Conditional Quantiles, Risk Measures, and Applications

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Abstract

Conditional quantiles are widely adopted in several distinguish areas, such as probability theory, statistics, decision theory, econometrics and, specially, mathematical finance. However, despite being largely employed in multiple fields, a comprehensive and rigorous investigation of their statistical and mathematical properties is still scarce in the literature. In the first half of this thesis, this gap in the literature is filled. We analyze conditional quantiles as an one-parameter family of operators and organize the results in parallel to the usual properties of the expectation operator.

In the remainder of this thesis, we focus on an important class of problems that is intimately connected to conditional quantiles: the characterization of convex risk measures. We follow the techniques and employ some of the results of the first half to derive a compendium of representations for convex and conditionally law-invariant risk measures in the static, finite and continuous-time dynamic frameworks.

Keywords: Conditional Quantiles; Conditional Risk Measures; Dynamic Risk Measures.

Resumo

Quantis condicionais são amplamente utilizados nas mais distintas áreas, como teoria da probabilidade, estatística, teoria da decisão, econometria e, especialmente, finanças matemáticas. Contudo, embora sejam largamente empregados em diferentes campos, uma investigação compreensiva e rigorosa de suas propriedades matemáticas e estatísticas ainda é escassa na literatura. Na primeira metade desta tese, tal buraco na literatura é preenchido. Analisamos quantis condicionais como uma família a um parâmetro de operadores e organizamos os resultados em paralelo às propriedades usuais do operador esperança.

No restante desta tese, focamos numa importante classe de problemas que está intimamente conectada aos quantis condicionais: a caracterização de medidas convexas de risco. Seguimos as técnicas e empregamos alguns dos resultados da primeira metade na derivação de um conjunto de representações para medidas de risco convexas e condicionalmente invariantes por lei no caso estático, dinâmico discreto e em tempo contínuo.

Palavras-chaves: Quantis Condicionais; Medidas de Risco Condicionais; Medidas de Risco Dinâmicas.

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Contents

Chapter 1

Introduction

Conditional quantiles are ubiquitous in several distinct theoretical and applied fields. Their applications range from decision theory (Mendelson (1987), Manski (1988),Chambers (2009), and Rostek (2010)), finance (Duffie and Pan (1997), McNeil et al. (2005) and Jorion (2007)), econometrics (Koenker and Bassett (1978) and Koenker et al. (2017)), among others. In mathematical finance, in particular, the usage of quantiles to model and assess the risk in the market is so frequent that they are also referred to as *value-at-risk*. [1] Beyond being an important and widespread risk measure, Kusuoka (2001), Delbaen (2002) and Frittelli and Rosazza Gianin (2005) demonstrated that value-at-risk (quantiles) are, in fact, the building blocks of the most comprehensive classes of risk measures, convex and coherent risk measures. Nowadays the representation of risk measures in terms of conditional quantiles is very active and widely studied in mathematics for the conditional and dynamic frameworks – see e.g. Madan et al. (2017) and Dela Vega and Elliott (2021).

Nevertheless, there is no reference in the literature investigating and formalizing the properties of conditional quantiles in their most general framework. Furthermore, there is also room for analyzing the consequence of its properties in conditional risk assessment, as well as in the representation of dynamic convex risk measures and their penalty functions. With this in mind, this thesis aims at filling these gaps in the following chapters:

- Our second chapter focuses on a mathematical rigorous definition of the quantile of a given random variable conditional to any sub-σ-algebra and its properties. The chapter parallels some of the known results for conditional expectation operator, and presents conditional quantiles as an one-parameter family of non-linear operators acting on the space of measurable random variables. Measurability, invariance in L^p-spaces and conditions for additivity are derived. Moreover, a generalization of Jensen's inequality for conditional quantiles is obtained, as well as Fatou's Lemma and continuity results with respect to different topologies. Conditions for the interchanging of the differential operator and the conditional quantile operator are also given. Finally, the problem of iteration of conditional quantiles with respect to different σ-algebras is addressed. This chapter is a preprint in SSRN 3924597 (submitted for publication), and it is a collaboration with Jorge P. Zubelli, Luciano I. de Castro and Antonio F. Galvao.

- In the third chapter, we generalize existing representation theorems for unconditional risk measures to the conditional case, for both static and dynamic settings. First, we derive a

[1] McNeil et al. (2005) and the references therein have a comprehensive discussion about the historical developments of risk management, specially value-at-risk, its central role in some recent regulatory accords, as Basel Accords, as well as its importance to financial institutions.

series of equivalent representations for conditionally law-invariant convex risk measures and their penalty functions in the static and conditional framework. These characterizations are expressed in terms of integrals of conditional quantiles, conditional average value-at-risk, random concave distortions of the conditional probability measures, as well as random Choquet's integrals of transition capacities. Second, we delve into one-step law-invariant and iterative dynamic risk measures to reconcile their time-consistency and relevance by weakening law-invariance. Representation theorems for this family in a finite-time setting are also explored. Then, we apply the results to describe the features of a suitable class of conditionally law-invariant convex risk measures based on utility functions. Finally, we conclude analyzing some further application of the dynamic and discrete-time representations to the understanding of the limiting continuous-time risk assessment as g-expectations. This chapter is a preprint (submitted for publication) and a collaboration with Jorge P. Zubelli, Luciano I. de Castro and Antonio F. Galvao.

1.1 Notation

Throughout this article, $(\Omega, \mathcal{F}, \mathsf{P})$ will be a probability space and $\mathcal{G}$ and $\mathcal{H}$ will be σ-algebras satisfying $\mathcal{F} \supset \mathcal{G} \supset \mathcal{H}$. We denote by $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$ the set of measurable maps $\mathsf{X} : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, where $\mathcal{B}(\mathbb{R}^n)$ stands for the Borel σ-algebra of the Euclidean space $\mathbb{R}^n$. If $n = 1$, we will simplify this notation to $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$. The set $\mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P}) \subset \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, for any $p \in [1, +\infty]$, corresponds to the random variables such that $\|f\|_p < +\infty$. We write $a.s.$ for almost surely and, for all $\mathsf{X} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, supp$\mathsf{X}$ means the support of the probability measure P_X, where P_X is the measure on $\mathbb{R}$ induced by X. Given a pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, if $x_i \leqslant y_i$, for all $i \in \{1, \ldots, n\}$, we say that x is smaller than y, denoting it by $x \leqslant y$.

Because the measurable space $(\Omega, \mathcal{F}, \mathsf{P})$ is fixed, $\mathcal{P} = \{\mathsf{Q} : \mathcal{F} \to [0, 1] \text{ probability measure} : \mathsf{Q} \ll \mathsf{P}\}$ stands for the set of probability measures in $\mathcal{F}$ that are absolutely continuous with respect to P, and $\mathcal{P}_\mathcal{G} = \{\mathsf{Q} \in \mathcal{P} : \mathsf{Q}|_\mathcal{G} = \mathsf{P}|_\mathcal{G}\}$ denotes the subset of those probabilities in $\mathcal{P}$ that coincides with P when restricted to $\mathcal{G}$. The symbols E^Q and E will be used referring to the expected value computed using Q and P, respectively. With this assumption, for any $\mathsf{Q} \in \mathcal{P}_\mathcal{G}$, then $\mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P}) = \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{Q})$ and $\mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \subset \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{Q})$.

If $(\mathsf{T}, \mathcal{T})$ is a first-countable topological space, $(\mathsf{M}, \mathcal{M})$ a topological space with a partial order $\leqslant$, and $f : \mathsf{T} \to \mathsf{M}$ a function, then f is said to be lower (or upper) semicontinuous provided that, for all sequence $(x_n)_{n \in \mathbb{N}} \subset \mathsf{T}$, such that $x_n \underset{\mathcal{T}}{\to} x$, then $f(x) \leqslant \liminf_{n \in \mathbb{N}} f(x_n)$ (or $\limsup_{n \in \mathbb{N}} f(x_n) \leqslant f(x)$). For any $\mathsf{A} \subset \mathbb{R}^n$, we denote the set of continuous (and bounded) functions $f : \mathsf{A} \to \mathbb{R}$ by $\mathcal{C}(\mathsf{A})$ (and $\mathcal{C}_b(\mathsf{A})$). A map, $f : \mathsf{A} \subset \mathbb{R} \to \mathbb{R}$, is non-decreasing if it is monotone and $f(x) \leqslant f(y)$, for any $x \leqslant y$. Similarly, it is non-increasing if $(-f)$ is non-decreasing. Finally, for any family of random variables, $\{\mathsf{X}_\lambda\}_{\lambda \in \Lambda} \subset \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \bar{\mathbb{R}})$, we denote the essential supremum and infimum of a family of random variables with respect to P by $\operatorname{esssup}_{\lambda \in \Lambda} \mathsf{X}_\lambda$ and $\operatorname{essinf}_{\lambda \in \Lambda} \mathsf{X}_\lambda$, as defined in Peskir and Shiryaev (2006).

