On an upper bound of the degree of polynomial identities regarding linear recurrence sequences

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Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. There are several interesting identities involving this sequence such as $F_n^2 + F_{n+1}^2 = F_{2n+1}$, for all $n \geq 0$. Inspired by this naive identity, in 2012, Chaves, Marques and Togbé proved that if $(G_m)_m$ is a linear recurrence sequence (under weak assumptions) and $G_n^s + \cdots + G_{n+k}^s \in (G_m)_m$, for infinitely many positive integers n, then s is bounded by an effectively computable constant depending only on k and the parameters of G_m. In this paper, we generalize this result, proving, in particular, that if $(G_m)_m$ and $(H_m)_m$ are linear recurrence sequences (also under weak assumptions), $R(z) \in \mathbb{C}[z]$ is a monic polynomial, and $\epsilon_0 R(G_n) + \epsilon_1 R(G_{n+1}) + \cdots + \epsilon_{k-1} R(G_{n+k-1}) + R(G_{n+k})$ belongs to $(H_m)_m$, for infinitely many positive integers n, then the degree of $R(z)$ is bounded by an effectively computable constant depending only on the upper and lower bounds of the ϵ_i's and the parameters of G_m (but surprisingly not on k).