Nonlinear $*$-Jordan-Type Derivations on Alternative $*$-Algebras

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Introduction

In this work we will present the results obtained by the authors of the paper [1].

Preliminaries

Let A and B two algebras with identities 1_A and 1_B respectively, and $\varphi : A \to B$ a map. We have the following concepts:

(i) φ **preserves product** if $\varphi(ab) = \varphi(a)\varphi(b)$, for all elements $a, b \in A$;

(ii) φ **preserves Jordan product** if $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$, for any elements $a, b \in A$;

(iii) φ is **additive** if $\varphi(a + b) = \varphi(a) + \varphi(b)$, for any $a, b \in A$;

(iv) φ is **isomorphism** if φ is a bijection additive that preserves products and scalar multiplication;

(v) φ is **unital** if $\varphi(1_A) = 1_B$.

An algebra A is called **$*$-algebra** if A is endowed with a involution. By involution, we mean a mapping $* : A \to A$ such that $(a + b)^* = a^* + b^*$, $(a^*)^* = a$, $(ab)^* = b^*a^*$, for all $a, b \in A$. An element $s \in A$ satisfying $s^* = s$ is called **symmetric element** of A. Let A and B two $*$-algebras and $\varphi : A \to B$ a map. We have the following definitions:

(i) φ **preserves involution** if $\varphi(a^*) = \varphi(a)^*$, for all element $a \in A$;

(ii) φ is **$*$-isomorphism** if φ is an isomorphism that preserves involution;

(iii) φ is **$*$-additive** if it preserves involution and it is additive.

Definition 1. *An algebra A not necessarily associative or commutative is called an* **alternative algebra** *if it satisfies the identities $a(ab) = a^2b$ and $(ba)a = ba^2$, for all elements $a, b \in A$. An alternative algebra A is called* **prime** *if for any elements $a, b \in A$ satisfying the condition $aAb = 0$, then either $a = 0$ or $b = 0$.*

Let A be an alternative $*$-algebra over the ground field $\mathbb{C}$. For any $a, b \in A$ denote a new product of a and b by $a \bullet b = ab + ba^*$, end this new product $\bullet$ is usually known as Jordan $\bullet$-product.

Definition 2. *A not necessarily linear map $\varphi : A \to A$ is said to be a* **nonlinear $*$-Jordan derivation** *if*

$$\varphi(a \bullet b) = \varphi(a) \bullet b + a \bullet \varphi(b),$$

for all $a, b \in A$. Similarly, a map $\varphi : A \to A$ is said to be a **nonlinear $*$-Jordan triple derivation** *if*

$$\varphi(a \bullet b \bullet c) = \varphi(a) \bullet b \bullet c + a \bullet \varphi(b) \bullet c + a \bullet b \bullet \varphi(c),$$

for all $a, b, c \in A$ where $a \bullet b \bullet c := (a \bullet b) \bullet c$ (we should be aware that $\bullet$ is not necessarily associative).

Suppose that $n \geq 2$ is a fixed positive integer. Accordingly, a nonlinear $*$-Jordan n-derivation is a map

$$\varphi(a_1 \bullet a_2 \bullet \ldots \bullet a_n) = \sum_{k=1}^{n} a_1 \bullet \ldots \bullet a_{k-1} \bullet \varphi(a_k) \bullet \ldots \bullet a_n,$$

for all $a_1, a_2, \ldots a_n \in A$, where

$$a_1 \bullet a_2 \ldots \bullet a_n = (\ldots((a_1 \bullet a_2) \bullet a_3) \ldots \bullet a_n).$$

By the definition, it is clear that every $*$-Jordan derivation is a $*$-Jordan 2-derivation and every $*$-Jordan triple derivation is a $*$-Jordan 3-derivation. It is obvious that every nonlinear $*$-Jordan derivation on any $*$-algebra is a $*$-Jordan n-derivation.