Chapter 2

Conditional Quantiles: An Operator-Theoretical Approach

2.1 Introduction

Quantiles are of fundamental importance in several fields of theoretical and applied work such as statistics, biostatistics, economics, finance, and decision theory, among others.[1] Since the seminal paper Koenker and Bassett (1978), quantile regression has become an important tool in statistical analysis for estimating conditional quantile functions models (see, e.g., Koenker (2005) and Koenker et al. (2017)). Quantile regression provides a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution. Quantiles are also important in decision theory. There has been increasing theoretical, empirical, and experimental interest in decision under uncertainty using quantile preferences (QP). This preference has been characterized in Manski (1988), who studied properties of a quantile model for individual's behavior.[2] Mendelson (1987) introduced the concept of quantile-preserving spread, which is a notion of risk aversion for the quantile model that establishes a parallelism with mean-preserving spreads in the standard expected utility framework.

Although conditional quantiles have been largely employed in multiple fields, the literature still lacks a systematic investigation of their statistical and mathematical properties. This work fulfills this gap. We employ an operator theoretical view to define the τ-conditional quantiles, which enables us to enlarge the theory and rigorously establish results that have not been formalized before. To use this approach, we first define the τ-conditional quantile random set as the set of solutions of an optimization problem using the check function as the objective function, as proposed in Koenker and Bassett (1978). This definition allows computation of conditional quantiles of any finite random variable conditional on any σ-algebra. By adjusting the argument in Valadier (1984) for conditional medians, we provide the measurability of conditional quantile random sets. From this, we define the right and left-conditional quantiles as well as demonstrate their measurability.

Next, we show that, when restricted to L^p-spaces, conditional quantiles take value on a smaller L^p-space. Consequently, it is possible to view them as an one parameter family of non-linear operators mapping distinct L^p-spaces. Moreover, three equivalent definitions to conditional quantiles

[1] Quantiles have also been used in practical decision making in banking and investment (in the form of Value-at-Risk, see, e.g., Duffie and Pan (1997) and Jorion (2007)), goal-reaching problems and in mining, oil and gas industries (in the form of "probabilities of exceeding", see, e.g., Apiwatcharoenkul et al. (2016) and Fanchi and Christiansen (2017)).

[2] More recently, QP have been formally axiomatized by Chambers (2009), and Rostek (2010).

are offered. Then, after defining and establishing their measurability, basic properties enjoyed by conditional quantiles are investigated, such as monotonicity, idempotency, and independence.

We generalize several known properties of unconditional quantiles to the conditional case. First, we investigate invariance properties and provide conditions for additivity of quantiles, that is, the conditional quantile operator of a sum of random variables equals to the sum of the conditional quantile of each random variable. We show that the concept of conditional comonotonicity introduced in Jouini and Napp (2004) may be useful to the conditional case. Under $\mathcal{G}$-comonotonicity, we show that conditional quantiles are additive. This result extends some findings of Embrechts et al. (2003). Furthermore, we show that positive homogeneity and translational invariance can be used to establish additivity for each quantile. Then, we generalize the property of invariance with respect to monotone transformation from the unconditional (see, e.g., Koenker, 2005) to the conditional case. Finally, we use the operator properties of conditional quantiles and the subdifferentiability of concave and convex functions to present a simple proof for Jensen's inequality for conditional quantiles.[3]

The next natural aspect of a non-linear operator to be dissected is its continuity. We investigate the continuity of conditional quantiles as operators with respect to different topologies. We start by describing a novel Fatou's lemma for conditional quantiles, proving that it holds under less stringent assumptions than its conditional expected value counterpart. As a direct consequence of this Fatou's lemma, we obtain conditions for the continuity of conditional quantiles with respect to almost sure convergence. Moreover, we provide conditions for continuity of conditional quantiles in L^p spaces. We also revisit some of the main theorems regarding the continuity of quantiles with respect to weak convergence and enlarge it in the context of conditional weak convergence, as proposed in Sweeting (1989). Overall, the results on continuity have important practical implications, as for instance, showing the convergence of quantiles under almost sure convergence of random variables.

We then investigate the differentiability properties of conditional quantiles. One of the most useful properties of the expected value is its ability of exchanging the order of the integration and differentiation, the well known Leibniz's rule.[4] The interchange of integration and differentiation has been extensively used in applications, for example, in deriving statistical properties of the maximum likelihood estimator (see, e.g., Ferguson, 1996). We extend Leibniz's rule to quantiles and establish a novel differentiability property that allows one to exchange the quantile and the derivative. In particular, we first show the validity of Leibniz' rule for monotone functions. Second, we extend this result to the case of separable functions.

Finally, we examine the analogue for the law of iterated expectations (LIE) for conditional quantiles.[5] We show that the law of iterated quantiles does not hold in general, that is the LIE does not extend to quantiles. Nevertheless, we characterize the maximum set of random variables for which this law holds, and investigate its consequences for the infinite composition of conditional quantiles.

The theory developed in this work may have important developments and applications in practice. The results are theoretically important because they provide grounds for subsequent research on statistical and mathematical analysis of conditional quantiles. From a practical point of view, the results might be useful in decision theory studies with quantile preferences, as well as establishing

[3] Merkle (2005) establishes an analogue of the Jensen's inequality for medians. Using a similar approach, Zhao et al. (2021) strengthens these inequalities.

[4] There are required conditions to achieve such a result. In the expectation case, interchanging a derivative with an expectation (an integral) can be established by applying the dominated convergence theorem. Intuitively, the conditions say that the derivative of the function of interest must be bounded by another function whose integral is finite.

[5] The LIE is also known as the law of total expectation or the tower property of conditional expectations.

statistical properties of quantile regression models.

The remainder is organized as follows. Section 2.2 presents definitions and basic properties of quantiles. In Section 2.3, we study the invariance properties of conditional quantiles. Section 2.4 provides continuity results. Section 2.5 deals with differentiability of conditional quantiles and establishes a result that allows for interchanging the derivative and the operator. Section 2.6 investigates the composition of quantiles. Finally, Section 2.7 concludes. We relegate all proofs to the Appendix A.

2.2 Conditional quantiles: definitions and basic properties

This section introduces the main definitions, proves the measurability of the objects and establishes their basic properties. In Section 2.2.1, we define conditional quantile random sets as well as right and left-quantiles. Besides, the measurability of each object is derived. Section 2.2.2 demonstrates how we can visualize conditional quantiles as an one-parameter family of operators acting on L^p-spaces. Finally, three equivalent ways to define conditional quantiles and a set of their basic properties are provided in Section 2.2.3.

2.2.1 Definition and measurability

Given a random variable in a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, we want to define the conditional quantile map $\mathsf{Q}_\tau[X|\mathcal{G}] : (\Omega, \mathcal{G}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Intuitively, each realization of $\mathsf{Q}_\tau[X|\mathcal{G}]$ should give the worst value y such that the conditional probability satisfies $\mathsf{P}[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau$, i.e. $\mathsf{Q}_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : \mathsf{P}[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau\}$. If $\mathcal{G} = \sigma(Y)$, X and Y are simple random variables, $X = \sum_{j=1}^n x_i \mathbb{1}_{A_i}$ and $Y = \sum_{j=1}^n y_i \mathbb{1}_{B_i}$, with $\mathsf{P}(B_i) > 0$ for every i, then $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$ is easily computed from Bayes' formula. Consequently, the definition of $\mathsf{Q}_\tau[X|\mathcal{G}]$ in this condition is trivial, as well as its measurability. However, for a general $\mathcal{G}$, this definition depends on transition kernels, $\mathsf{P}[X \in \cdot\,|\mathcal{G}]$, not easily computed, satisfying the following.

1. The map $\mathsf{P}[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is so that, for all $\omega \in \Omega$, then:

$$A \in \mathcal{B}(\mathbb{R}) \mapsto \mathsf{P}[X \in A|\mathcal{G}](\omega) \text{ is a probability measure.}$$

2. Fixed $A \in \mathcal{B}(\mathbb{R})$, then:

$$\omega \in \Omega \mapsto \mathsf{P}[X \in A|\mathcal{G}](\omega) \text{ is } \mathcal{G}\text{-measurable.}$$

3. For all $G \in \mathcal{G}$ and $A \in \mathcal{B}(\mathbb{R})$, then:

$$\mathsf{P}[\{X \in A\} \cap G] = \int_G \mathsf{P}[X \in A|\mathcal{G}](\omega) d\mathsf{P}|_{\mathcal{G}}(\omega),$$

where $\mathsf{P}|_{\mathcal{G}} : \mathcal{G} \to [0, 1]$ denotes the restriction of the probability measure to the sub-σ-algebra $\mathcal{G}$.

