The finiteness of associated primes of generalized cohomology modules: A dimensional approach

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Abstract

This work is addressed to the study of finiteness of associated primes of ordinary and generalized local cohomology modules. Our approach is focused on the dimension of the base ring.

Introduction

The finiteness of the set of associated primes of local cohomology modules is a long-standing topic in commutative algebra.The cornerstone of this problem rest in a conjecture made by C. Huneke in [2] at Sundance conference in 1990:

Let R be a local Noetherian ring, let I be an ideal of R and M be a finite R- module. Then, for all $i \geq 0$, the set of associated primes of $H_I^i(M)$ is finite.

In [1], Hartshorne proposed the following problem:

Is $H_I^i(M)$ I-cofinite for all $i \geq 0$ and for all finite modules M?

Here, an R-module N is I-cofinite if $\operatorname{Supp}_R N \subset V(I)$ and $\operatorname{Ext}_R^i(R/I, N)$ is a finite R-module for all i. In a such case, $\operatorname{Ass}_R N$ is finite. This establishes a direct connection between the both conjectures.

In this work, we present some new positive cases of the natural generalized version of Hartshorne's problem:

Is $H_I^i(M, N)$ I-weakly cofinite whenever M is finite and N is weakly Laskerian?

The main goal

Find new positive cases of Huneke's conjecture and generalized Hartshorne's problem when the base ring has dimension up to six.

Results

Henceforth, R is a commutative, unital and Noetherian ring. We begin with the following definition:

Definition 1. Let M be an R-module.

(i) We say that M is *weakly Laskerian* if $\operatorname{Ass}_R M/N$ is finite, for each submodule N of M.

(ii) We say that M is *I-weakly cofinite* if $\operatorname{Supp}_R M \subset V(I)$ and $\operatorname{Ext}_R^i(R/I, M)$ is weakly Laskerian for all $i \geq 0$.

It follows that every I-weakly cofinite module has finitely many associated primes. From now on, M is a finite R-module and N is a weakly Laskerian R-module.

Three-dimensional case

Theorem 1. *Let R is a three-dimensional ring. If R is a UFD, then $H_I^i(M, N)$ is I-weakly cofinite for all i.*

Theorem 2. *Let R be a three-dimensional semilocal ring. Then, $H_I^i(M, N)$ is I-weakly cofinite for all i.*

The four-dimensional case

Theorem 3. *Let R be a four-dimensional semilocal ring. If $\operatorname{ht} I \geq 2$, then $H_I^i(M, N)$ is I-weakly cofinite for all i.*

Theorem 4. *Let R be a four-dimensional semilocal ring. If R is a UFD, then $H_I^i(M, N)$ is I-weakly cofinite for all i.*

Theorem 5. *Let R be a four-dimensional semilocal ring. Suppose that $\operatorname{pdim}_R M < \infty$. If I contains an N-regular sequence of length 2, then $H_I^i(M, N)$ is I-weakly cofinite for all i.*

The five- dimensional case

Theorem 6. *Let R be a five-dimensional local ring. Suppose that R is a UFD and that $\operatorname{pdim}_R M < \infty$.*

(i) *If $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R N = \emptyset$, then $H_I^i(N)$ is I-weakly cofinite for all i, whenever $\operatorname{ht} I \neq 1$ or if I is a principal ideal.*

(ii) *If $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R N = \emptyset$ and $\operatorname{Ann}_R N \neq 0$, then $\operatorname{Ass}_R H_I^i(N)$ is finite for all i.*

(iii) *Suppose that $\operatorname{Ann}_R N \neq 0$. If $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R M = \emptyset$ or $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R N = \emptyset$, then $\operatorname{Ass}_R H_I^i(M, N)$ is finite for all $i \neq 2$. Moreover, if $\operatorname{ht} I \neq 1$ or if I is a principal ideal, then $H_I^i(M, N)$ is I-weakly cofinite for all i and, in particular, $\operatorname{Ass}_R H_I^i(M, N)$ is finite.*

The six-dimensional case

Theorem 7. *Let $(R, \mathfrak{m})$ be a six-dimensional regular local ring containing a field and N a weakly Laskerian R-module such that $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R N = \emptyset$ and $\operatorname{Ann}_R N \not\subset \mathfrak{m}^2$. Suppose that I is an unmixed ideal. Then $\operatorname{Ass}_R H_I^i(N)$ is finite for all $i \neq 2$.*

In addition, let M be a finite R-module module with $\operatorname{pdim}_R M = 1$. Assume that M is perfect and that the following conditions hold:

(i) $\operatorname{Ass}_{\widehat{R_Q}} \widehat{M_Q} = \{\hat{J} \mid J \in \operatorname{Ass}_{R_Q} M_Q\}$ *for each prime ideal Q of R.*

(ii) $\operatorname{Min}_R R/I \cap \operatorname{Supp}_R M = \emptyset$.

(iii) *For each $Q \in \operatorname{Ass}_R M$, the ideal $I + Q$ does not have any minimal prime ideal of height equal to 4 or 5.*

(iv) $\operatorname{Ann}_R M \not\subset \mathfrak{m}^2$.

Then, $\operatorname{Ass}_R H_I^i(M, N)$ is finite for all $i \neq 2, 3$.

Example 1. *Let k be a field and consider $S = k[x_1, x_2, x_3, x_4, x_5, x_6]$. Let $R = S_{\mathfrak{m}}$, where $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5, x_6)$. Set $M = R/(x_5)$ and $N = R/J$, where $J = (x_1x_2, x_5)R$. Then, $\operatorname{Ass}_R H_I^i(M, N)$ is finite for $i \neq 2, 3$.*

References

[1] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9**, (1970) 145–164.

[2] C. Huneke, *Problems on local cohomology, in: Free resolutions in commutative algebra and algebraic geometry*, (Sundance, Utah, 1990), Research Notes in Mathematics 2, Jones and Bartlett Publishers, 93–108, (1992).

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