We consider an alternative algebra A with identity 1_A. Fix a nontrivial symmetric idempotent element $e_1 \in A$ and denote $e_2 = 1_A - e_1$. It is easy to see that $(e_k a)e_j = e_k(ae_j)$, $k, j = 1, 2$ and for all $a \in A$. Then A has a Peirce decomposition

$$A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$$

where $A_{kj} := e_k A e_j$, $k, j = 1, 2$ satisfying the following multiplicative relations

(i) $A_{kj}A_{jl} \subseteq A_{kl}$, $k, j, l = 1, 2$;

(ii) $A_{kj}A_{kj} \subseteq A_{jk}$, $k, j = 1, 2$;

(iii) $A_{kj}A_{ml} = \{0\}$, if $j \neq m$ and $(k, j) \neq (m, l)$, $(k, j, m, l = 1, 2)$;

(iv) $x_{kj}^2 = 0$, for all $x_{kj} \in A_{kj}$, $k, j = 1, 2$ and $k \neq j$.

Main Result

Theorem 1. *Let A be a unital alternative $*$-algebra with the unit 1_A. Assume that A contains a nontrivial symmetric idempotent element e_1 which satisfies*

$$(\heartsuit) \quad xA \cdot e_1 = 0 \ \text{implies} \ x = 0;$$

$$(\spadesuit) \quad xA \cdot e_2 = 0 \ \text{implies} \ x = 0.$$

If a map $\varphi : A \to A$ satisfies

$$\varphi(a_1 \bullet a_2 \bullet \ldots \bullet a_n) = \sum_{k=1}^{n} a_1 \bullet \ldots \bullet a_{k-1} \bullet \varphi(a_k) \bullet a_{k+1} \bullet \ldots \bullet a_n,$$

for all $a_{n-1}, a_n \in A$ and $a_i = 1_A$, for all $i \in \{1, \ldots, n-2\}$, then φ is an additive $*$-derivation.

Corollary 1. *Let A be a unital alternative $*$-algebra with the unit 1_A. Assume that A contains a nontrivial symmetric idempotent element e_1 which satisfies $(\heartsuit)$ and $(\spadesuit)$. Then φ is a nonlinear $*$-Jordan-type derivation on A if and only if φ is an additive $*$-derivation.*

Corollary 2. *Let A be a prime unital alternative $*$-algebra with the unit 1_A. Assume that A contains a nontrivial symmetric idempotent element e_1. If a map $\varphi : A \to A$ satisfies*

$$\varphi(a_1 \bullet a_2 \bullet \ldots \bullet a_n) = \sum_{k=1}^{n} a_1 \bullet \ldots \bullet a_{k-1} \bullet \varphi(a_k) \bullet a_{k+1} \bullet \ldots \bullet a_n,$$

for all $a_{n-1}, a_n \in A$ and $a_i = 1_A$ for all $i \in \{1, \ldots, n-2\}$, then φ is an additive $*$-derivation.

Applications

A complete normed alternative complex $*$-algebra A is called of **alternative C^*-algebra** if it satisfies the condition: $||a^*a|| = ||a||^2$, for all $a \in A$. An alternative C^*-algebra A is called of **alternative W^*-algebra** if it is a dual Banach space and a prime alternative W^*-algebra is called alternative W^*-factor.

Theorem 2. *Let A be an alternative W^*-factor. If a map $\varphi : A \to A$ satisfies*

$$\varphi(a_1 \bullet a_2 \bullet \ldots \bullet a_n) = \sum_{k=1}^{n} a_1 \bullet \ldots \bullet a_{k-1} \bullet \varphi(a_k) \bullet a_{k+1} \bullet \ldots \bullet a_n,$$

for all $a_{n-1}, a_n \in A$ and $a_i \in \{1, \ldots, n-2\}$, then φ is an additive $*$-derivation.

Corollary 3. *Let A be an alternative W^*-factor. Then φ is a nonlinear $*$-Jordan-type derivation on A if and only if φ is an additive $*$-derivation.*

Referências

[1] De Oliveira Andrade, A. J., Moraes, G. C., Ferreira, R. N., and Ferreira, B. L. M. Nonlinear $*$-jordan-type derivations on alternative $*$-algebras. *Siberian Electronic Mathematical Reports 19*, 1 (2022), 125–137.