The existence of such kernel is guaranteed by the disintegration theorem – Durrett (2019). As an immediate consequence of this definition, we obtain that $\mathsf{E}[f(X)|\mathcal{G}](\omega) = \int f(x)\mathsf{P}[X \in dx|\mathcal{G}](\omega)$ a.s., for all $f \in \mathcal{B}(\mathbb{R})$-measurable and bounded – see Le Gall (2006). Even though $\mathsf{P}[X \in A|\mathcal{G}] = \mathsf{E}[\mathbb{1}_{X \in A}|\mathcal{G}]$ for all $A \in \mathcal{B}(\mathbb{R})$ a.s., $\mathsf{P}[X \in \cdot\,|\mathcal{G}](\omega)$ has the advantage of being a probability measure on $\mathcal{B}(\mathbb{R})$. Moreover, given two transition kernels, $P[X \in \cdot\,|\mathcal{G}]$ and $\bar{P}[X \in \cdot\,|\mathcal{G}]$, Le Gall (2006) shows that there is a set $\Omega' \subset \Omega$, with full measure, such that:

$$P[X \in A|\mathcal{G}](\omega) = \bar{P}[X \in A|\mathcal{G}](\omega), \text{ for all } A \in \mathcal{B}(\mathbb{R}) \text{ and } \omega \in \Omega'.$$

From now on, we assume that for all fixed X we are using the same version of $P[X \in \cdot\,|\mathcal{G}]$, unless otherwise stated.

Instead of defining the τ-conditional quantile directly by $Q_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : P[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau\}$, in this work, we will first adopt the optimization problem definition, similar to that in Koenker and Bassett (1978). This method will lead us to conditional quantile random sets. By proving that these new random sets are $\mathcal{G}$-measurable, we finally define the left and right conditional quantiles simply as their composition with some specific measurable maps. As an advantage, the previous approach allows us to readily derive the measurability of both right and left conditional quantiles at once.

Valadier (1984) shows that it is possible to define the conditional median of a random variable with respect to some σ-algebra as a compact random set, proving its measurability with respect to the σ-algebra $\mathcal{B}(\mathcal{K})$ over the set of compact sets of the real line $\mathcal{K}$; more precisely,

$$\mathcal{B}(\mathcal{K}) = \sigma\left(\{K \in \mathcal{K} : K \cap G \neq \emptyset\} : \; G \subset \mathbb{R} \text{ open}\right),$$

see Molchanov (2017) for more details on $\mathcal{B}(\mathcal{K})$. Based on that, we define the τ-conditional quantile random set as the set of solutions of the following convex problem, where $\rho_\tau : \mathbb{R} \to \mathbb{R}$ stands for $\rho_\tau(x) := (\tau - 1)x\mathbb{1}_{[x<0]} + \tau x \mathbb{1}_{[x \geqslant 0]}$, also known as the *check function* – see Koenker and Bassett (1978).

Definition 2.2.1. *Given $X \in L^0(\Omega, \mathcal{F}, P)$ in a probability space $(\Omega, \mathcal{F}, P)$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that the conditional law of X given $\mathcal{G}$ is $P[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$, the τ-quantile random set of X conditional to $\mathcal{G}$ is a map $\Gamma_\tau[X|\mathcal{G}] : (\Omega, \mathcal{G}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ satisfying:*

$$\Gamma_\tau[X|\mathcal{G}](\omega) = \underset{y \in \mathbb{R}}{\operatorname{argmin}} \int \big(\rho_\tau(x - y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega), \; \forall \omega \in \Omega. \tag{2.1}$$

It is worth noting that if we choose another representative for the transition kernel $P[X \in \cdot\,|\mathcal{G}]$, then the τ-quantile random set associated to each representative coincides in a set of full probability measure. Therefore, the above definition is unique up to a modification on a set of zero measure.

Our first result guarantees that the above map is well-defined as a compact random set, that is, takes compact values and is measurable. We adapt the proof given in Valadier (1984) for $\tau = \frac{1}{2}$.

Proposition 2.2.2. *Given $X \in L^0(\Omega, \mathcal{F}, P)$ in a probability space $(\Omega, \mathcal{F}, P)$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that its transition kernel is given by $P[X \in \cdot\,|\mathcal{G}]$, then $\Gamma_\tau[X|\mathcal{G}](\omega)$ is nonempty and compact for all $\omega \in \Omega$. Moreover, the map $\Gamma_\tau[X|\mathcal{G}] : (\Omega, \mathcal{G}) \to (\mathcal{K}, \mathcal{B}(\mathcal{K}))$ is measurable.*

It is trivial to show that the maps $\inf : (\mathcal{K}, \mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ and $\sup : (\mathcal{K}, \mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ are measurable.[6] Since $(\Gamma_\tau[X|\mathcal{G}])_{\tau \in (0,1)}$ is a family of measurable compact random sets, we can define the left and right conditional quantile as the composition of inf and sup with $\Gamma_\tau[X|\mathcal{G}]$:

Definition 2.2.3. *Given $X \in L^0(\Omega, \mathcal{F}, P)$ in a probability space $(\Omega, \mathcal{F}, P)$, $\tau \in (0,1)$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that the conditional law of X given $\mathcal{G}$ is $P[X \in \cdot\,|\mathcal{G}] : \Omega \times \mathcal{B}(\mathbb{R}) \to [0,1]$, the τ-quantile, $\tau \in (0,1)$, of X conditional to $\mathcal{G}$ is:*

$$Q_\tau[X|\mathcal{G}](\omega) = \inf \Gamma_\tau[X|\mathcal{G}](\omega), \; \forall \omega \in \Omega. \tag{2.2}$$

[6]Recall that, for $K \in \mathcal{K}$, $|\inf K| = |\sup K| = +\infty$ if, and only if, $K = \emptyset$. Since $\Gamma_\tau[X|\mathcal{G}]$ is non-empty, the composition of these maps generates an $\mathbb{R}$-valued random variable.

We also define the right conditional quantile as:

$$\mathsf{Q}_{\tau+}[\mathsf{X}|\mathcal{G}](\omega) = \sup \Gamma_\tau[\mathsf{X}|\mathcal{G}](\omega), \ \forall \omega \in \Omega. \tag{2.3}$$

The above definition includes the unconditional left and right-quantiles, when $\mathcal{G} = \{\emptyset, \Omega\}$. In this case, we omit the trivial σ-algebra and simply refer to $\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[\mathsf{X}|\mathcal{G}]$ as $\mathsf{Q}_\tau[\mathsf{X}]$ and $\mathsf{Q}_{\tau+}[\mathsf{X}]$, respectively.

Observe now that measurability of the right and left conditional quantile are easily derived from the fact that they are composition of measurable maps.

Proposition 2.2.4. *Given $\mathsf{X} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ in a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and a σ-algebra $\mathcal{G} \subset \mathcal{F}$, such that its transition kernel is given by $\mathsf{P}[\mathsf{X} \in \cdot\, |\mathcal{G}]$, $\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}] : \Omega \to \mathbb{R}$ and $\mathsf{Q}_{\tau+}[\mathsf{X}|\mathcal{G}] : \Omega \to \mathbb{R}$ are well-defined and $\mathcal{G}$-measurable random variables.*

To illustrate the concepts defined along this section, we offer some examples in the Appendix A. The first is designed to demonstrate how to determine the conditional quantile in a very simple framework, with auxiliary graphs. The second, on the other hand, characterize a particular conditional quantile random set and, from it, obtain the associated left and right quantiles. Finally, the third shows a concrete example where the conditional quantile of the sum of two variables equals the sum of each individual quantile, which will be revisited in Section 2.3.1. See more details in the Appendix A.

Proposition 2.2.4 allows us to define maps, $\mathsf{Q}_\tau[\cdot|\mathcal{G}]$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$, over $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ and taking value on $\mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, which compute the τ left and right quantile of X conditional to $\mathcal{G}$, for all $\tau \in (0, 1)$. Moreover, since, for any given $\mathsf{X} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, two transition kernels agree a.s.[7], we obtain that both maps are well-defined a.s. Therefore, from now on, we will visualize $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$ and $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$ as non-linear operators, and we will derive their properties in the subsequent sections.

2.2.2 Conditional quantiles as operators

As we showed previously, it is possible to define a one parameter family of non-linear operators acting on $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ and taking values on $\mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, for each $\tau \in (0, 1)$. The next proposition investigates the properties of these operators when restricted to the space $\mathsf{L}^\mathsf{p}(\Omega, \mathcal{F}, \mathsf{P})$.

Proposition 2.2.5.

