The Chromatic Number of some Latin squares based on groups

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Abstract

There are several graphs whose vertex set can be associated with many algebraic structures and whose edges reflect the nature of the structure in some way. We study the graph associated with the Cayley table of a finite group considering this as a Latin square. In particular, in this poster, we show the relation between the Chromatic number of the Latin squares based on a finite group and the existence of a nontrivial cyclic Sylow $2-$subgroup in the group. Recently, many authors have investigated this relationship and some important results have been obtained for abelian and dihedral groups (see [1] e [4]). Our research seeks to extend such results to other important families of groups.

Introduction

Let n be a positive integer, let $[n] = \{0, 1, 2, \cdots, n-1\}$, and let L be a **Latin square** of order n, which we define as a $n \times n$ array in which each row and each column is a permutation of some set of n symbols indexed by $[n]$. For example, the following is a Latin square of order 3.

$$L = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & 0 \\ \hline 2 & 0 & 1 \\ \hline \end{array}$$

We define a **partial transversal** of L as a collection of cells that intersects each row, each column, and each symbol class at most once. A **transversal** of L (latin square of order n) is a partial transversal of size n and a **near transversal** is a partial transversal of size $n-1$.

Two Latin squares L and L' of the same order are orthogonal if, superimposing them, each possible ordered pair of symbols occurs exactly once. It is well known that a Latin square possesses an orthogonal mate if and only if it can be partitioned into transversals. But when L does not have an orthogonal mate, can we still efficiently partition its cells into a transversal and partial transversals, or into partial transversals? In order to try to answer this question Brualdi conjected that every Latin square possesses a near transversal. Even mores, in 1967 Ryser conjected that every latin square of odd order possesses a transversal.

To study the decompositon of a Latin square into transversals or partial tranversals we will approach this problem from its equivalent version in graphs. Associated with every latin square L is a strongly regular graph $\Gamma(L)$ defined on vertex set $\{(r, c, L_{r,c}) : r, c \in [n]\}$ with $(r_1, c_1, s_1) \sim (r_2, c_2, s_2)$ if and only if one of $r_1 = r_2$, *or* $c_1 = c_2$, *or* $s_1 = s_2$ holds. It is straightforward to check that partial transversals of L correspond to independent sets in $\Gamma(L)$.

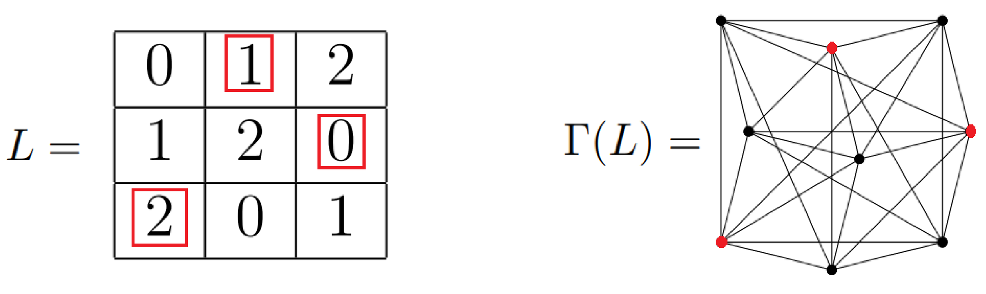

Figure 1: Latin square of order 3 with their associated graph and some independent set

Thus, the **graph chromatic number** $\chi(\Gamma(L))$ is the minimum number of partial transversals needed to cover all of the cells in L, and define the **Latin square chromatic number** $\chi(L)$ as these number.

Clearly $\chi(L) \geqslant n$. To obtain an upper bound of $\chi(L)$, observe that the associated graph, $\Gamma(L)$, is a connected graph with order $\Delta = 3(n-1)$ then by the Brooks' theorem follows that $\chi(L) \leq 3n - 3$. This upper bound is far from tight. Indeed, a recent conjecture due to Cavenagh proposes that the chromatic number of a latin square can differ from its order by at most 2, i.e.,

Conjecture 1 (Cavenagh's conjecture, 2015)**.** *Let L be a Latin square of order n. Then*

$$\chi(L) \leq \begin{cases} n+1, & n \text{ is odd} \\ n+2, & n \text{ is even} \end{cases}$$

The chromatic number of a Latin square based on group

But, how can Latin squares and the chromatic number of a Latin square be related to finite groups? First, note that the chromatic number of $\Gamma := \Gamma(L)$ is not affected by relabelling the rows, columns, or symbol classes of L, nor is it affected by applying a fixed permutation to each of the triples $(r, c, s) \in V(\Gamma)$. Thus, $\chi(L)$ is a main class invariant, and it makes sense in this context to speak of the Cayley table of a group G, which we denote by $L(G)$. We write $\chi(G)$ for denote the chromatic number of $L(G)$ and $\Gamma(G)$ for the latin square graph $\Gamma(L(G))$.

The groups for which $\chi(G) = n$ were recently characterized by Bray, Evans, and Wilcox:

Theorem 1 (Bray, Evans, and Wilcox, 2009)**.** *Let G be a group of order n. Then the following conditions are equivalent:*

i) $\chi(G) = n$,

ii) $\chi(G) \leqslant n+1$,

iii) $L(G)$ *has a transversal,*

iv) $Syl_2(G)$ *is either trivial or non-cyclic.*

Observe that, if $Syl_2(G)$ is trivial then G has odd order. Therefore prove the Cavenagh's conjecture in the abelian group category is equivalent to prove:

Theorem 2 (Goddyn, Halasz, Mahmoodian, 2019)**.** *Let G be an Abelian group of order n. Then*

$$\chi(G) = \begin{cases} n & Syl_2(G) \text{ is either trivial or non-cyclic,} \\ n+2 & \text{otherwise} \end{cases}$$

Our research

In 2015, Shokri, proved that the Cavenagh's conjecture is valid for the Dihedral groups D_p when $p > 3$ is a prime number. The first objective of our research is to generalize this result to the general context of dihedral groups. Other open problem that we deal with in our research is to improve the upper bound for $\chi(G)$ given by $\chi(H)\chi(G/H)$ when G is a finite group and let $H \lhd G$ be a normal subgroup.

References

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