely approximation by known figures. Then, over the years, it is clear that the integration process also has a strong connection with the derivative.
Its development goes through Riemann, in the 1850s, who separates such concepts again using limits and sums, and is equivalent to the concept presented by Darboux, when we are working with bounded functions, which uses the concept of upper and lower integrals of a bounded function in an interval.

RIEMANN INTEGRAL

When considering all functions in an interval where the integration process could be defined, we have that:
A function $f : I \to \mathbb{R}$ is said to be *Riemann-Integrble* on I if there exists a number $A \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a number $\gamma_\epsilon > 0$ such that if $\dot{P} := (I_i, t_i)_{i=1}^n$ any tagged partition of I such that $l(I_i) < \gamma_\epsilon(t_i)$, for $i = 1, ..., n$, then

$$|S(f; \dot{P}) - A| \leq \epsilon$$

We can call the number A the limit of the Riemann Sum $S(f, \dot{P})$ when the trend norm tends to zero, that is, $||\dot{P}||\ rightarrow 0$, and if $f \in \mathbb{R}_{[a,b]}$, then A is said to be the Riemann integral of f over $[a, b]$, and we write:

$$L = \int_a^b f \ ou \ \int_a^b f(x)dx$$

LEBESGUE INTEGRAL

At the beginning of the 20th century, Lebesgue proposed a new, more general concept of integral that could integrate a greater number of functions, which solves several problems related to integrals, such as the problem of the validity of the Fundamental Theorem of Calculus, since according to Lebesgue, for the fundamental theorem to be valid, it is necessary that the function has a bounded derivative.

Theorem: A function $f : [a, b] \to \mathbb{R}$ is Lebesgue Integrable if and only if it is absolutely integrable.

Exemple 1: Defining $f : [0, 1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} t^2 \cos\left(\dfrac{\pi}{t^2}\right), \ t \in (0, 1]; \\ 0, \ t = 0. \end{cases}$$

Calculating f':

$$f'(t) = \begin{cases} 2t \cos\left(\dfrac{\pi}{t^2}\right) + \dfrac{2\pi}{t} \sin\left(\dfrac{\pi}{t^2}\right), \ t \in (0, 1]; \\ 0, \ t = 0. \end{cases}$$

The function f' is not Lebesgue Integrable because it is not absolutely integrable.

BASIC CONCEPTS

The Henstock-Kurzweil integral naturally pays more attention to markings than more traditional models of integration, then, this concept is constructed by allowing the $\gamma_\epsilon > 0$ used in Riemann's definition to be any positive function, this allows a wider class of functions to be integrable. That $\gamma_\epsilon > 0$ is called *Gauge*, and we have the following definitions which can be found in [1]:

If $I : [a, b] \subset \mathbb{R}$, then a function $\delta : I \to \mathbb{R}$ is said to be a gauge on I_s if $\delta(t) > 0$, for all $t \in I$. The interval around $t \in I$ controlled by a gauge δ is the interval

$$B[t, \delta(t)] := [t - \delta(t), t + \delta(t)]$$

Let $I \subset [a, b]$ and let $\dot{\mathrm{P}}= \{(I_i, t_i)\}_{i=1}^n$ be a tagged partition. If δ is a gauge on I, then we say that $\dot{\mathrm{P}}$ is δ-fine if

$$I_i \subset [t_i - \delta(t_i), t_i + \delta(t_i)], \text{ for all } i = 1, ..., n$$

Cousin's Theorem: If $I := [a, b]$ is a nondegenerate compact interval in $\mathbb{R}$ and δ is a gauge on I, then there exists a partition of I that is δ-fine.

HENSTOCK-KURZWEIL INTEGRAL

When considering the basic concepts, we have that the Henstock-Kurzweil Integral is given by the following definition:
A function $f : I \to \mathbb{R}$ is said to be Henstock-Kurzweil-Integrable on I if there exists a number $B \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a gauge γ_ϵ on I such that if $\dot{P} := (I_i, t_i)_{i=1}^n$ is any tagged partition of I such that $l(I_i) < \gamma_\epsilon(t_i)$ for $i = 1, ..., n$, then

$$|S(f; \dot{P}) - B| \leq \epsilon$$

The existence of the gauge function in the definition of the

Henstock-Kurzweil integral motivates its generality, and is the main difference from the Riemann integral.

Example 2: The function f' defined in example 1 is Henstock-Kurzweil integrable. This follows from the Fundamental Theorem. Since f as a primitive of f', then $\int_0^1 f' = f(1) - f(0) = -1$.

CONCLUSION

The advances made in the theory of integration were due to attempts to generalize the concept of integral approached by Riemann and Lebesgue, the construction of the Henstock-Kurzweil Integral started from the investigation of an integration process with the objective of reconstructing the function using the derivative, through the concepts of Riemann and Darboux, and it is responsible for covering a class of functions broader than those of Riemann and Lebesgue, without the need to work with measure theory, as is necessary in Lebesgue-Integrable functions.

REFERENCES

1. BARTLE, R. G., A Modern Theory of Integration, American Mathematical Society, 2001.