1. *$\mathsf{X} \in \mathsf{L}^\mathsf{p}(\Omega, \mathcal{F}, \mathsf{P})$, $\mathsf{p} \in [1, +\infty)$, if, and only if, $\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}] \in \mathsf{L}^\mathsf{p}(\Omega, \mathcal{G}, \mathsf{P})$, for all $\tau \in (0, 1)$, $\mathsf{s} \mapsto \mathsf{E}[|\mathsf{Q}_\mathsf{s}[\mathsf{X}|\mathcal{G}]|^\mathsf{p}]$ is left-continuous with right-limits and:*

$$\int_0^1 \mathsf{E}[|\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}]|^\mathsf{p}] d\tau < +\infty.$$

2. *$\mathsf{X} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ if, and only if, $\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}] \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$, for all $\tau \in (0, 1)$, and:*

$$\sup_{\tau \in (0,1)} \|\mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}]\|_\infty < +\infty.$$

3. *If $\mathsf{X} \in \mathsf{L}^\mathsf{p}(\Omega, \mathcal{F}, \mathsf{P})$, for $\mathsf{p} \in [1, +\infty)$, then $\tau \in (0, 1) \mapsto \mathsf{Q}_\tau[\mathsf{X}|\mathcal{G}] \in \mathsf{L}^\mathsf{p}(\Omega, \mathcal{G}, \mathsf{P})$ is left-continuous with right-limits, as a curve in $\mathsf{L}^\mathsf{p}(\Omega, \mathcal{G}, \mathsf{P})$.*

It is worth noting that, as an immediate consequence of the Proposition 2.2.5, we obtain $\mathsf{Q}_\tau[\cdot|\mathcal{G}] : \mathsf{L}^\mathsf{p}(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\mathsf{p}(\Omega, \mathcal{G}, \mathsf{P})$ for all $\tau \in (0, 1)$ and $\mathsf{p} \in [1, +\infty]$.

[7] There is a set of full probability measure such that $\mathsf{P}[\mathsf{X} \in \mathsf{A}|\mathcal{G}] = \bar{\mathsf{P}}[\mathsf{X} \in \mathsf{A}|\mathcal{G}]$ for all $\mathsf{A} \in \mathcal{B}(\mathbb{R})$.

2.2.3 Basic properties

We now show that the τ-quantile operator coincides with our first conjecture for a conditional quantile as well as admits several distinct representations.

Theorem 2.2.6. *The following equalities hold:*

1. $Q_\tau[X|\mathcal{G}] = \inf\{Y \in L^0(\Omega, \mathcal{G}, P) : P[X \leqslant Y|\mathcal{G}] \geqslant \tau\}$ *pointwise.*

2. $Q_\tau[X|\mathcal{G}] = \min\{\operatorname{argmin}_{y\in\mathbb{R}} E[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}]\}$ *a.s.*

3. $Q_\tau[X|\mathcal{G}] = \inf\{y \in \mathbb{R} : P[X \leqslant y|\mathcal{G}] \geqslant \tau\}$ *pointwise.*

The result in item 1 of Theorem 2.2.6 means that if $Y \in L^0(\Omega, \mathcal{G}, P)$, then we understand $P[X \leqslant Y|\mathcal{G}] \geqslant \tau$ as $P[X \leqslant Y(\omega)|\mathcal{G}](\omega) \geqslant \tau$, for all $\omega \in \Omega$, and the infimum is pointwise. Furthermore, item 2 of Theorem 2.2.6 assumes continuous sample paths of the objective function in the minimization problem. Notice that $(E[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}])_{y\in\mathbb{R}}$ is a stochastic process satisfying:

$$E[|E[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}] - E[\rho_\tau(X-z) - \rho_\tau(X)|\mathcal{G}]|^p] \leqslant \left(\frac{1}{2} + \left|\frac{1}{2} - \tau\right|\right)^p |z-y|^p,$$

for all $p > 1$ and $y, z \in \mathbb{R}$. Therefore, Kolmogorov's theorem guarantees that exists a modification of this process with continuous sample paths (see Le Gall, 2013). Consequently, there is no loss of generality by imposing the continuity condition, and from now on, for each $X \in L^0(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ given, we assume that the sample paths of $(E[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}])_{y\in\mathbb{R}}$ are continuous.

We may combine Theorem 2.2.6 and Proposition 2.2.5 to obtain yet another equivalent characterization of conditional quantile as the minimal solution of an optimization problem in $L^p(\Omega, \mathcal{F}, P)$.

Proposition 2.2.7. *For all $\tau \in (0,1)$ and $p \in [1, +\infty]$, the τ-conditional quantile operator, $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$ satisfies:*

$$Q_\tau[X|\mathcal{G}] = \inf\{Z \in L^p(\Omega, \mathcal{G}, P), Z \in \operatorname*{argmin}_{Y\in L^p(\Omega,\mathcal{G},P)} E[\rho_\tau(X-Y)]\} \text{ a.s..}$$

Moreover, the optimization problem in Theorem 2.2.6 item 1 can be restricted to $L^p(\Omega, \mathcal{G}, P)$.

Remark 2.2.8. *The infimum is understood as the essential infimum of a family of random variables as in Peskir and Shiryaev (2006).*

The results previously derived show that, for all $\tau \in (0,1)$ and $p \in [1, +\infty] \cup \{0\}$, the τ-conditional quantile is an invariant operator with respect to the regularity of the space, i.e. $Q_\tau[\cdot|\mathcal{G}](L^p(\Omega, \mathcal{F}, P)) = L^p(\Omega, \mathcal{G}, P)$. Indeed, if $X \in L^p(\Omega, \mathcal{G}, P)$, then $E[\rho_\tau(X-X)] = 0$. Since, for all $Y \in L^p(\Omega, \mathcal{G}, P)$, $E[\rho_\tau(X-Y)] \geqslant 0$, with strict inequality when $X \neq Y$ in a non-negligible set, we conclude that X is the minimizer in Proposition 2.2.7. Therefore, $Q_\tau[X|\mathcal{G}] = X$. Together with Proposition 2.2.5, we get $Q_\tau[\cdot|\mathcal{G}](L^p(\Omega, \mathcal{F}, P)) = L^p(\Omega, \mathcal{G}, P)$. To further analyze its properties as an operator, next section investigates the conditions under which it is additive, as well as its other invariance properties.

The characterizations in Theorem 2.2.6 are the basis for quantile regression. When estimating a linear or nonlinear quantile model, one simply replaces the population expectation in item 2 of the theorem with the corresponding sample average and uses linear programming to solve the optimization problem (see, e.g., Koenker (2005) for details).

As an example of how each characterization may play an important role in the theory, we apply the results in Theorem 2.2.6 to provide a set of basic properties for the conditional quantile.

Proposition 2.2.9. *Let $X \in L^0(\Omega, \mathcal{F}, P)$ be fixed. We have the following:*

1. *For each $\omega \in \Omega$, the map $s \in (0,1) \mapsto Q_s[X|\mathcal{G}](\omega)$ is non-decreasing, left-continuous with right-limits. Moreover, $Q_{\tau+}[X|\mathcal{G}](\omega) = \lim_{s\downarrow\tau} Q_s[X|\mathcal{G}](\omega)$ and it can be characterized as $Q_{\tau+}[X|\mathcal{G}](\omega) = \sup\{y \in \mathbb{R} : P[X \leqslant y|\mathcal{G}](\omega) \leqslant \tau\}$, for all $\omega \in \Omega$.*

2. *For every $\tau \in (0,1)$, then $Q_\tau[X|\mathcal{G}] \in \operatorname{supp} X$ a.s.*

3. *(Monotonicity) If $X \leqslant Y$ a.s., then, for all $\tau \in (0,1)$, $Q_\tau[X|\mathcal{G}] \leqslant Q_\tau[Y|\mathcal{G}]$ a.s.*

4. *If Y is independent of $\mathcal{G}$, then $Q_\tau[Y|\mathcal{G}] = Q_\tau[Y]$ a.s., for all $\tau \in (0,1)$.*

5. *(Invariance) If X is $\mathcal{G}$-measurable, then $Q_\tau[X|\mathcal{G}] = X$ a.s.*

6. *If $g \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$, then the following holds a.s.:*

$$E[g(X)|\mathcal{G}] = \int_0^1 g(Q_\tau[X|\mathcal{G}])d\tau.$$

2.3 Invariance properties of conditional quantiles

This section investigates the invariance properties of conditional quantiles, i.e. which transformations commute with the operator. As in the unconditional setting, we are able to demonstrate that there are conditions that guarantee additivity (Section 2.3.1) and monotone invariance for this family of operators (Section 2.3.2). In addition, in Section 2.3.3 we use these results to demonstrate a Jensen's inequality for conditional quantiles.

2.3.1 Conditions for additivity

We provide conditions under which the conditional quantile operator of a sum of random variables equals to the sum of the conditional quantile of each random variable. First, we show how the concept of $\mathcal{G}$-comonotonicity introduced in Jouini and Napp (2004) may be used to obtain this result for the sum of $\mathcal{G}$-comonotonic random variables. This is similar to the works Dhaene et al. (2002) and Embrechts et al. (2003) for the unconditional quantile. Second, we generalize the concept of translational invariance and positive homogeneity for the conditional quantile operator.

We begin with an extension of the notion of comonotonicity of a random vector appropriated to the conditional case. Comonotonicity plays a key role in the additivity of quantiles, as it can be seen in Dhaene et al. (2002) for the unconditional quantile. Equipped with the definition of conditional comonotonicity, we show that it is, in fact, a sufficient condition for the additivity of the conditional quantile of a sum of random variables. Hence, we first define what are comonotonic sets.

Definition 2.3.1. *A set $A \subset \mathbb{R}^n$ is comonotonic if for all $x, y \in A$, then either $x \leqslant y$ or $y \leqslant x$.*

Dhaene et al. (2002) present and discuss in details the concept of comonotonicity as well as its consequences to quantiles. In the conditional framework, we define $\mathcal{G}$-conditional random vector as:

Definition 2.3.2. *Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra, and $X \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$. X is a $\mathcal{G}$-comonotonic random vector, if* $\operatorname{supp} P[X \in \cdot\,|\mathcal{G}](\omega)$ *is comonotonic almost surely on Ω.*

This definition is similar to one presented in Jouini and Napp (2004) and Cheung (2007), with the exception that we assume comonotonic support of the conditional law to be an almost surely property instead of pointwise, as they do. We adopted a different definition because changes in the conditional law representative would not alter the conditional comonotonicity property, since any representative would agree almost surely.

When $\mathcal{G} = \{\emptyset, \Omega\}$, we may see that Definition 2.3.2 coincides with the concept of comonotonic random variables as introduced (for $n = 2$) in Schmeidler (1986), and further explored in Dhaene et al. (2002). Observe that, in this case, there is a set of full probability measure such that, for all $A \in \mathcal{B}(\mathbb{R}^n)$, then $\mathsf{P}[X \in A|\mathcal{G}](\omega) = \mathsf{P}[X \in A]$. Thus, supp $\mathsf{P}[X \in \cdot\,|\mathcal{G}](\omega) =$ supp $\mathsf{P}[X \in \cdot\,]$ a.s., so that X is $\mathcal{G}$-comonotonic if, and only if, it is comonotonic, as it appears for instance in Dhaene et al. (2002). Furthermore, notice also that comonotonicity implies $\mathcal{G}$-comonotonicity, for all σ-algebra $\mathcal{G} \subset \mathcal{F}$, whereas the opposite implication is false – see a counterexample in the third example in Section 1 of Appendix A.

The first result of this section is a characterization of $\mathcal{G}$-comonotonic vectors similar to Cheung (2007, Lemma 2). We simply modified the results to hold a.s., agreeing to our definition of $\mathcal{G}$-comonotonicity. Before stating the result, it is worth noting that we assume that the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is rich enough to support an uniform random variable on $(0, 1)$, which is equivalent to P being atomless – see Proposition A.27 in Föllmer and Schied (2002) for a thoroughly discussion on the consequences of such assumption in the probability space.

Lemma 2.3.3. *Let $X \in L^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$, such that $X = (X_1, \ldots, X_n)$. Then the following statements are equivalent:*

1. *X is $\mathcal{G}$-comonotonic.*

2. *There exists a set $\Omega' \in \mathcal{G}$, such that $\mathsf{P}[\Omega'] = 1$, and on Ω':*

$$\mathsf{P}[X \leqslant x|\mathcal{G}](\omega) = \min_{i \in \{1,\ldots,n\}} \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega), \text{ for all } x \in \mathbb{R}^n.$$

3. *There exists a uniform random variable $U : \Omega \to (0, 1)$ and $\Omega' \subset \Omega$, with $\mathsf{P}[\Omega'] = 1$, such that for all $\omega \in \Omega'$ and any set $A \in \mathcal{B}(\mathbb{R}^n)$:*

$$\mathsf{P}[X \in A|\mathcal{G}](\omega) = \mathsf{P}[(Q_U[X_1|\mathcal{G}](\omega), \ldots, Q_U[X_n|\mathcal{G}](\omega)) \in A].$$

As a consequence of the characterization obtained above, we derive the following generalization of Embrechts et al. (2003, Proposition 3.1):

Theorem 2.3.4. *Let $\psi : \mathcal{X} \subset \mathbb{R}^m \times \mathcal{Y} \subset \mathbb{R}^n \to \mathbb{R}$ be a function such that, for all $x \in \mathcal{X}$, then $y \in \mathcal{Y} \mapsto \psi(x, y)$ is non-decreasing, left-continuous with right-limits in each argument. Then, for any $\mathcal{G}$-comonotonic random vector $Y = (Y_1, \ldots, Y_n)$, whose support lies in $\mathcal{Y}$, and $X \in L^0(\Omega, \mathcal{G}, \mathsf{P}; \mathbb{R}^m)$, with support in $\mathcal{X}$, and $\tau \in (0, 1)$ fixed:*

$$Q_\tau[\psi(X, Y)|\mathcal{G}] = \psi\left(X,\, Q_\tau[Y_1|\mathcal{G}], \ldots, Q_\tau[Y_n|\mathcal{G}]\right), \text{ a.s..}$$

In Appendix A, we show how the previous result may be employed to completely characterize the random variables that appear in a quantile regression framework. Moreover, as a corollary, we also obtain that the τ-conditional quantile of the sum of the components of a $\mathcal{G}$-comonotonic random vector equals the sum of the individual quantiles a.s.

Corollary 2.3.5. *Let* $X = (X_1, \ldots, X_n)$ *be a* $\mathcal{G}$*-comonotonic random vector. Then, for all* $\tau \in (0,1)$*:*

$$Q_\tau\left[\sum_{i=1}^{n} X_i \middle| \mathcal{G}\right] = \sum_{i=1}^{n} Q_\tau\left[X_i|\mathcal{G}\right], \text{ a.s.,}$$

and provided that $X_i \geqslant 0$ *a.s.,* $\forall i \in \{1, \ldots, n\}$*, then:*

$$Q_\tau\left[\prod_{i=1}^{n} X_i \middle| \mathcal{G}\right] = \prod_{i=1}^{n} Q_\tau\left[X_i|\mathcal{G}\right], \text{ a.s..}$$

Instead of using $\mathcal{G}$-comonotonicity, the following result provides a stronger version of positive homogeneity and translational invariance to establish additivity for each quantile.

Theorem 2.3.6. *If* $a, b \in L^0(\Omega, \mathcal{G}, P)$*, then a.s.:*

$$Q_\tau[a + bX|\mathcal{G}] = a + bQ_\tau[X|\mathcal{G}]\mathbb{1}_{\{b\geqslant 0\}} + bQ_{(1-\tau)+}[X|\mathcal{G}]\mathbb{1}_{\{b<0\}}.$$

Theorem 2.3.6 also holds for $Q_{\tau+}[\cdot|\mathcal{G}]$, simply changing τ and $(1-\tau)+$ by $\tau+$ and $1-\tau$, respectively.

2.3.2 Equivariance to monotone transformations

Continuing the discussion above on invariance properties of quantiles, we generalize their invariance to monotone transformation from the unconditional – given in Koenker (2005) – to the conditional case. To accomplish this, we first provide a simple version of invariance for conditional quantiles with respect to monotone transformations. Then, we extend it to the case of monotone functions with measurable parameters.

Proposition 2.3.7. *Suppose* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies that, for a given* $x \in \mathbb{R}$*,* $y \in \mathbb{R} \mapsto g(x, y)$ *is non-decreasing and left-continuous. Then, for all* $\tau \in (0,1)$ *fixed and* $Y \in L^0(\Omega, \mathcal{F}, P)$*,*

$$Q_\tau[g(x, Y)|\mathcal{G}] = g\left(x, Q_\tau[Y|\mathcal{G}]\right) \text{ a.s..}$$

If $y \in \mathbb{R} \mapsto g(x, y)$ *is non-increasing and left-continuous for a given* $x \in \mathbb{R}$*, then:*

$$Q_\tau[g(x, Y)|\mathcal{G}] = g\left(x, Q_{(1-\tau)+}[Y|\mathcal{G}]\right) \text{ a.s..}$$

Moving forward with the discussion on how $\mathcal{G}$-measurable parameters influence the invariance properties, we now extend Proposition 2.3.7 to monotone and left-continuous functions with measurable parameters.

Proposition 2.3.8. *Suppose that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies*

1. *For each* $x \in \mathbb{R}$*,* $y \in \mathbb{R} \mapsto g(x, y)$ *is non-decreasing and left-continuous.*
2. *For each* $y \in \mathbb{R}$*,* $x \in \mathbb{R} \mapsto g(x, y)$ *is* $\mathcal{B}(\mathbb{R})$*-measurable.*

Then, for all $X \in L^0(\Omega, \mathcal{G}, P)$*,* $Y \in L^0(\Omega, \mathcal{F}, P)$ *there is a set* $\Omega' \in \mathcal{G}$*, with* $P[\Omega'] = 1$*, such that for all* $\tau \in (0,1)$ *and* $\omega \in \Omega'$

$$Q_\tau[g(X, Y)|\mathcal{G}](\omega) = g\left(X, Q_\tau[Y|\mathcal{G}](\omega)\right),$$
$$Q_{\tau+}[g(X, Y)|\mathcal{G}](\omega) = g\left(X, Q_{\tau+}[Y|\mathcal{G}](\omega)\right).$$

If in item 1 g *is non-increasing and left-continuous, and item 2 holds true, then:*

$$Q_\tau[g(X, Y)|\mathcal{G}](\omega) = g\left(X, Q_{(1-\tau)+}[Y|\mathcal{G}](\omega)\right),$$
$$Q_{\tau+}[g(X, Y)|\mathcal{G}](\omega) = g\left(X, Q_{(1-\tau)}[Y|\mathcal{G}](\omega)\right).$$

We highlight that the invariance properties obtained along this section are generalizations of known results to conditional quantiles. Indeed, Koenker and Bassett (2010) presents Proposition 2.3.7 for the unconditional setup. We generalize it to the conditional case for all σ-algebra. Nevertheless, Proposition 2.3.8 extends this invariance to functions with measurable parameters. Moreover, if $\mathsf{Q}_\tau[g(X,Y)|\mathcal{G}] = \mathsf{Q}_{\tau+}[g(X,Y)|\mathcal{G}]$, then it is possible to show that Proposition 2.3.8 is a direct consequence of Proposition 2.3.7 and Corollary 2.4.3, by proving the invariance along parameters which are simple random variables, approximating the real parameter by them and, then, using the continuity of g and Corollary 2.4.3.

As an application of these results, in Section 2.5 we will apply the invariance of this family of operators to derive important rules regarding the exchange in the order of derivative operator and conditional quantiles, reinforcing that the operator-theoretical approach to this family may lead to the understanding and enlargement of the properties enjoyed by them.

2.3.3 Jensen's inequality

This section derives Jensen's inequality for the one parameter family of operators. As a consequence of Theorem 2.3.6, we are able to establish a version of Jensen's inequality for quantiles using the same approach adopted in the proof of this inequality for conditional mean, Lemma 1.23 in Shreve (2004).

Theorem 2.3.9 (Jensen's Inequality for Quantiles)**.** *Let $u : \mathbb{R} \to \mathbb{R}$ be a function and $\tau \in (0,1)$.*

1. *If u is concave and $\tau \in (0, \frac{1}{2}]$, then:*

$$\mathsf{Q}_\tau[u(X)|\mathcal{G}] \leqslant u\big(\mathsf{Q}_\tau[X|\mathcal{G}]\big), \text{ a.s..}$$

2. *If u is convex and $\tau \in (\frac{1}{2}, 1)$, then:*

$$u\big(\mathsf{Q}_\tau[X|\mathcal{G}]\big) \leqslant \mathsf{Q}_\tau[u(X)|\mathcal{G}], \text{ a.s..}$$

3. *If u is convex and $\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}] = \mathsf{Q}_{\frac{1}{2}+}[X|\mathcal{G}]$, then:*

$$u\big(\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}]\big) \leqslant \mathsf{Q}_{\frac{1}{2}}[u(X)|\mathcal{G}], \text{ a.s..} \tag{2.4}$$

Conversely, if (2.4) holds for all u convex, then $\mathsf{Q}_{\frac{1}{2}}[X|\mathcal{G}] = \mathsf{Q}_{\frac{1}{2}+}[X|\mathcal{G}]$ a.s.

We remark that Merkle (2005) establishes an analogue of Jensen's inequality for medians. Recently, Zhao et al. (2021) strengthen these inequalities using a similar approach to the one used in Merkle (2005). Nevertheless, the results in Theorem 2.3.9 are obtained through a simple operator-like argument, providing a generalization for conditional medians and demonstrating the necessity and sufficiency of continuity of conditional quantiles at $\tau = \frac{1}{2}$ for Jensen's inequality. Besides that, it is trivial to see that Theorem 2.3.9 item 1 and 2 also hold for $\mathsf{Q}_{\tau+}[\cdot|\mathcal{G}]$ operator when $\tau \in (0, \frac{1}{2})$ and $\tau \in [\frac{1}{2}, 1)$, respectively.

Recall that if $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is regular enough, for example in $\mathcal{C}_b(\mathbb{R} \times \mathbb{R})$, then, for all $X \in L^0(\Omega, \mathcal{G}, P)$ and $Y \in L^0(\Omega, \mathcal{F}, P)$, the expected value of $g(X,Y)$, on a full probability set, with respect to $\mathcal{G}$ is equivalent to $\mathsf{E}[g(X,Y)|\mathcal{G}](\omega) = \int g(X(\omega), y) P[Y \in dy|\mathcal{G}](\omega)$. In other words, we may interpret X as a parameter that does not affect the computation of the conditional expectation. Based on this, we generalize the former Jensen's inequality as following.

Corollary 2.3.10. *If $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:*

1. *For each* $x \in \mathbb{R}$, $y \in \mathbb{R} \mapsto u(x, y)$ *is concave.*

2. *For each* $y \in \mathbb{R}$, $x \in \mathbb{R} \mapsto u(x, y)$ *is continuous.*

3. *The function* $u'_{2,+}\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $u'_{2,+}(x, y) = \lim_{h \downarrow 0} \frac{u(x,y+h)-u(x,y)}{h}$, *is* $\mathcal{B}(\mathbb{R} \times \mathbb{R})$-*measurable.*

Then, for all $\tau \in (0, \frac{1}{2}]$, $X \in L^0(\Omega, \mathcal{G}, P)$, *and* $Y \in L^0(\Omega, \mathcal{F}, P)$ *fixed:*

$$Q_\tau[u(X, Y)|\mathcal{G}] \leqslant u(X, Q_\tau[Y|\mathcal{G}]), \text{ a.s..}$$

If $y \in \mathbb{R} \mapsto u(x, y)$ *is convex, for all* $x \in \mathbb{R}$, *and items 2 and 3 remain true, then for all* $\tau \in (\frac{1}{2}, 1)$*:*

$$Q_\tau[u(X, Y)|\mathcal{G}] \geqslant u(X, Q_\tau[Y|\mathcal{G}]), \text{ a.s..}$$

Finally, if $\tau = \frac{1}{2}$ *and* $Q_{\frac{1}{2}+}[Y|\mathcal{G}] = Q_{\frac{1}{2}}[Y|\mathcal{G}]$ *a.s., then:*

$$u(X, Q_{\frac{1}{2}}[Y|\mathcal{G}]) \leqslant Q_{\frac{1}{2}}[u(X, Y)|\mathcal{G}], \text{ a.s..}$$

2.4 Continuity

After establishing that conditional quantiles can be viewed as operators, and investigating conditions for their additivity, we now investigate their continuity properties. We start by describing a new Fatou's lemma for conditional quantiles, proving that it holds under less stringent assumptions than its conditional expected value counterpart. Then, as a direct consequence of our Fatou's lemma, we obtain conditions for the continuity of conditional quantiles with respect to almost sure convergence. Furthermore, since Proposition 2.2.5 guarantees that conditional quantiles are well-defined and invariant operators on L^p spaces, i.e. $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$ for all $p \in [1, +\infty]$, we provide conditions for the continuity of these operators under the L^p-topology. Finishing this section, we revisit and enlarge some of the main theorems regarding the continuity of quantiles with respect to weak convergence, now in the context of conditional weak convergence a.s. and its implications to conditional quantiles.

2.4.1 Fatou's lemma and almost sure continuity

Fatou's lemma for conditional expectation states that given a sequence of non-negative random variables, $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega, \mathcal{F}, P)$, then:

$$E[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} E[X_n|\mathcal{G}], \text{ a.s.} \tag{2.5}$$

As a consequence, if $\sup_{n\in\mathbb{N}} X_n < C$ a.s., then:

$$E[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} E[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} E[X_n|\mathcal{G}] \leqslant E[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}], \text{ a.s..} \tag{2.6}$$

In this section, we prove an analog to equation (2.5), substituting conditional expected value operator by conditional quantile. Moreover, we show that the requirement to be non-negative is not mandatory. However, to obtain an equation parallel to (2.6), we require the use of both $Q_\tau[\cdot|\mathcal{G}]$ and $Q_{\tau+}[\cdot|\mathcal{G}]$ operators. It is also possible to create an example where the last inequality in equation (2.6), with $Q_\tau[\cdot|\mathcal{G}]$ replacing $E[\cdot|\mathcal{G}]$, is false, i.e. $\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] > Q_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$.

We begin with an analog of Fatou's lemma for conditional quantiles.

Theorem 2.4.1. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of random variables in* $L^0(\Omega,\mathcal{F},P)$*, such that* $\inf_{n\in\mathbb{N}} X_n$ *and* $\sup_{n\in\mathbb{N}} X_n$ *are in* $L^0(\Omega,\mathcal{F},P)$*. For all* $\tau\in(0,1)$ *fixed, then:*

$$\begin{aligned} Q_\tau[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] &\leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \\ &\leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n|\mathcal{G}] \leqslant Q_{\tau+}[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}], \text{ a.s..} \end{aligned}$$

As we pointed out above, the previous result establishes a chain of inequalities similar to (2.6). Nevertheless, example 2.4.2 below shows that equation (2.6), with $Q_\tau[\cdot|\mathcal{G}]$ replacing $E[\cdot|\mathcal{G}]$, does not hold when using only the operator $Q_\tau[\cdot|\mathcal{G}]$. Indeed, the inequality $\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$ is not necessarily true, except in the trivial case when $Q_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}] = Q_{\tau+}[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$.

Example 2.4.2. *Let* $\tau\in(0,1)$*, and* $U\in L^0(\Omega,\mathcal{F},P)$ *be uniformly distributed in* $(0,1)$ *independently of* $\mathcal{G}$*. Define, for each* $n\in\mathbb{N}$*,*

$$X_n(\omega) = \frac{1}{n}\mathbb{1}_{U\in[0,\tau)} - \big(n(\omega-\tau)\big)^{\frac{1}{n}}\mathbb{1}_{U\in[\tau,1]} \quad \text{and} \quad X = -\mathbb{1}_{U\in(\tau,1]}. \tag{2.7}$$

Then it is immediate to see that $X_n \underset{n\to\infty}{\longrightarrow} X$ *pointwise,* $Q_\tau[X_n|\mathcal{G}] = 0$ *and* $Q_\tau[X|\mathcal{G}] = -1$*. Finally, this implies that* $\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] > Q_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}]$*.*

This example also demonstrates that discontinuities of quantile sample paths play a crucial role in convergence theorems. Indeed, as operators on $L^0(\Omega,\mathcal{F},P)$, the previous result implies that conditional quantiles are continuous with respect to a.s. convergence, provided that $\tau\in(0,1)$ is a continuity point for the sample path of the conditional quantile of the limiting random variable, $s\mapsto Q_s[X|\mathcal{G}]$, in a set of full probability measure.

Corollary 2.4.3. *Let* $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega,\mathcal{F},P)$*, such that* $X_n\to X\in L^0(\Omega,\mathcal{F},P)$ *a.s. Then, for each* $\tau\in(0,1)$ *such that* $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ *a.s., we have:*

$$Q_\tau[X|\mathcal{G}] = \lim_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}], \text{ a.s..}$$

Corollary 2.4.3 proves that, for each $\tau\in(0,1)$, $Q_\tau[\cdot|\mathcal{G}] : L^0(\Omega,\mathcal{F},P)\to L^0(\Omega,\mathcal{G},P)$ is a continuous operator, if restricted to random variables such $x\in\mathbb{R}\mapsto P[X\leqslant x|\mathcal{G}]$ is strictly increasing a.s. and $L^0(\Omega,\mathcal{F},P)$ and $L^0(\Omega,\mathcal{G},P)$ are considered with a.s. convergence of random variables. On the other hand, Theorem 2.4.1 shows that $Q_\tau[\cdot|\mathcal{G}]$ and $Q_{\tau+}[\cdot|\mathcal{G}]$ are, respectively, lower semicontinuous and upper semicontinuous operators on $L^0(\Omega,\mathcal{F},P)$, taking values in $L^0(\Omega,\mathcal{G},P)$, when both spaces are considered with a.s. convergence. Next, we demonstrate that a similar phenomenon occurs when these operators are restricted to $L^p(\Omega,\mathcal{F},P)$, $p\in[1,+\infty)$.

2.4.2 L^p continuity

As we proved in Proposition 2.2.5, the operator $Q_\tau[\cdot|\mathcal{G}]$ maps $L^p(\Omega,\mathcal{F},P)$ onto $L^p(\Omega,\mathcal{G},P)$, for $p\in[1,+\infty]$. Thus, it is natural to study continuity properties for the family of conditional quantiles operators with respect to L^p-convergence. Our next proposition shows that $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega,\mathcal{F},P)\to L^p(\Omega,\mathcal{G},P)$, for all $p\in[1,+\infty)$ and $\tau\in(0,1)$, is a lower semicontinuous operator with respect L^p-topology. Besides that, we show that the requirement for a.s. continuity of the sample paths of the conditional quantile of the limiting random variable, X, also guarantees L^p continuity of the operator $Q_\tau[\cdot|\mathcal{G}]$ at X. We close this section demonstrating that $Q_\tau[\cdot|\mathcal{G}] : L^\infty(\Omega,\mathcal{F},P)\to L^\infty(\Omega,\mathcal{G},P)$ is 1-Lipschitz, whereas for all $p\in[1,+\infty)$ the operator $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega,\mathcal{F},P)\to L^p(\Omega,\mathcal{G},P)$ is not Lipschitz.

Proposition 2.4.4. *For each* $p \in [1, +\infty)$ *and* $\tau \in (0,1)$, $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$ *is lower semicontinuous with respect to* L^p*-convergence, i.e. if* $X_n \xrightarrow[n\to\infty]{L^p} X$, *then* $\liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \in L^p(\Omega, \mathcal{G}, P)$ *and* $\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \in L^p(\Omega, \mathcal{G}, P)$ *with:*

$$Q_\tau[X|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}], \text{ a.s..}$$

Furthermore, for all $\tau \in (0,1)$ *where* $s \mapsto E[Q_s[X|\mathcal{G}]]$ *is continuous, then* $Q_\tau[X_n|\mathcal{G}] \xrightarrow{L^p} Q_\tau[X|\mathcal{G}]$.

When $p = +\infty$ the continuity condition for the family of conditional quantiles operator simplifies. Recall that, by Proposition 2.2.5 item 2, $Q_\tau[\cdot|\mathcal{G}] : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$. Using Theorem 2.3.6 and the monotonicity of $Q_\tau[\cdot|\mathcal{G}]$, Proposition 2.2.9 item 3, we are able to repeat the argument in Lemma 4.3 in Föllmer and Schied (2002) to derive that this family of operators are 1-Lipschitz and, hence, continuous.

Proposition 2.4.5. *For all* $\tau \in (0,1)$, $Q_\tau[\cdot|\mathcal{G}] : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ *is a continuous non-linear operator in* L^∞*-norm. Moreover,* $Q_\tau[\cdot|\mathcal{G}]$ *is 1-Lipschitz, i.e. for all* $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$, *then:*

$$\|Q_\tau[X|\mathcal{G}] - Q_\tau[Y|\mathcal{G}]\|_\infty \leqslant \|X - Y\|_\infty.$$

As claimed before, we now show that $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$ is not Lipschitz.

Example 2.4.6. *For all* $\tau \in (0,1)$ *and* $p \in [1, +\infty)$, *let* $(X_n)_{n\in\mathbb{N}} \subset L^p(\Omega, \mathcal{F}, P)$ *and* $X \in L^p(\Omega, \mathcal{F}, P)$ *be as in* (2.7). *By the dominated convergence theorem,* $X_n \to X$ *in* L^p *. Therefore, given a* $K \in \mathbb{R}_+$, *there exists a* $n_0 \in \mathbb{N}$ *such that, for all* $n \geqslant n_0$, $\|X - X_n\|_{L^p} < \frac{1}{K}$. *However,* $\|Q_\tau[X_n|\mathcal{G}] - Q_\tau[X|\mathcal{G}]\|_{L^p} = 1$, *for all* $n \in \mathbb{N}$. *Consequently, we conclude that, for all* $n \geqslant n_0$,

$$\|Q_\tau[X_n|\mathcal{G}] - Q_\tau[X|\mathcal{G}]\|_{L^p} > K\|X - X_n\|_{L^p},$$

and $Q_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$ *is not Lipschitz.*

2.4.3 Weak continuity

Instead of considering $L^0(\Omega, \mathcal{F}, P)$ with almost sure convergence, we may consider this space equipped with the convergence in distributions. Recall that a sequence of random variables, $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega, \mathcal{F}, P)$, is said to converge in distribution (or weakly) to $X \in L^0(\Omega, \mathcal{F}, P)$, $X_n \Rightarrow X$ (or $F_n \Rightarrow F$), if, and only if, the sequence of c.d.f, $(F_n)_{n\in\mathbb{N}}$, converges pointwisely at every continuity point of F, the c.d.f. of the limiting random variable. As we are dealing with the conditional framework, the suited concept for conditional convergence in distribution was the one initially proposed in Sweeting (1989):

Definition 2.4.7. *A sequence of random variables,* $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega, \mathcal{F}, P)$, *converges weakly to* $X \in L^0(\Omega, \mathcal{F}, P)$ *conditional to* $\mathcal{G}$ *almost surely,* $X_n \underset{\mathcal{G}}{\Rightarrow} X$ *a.s., if there exists a set* $\Omega' \in \mathcal{G}$, *with full probability measure, such that:*

$$P[X_n \in \cdot|\mathcal{G}](\omega) \Rightarrow P[X \in \cdot|\mathcal{G}](\omega), \text{ for all } \omega \in \Omega'.$$

Notice that, when restricted to $\mathcal{G} = \{\emptyset, \Omega\}$, $\mathcal{G}$-weak convergence a.s. reduces to weak convergence. Thus, the results derived in this section for the conditional framework might be immediately translated to the unconditional setup using weak convergence.

We begin this section proving that conditional right and left-quantiles, when viewed as operators on $L^0(\Omega, \mathcal{F}, P)$ and taking values on $L^0(\Omega, \mathcal{G}, P)$, are, respectively, upper and lower semicontinuous with respect to $\mathcal{G}$-weak convergence a.s. In the unconditional framework, this result was initially proposed in Chambers (2009). Finally, we conclude with conditions for the convergence of the quantiles for a monotone sequence with respect to first order stochastic dominance in the conditional setup.

Theorem 2.4.8. *For all* $\tau \in (0,1)$*, the operators* $Q_\tau[\cdot|\mathcal{G}] : L^0(\Omega, \mathcal{F}, P) \to L^0(\Omega, \mathcal{G}, P)$ *and* $Q_{\tau+}[\cdot|\mathcal{G}] : L^0(\Omega, \mathcal{F}, P) \to L^0(\Omega, \mathcal{G}, P)$ *are weakly lower and upper semicontinuous, respectively. Moreover,* $X_n \underset{\mathcal{G}}{\Rightarrow} X$ *a.s. if, and only if, there exists a set* $\Omega' \in \mathcal{G}$*, such that* $P(\Omega') = 1$ *and on it:*

$$Q_\tau[X|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}],$$

for all $\tau \in (0,1)$*.*

We conclude this section discussing how Theorem 2.4.8 may be used to determine the convergence of quantiles along monotone sequences of random variables. In order to prove a monotone convergence theorem, we must first define what it means for a random variable to converge from above or below. To achieve this, we use the *first order stochastic dominance* concept adapted to the conditional setting.

Definition 2.4.9. *(Conditional First Order Stochastic Dominance) Let* $X, Y \in L^0(\Omega, \mathcal{F}, P)$*, then* $X \succeq_{\mathcal{G}} Y$*, if there exists* $\Omega \in \mathcal{G}'$*, with full probability measure, such that:*

$$P[X \leqslant x|\mathcal{G}](\omega) \leqslant P[Y \leqslant x|\mathcal{G}](\omega), \text{ for all } x \in \mathbb{R} \text{ and } \omega \in \Omega'.$$

Notice that, in particular, if $X \geqslant Y$ a.s., then $X \succeq_{\mathcal{G}} Y$. Moreover, the definition of *conditional first order stochastic dominance* gives rise to a natural definition of monotone convergence for sequences of random variables. A sequence of random variables $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega, \mathcal{F}, P)$ is first order increasing if $X_{n+1} \succeq_{\mathcal{G}} X_n$ for all $n \in \mathbb{N}$. If this happens and, additionally, $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s., we write $X_n \uparrow_D X$. Similarly, a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ is first order decreasing if $X_{n+1} \preceq_{\mathcal{G}} X_n$ for all $n \in \mathbb{N}$. In this case, we write $X_n \downarrow_D X$ when $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s..

Equipped with the concepts defined above, we are now able to provide sufficient and necessary conditions that assure convergence for a conditional first order monotone sequence of random variables.

Proposition 2.4.10. *Let* $\Pi = \{X_n\}_{n\in\mathbb{N}}$ *be a conditional first order monotone sequence of random variables. Then,* $(X_n)_{n\in\mathbb{N}}$ *is* $\mathcal{G}$*-weakly convergent a.s. if, and only if, there exists a* $\Omega' \in \mathcal{G}$*, with full probability, and, for all* $\epsilon > 0$*, there is a* $m(\epsilon, \omega) > 0$ *such that:*

$$\sup_{\substack{X\in\Pi \\ \tau\in(\epsilon,1-\epsilon]}} |Q_\tau[X|\mathcal{G}](\omega)| \leqslant m(\epsilon,\omega) \quad \text{and} \quad \sup_{\substack{X\in\Pi \\ \tau\in[\epsilon,1-\epsilon)}} |Q_{\tau+}[X|\mathcal{G}](\omega)| \leqslant m(\epsilon,\epsilon),$$

for all $\omega \in \Omega'$*.*

Furthermore, if it is convergent, then there is a $X \in L^0(\Omega, \mathcal{F}, P)$ *and* $\Omega' \in \mathcal{G}$*, with full measure, such that on it:*

$$Q_\tau[X|\mathcal{G}] \leqslant \lim_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}], \text{ for all } \tau \in (0,1).$$

Finally, if $X_n \downarrow_D X$ *(or* $X_n \uparrow_D X$*) then* $Q_\tau[X_n|\mathcal{G}] \downarrow Q_\tau[X|\mathcal{G}]$ *a.s. (or* $Q_\tau[X_n|\mathcal{G}] \uparrow Q_\tau[X|\mathcal{G}]$ *a.s.) at every* $\tau \in (0,1)$ *such that* $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ *a.s.*

2.5 Interchanging quantiles and derivatives: Leibniz's rule

One of the most useful properties of the expected value is its ability of interchanging the order of integration and differentiation, also referred to as "Leibniz's rule". Nevertheless, there are required conditions to achieve such a result. Interchanging a derivative with an expectation (an integral) can be established by applying the dominated convergence theorem. Intuitively, the conditions say that the derivative of the function of interest must be bounded by another function whose integral is finite. The interchange of integration and differentiation has been extensively used in applications; for example, in deriving statistical properties of the maximum likelihood estimator (see, e.g., Ferguson, 1996).

Recall that Sections 2.3.1 and 2.3.2 state that quantiles are invariant to some transformations. In this section, we show that this property may be used extensively to prove differentiability of the quantile along a family of functions in the support of a random variable. Moreover, we are able to provide an example where the differentiation under the expectation sign fails, even though we may interchange the differentiation and the quantile functional for all $\tau \in (0,1)$. Section 2.5.1 deals with the Leibniz rule for monotone functions and it is related to Section 2.3.2. On the other hand, Section 2.5.2 uses Section 2.3.1 to separable functions to generate the same result.

2.5.1 Leibniz's rule for monotone functions

Recall that given a stochastic process $(X_t)_{t\in V}$, we call $(\bar{X}_t)_{t\in V}$ its modification if:

$$\mathsf{P}[X_t = \bar{X}_t] = 1, \text{ for all } t \in V.$$

Our first result is accomplished by applying Proposition 2.3.7 to obtain a differentiable modification of the process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x}\in V}$ in a neighbourhood V of x.

Theorem 2.5.1. *Let $\mathcal{X} \subset \mathbb{R}$, $\mathcal{Y} \subset \mathbb{R}$, $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ and $x \in \mathcal{X}$ such that:*

1. *There exists an open neighbourhood of $x \in V \subset \mathbb{R}$, such that $y \in \mathcal{Y} \mapsto h(\bar{x}, y)$ is non-decreasing and left-continuous for all $\bar{x} \in V \cap \mathcal{X}$.*
2. *$\bar{x} \mapsto h(\bar{x}, y)$ is differentiable at x, for all $y \in \mathcal{Y}$.*

Then, for all $\tau \in (0,1)$ and $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, whose support lies in $\mathcal{Y}$, the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ admits a modification differentiable a.s. at x so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x, Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, \mathsf{Q}_\tau[Y|\mathcal{G}]).$$

Moreover, if in condition 1 above h is non-increasing and left-continuous, then, the stochastic process $(\mathsf{Q}_\tau[h(\bar{x}, Z)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$, for all $\tau \in (0,1)$, admits a modification differentiable a.s. at x so that:

$$\frac{d}{dx}\mathsf{Q}_\tau[h(x, Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, \mathsf{Q}_{(1-\tau)+}[Y|\mathcal{G}]).$$

In many situations, we are interested in differentiating h under the Q_τ functional. Instead of exploiting the linearity, which is the key tool for the expectation, we may obtain such result just using again the invariance of conditional quantiles to monotone transformation.

Corollary 2.5.2. *Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function, $\tau \in (0,1)$ and $Y \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, so that its support is in $\mathcal{Y}$. If h satisfies:*

1. *There exists an open neighbourhood of* $x \in V \subset \mathbb{R}$, *such that* $y \in \mathcal{Y} \mapsto h(\bar{x}, y)$ *is non-decreasing and left-continuous, for all* $\bar{x} \in V \cap \mathcal{X}$.

2. $\bar{x} \in V \cap \mathcal{X} \mapsto h(\bar{x}, y)$ *is differentiable at* x, *for all* $y \in \mathcal{Y}$.

3. $y \in \mathcal{Y} \mapsto \frac{\partial h}{\partial x}(x, y)$ *is non-decreasing and left-continuous.*

Then, there is a modification of the stochastic process $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

If in item 3 above $\frac{\partial h}{\partial x}(x, \cdot)$ *is non-increasing and left-continuous, then there is a modification of the stochastic process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Moreover, if in condition 1 h *is non-increasing and left-continuous, then there is a modification of the stochastic process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Finally, if in addition to the previous change in item 1, in item 3 above $\frac{\partial h}{\partial x}(x, \cdot)$ *is non-increasing and left-continuous, then there is a modification of the process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

2.5.2 Leibniz's rule for separable functions

Instead of exploring the monotonicity property, we may apply the translational invariance and homogeneity of quantiles to investigate the interchanging of differentiation and Q_τ. In the following result, assuming a separability condition on h, we are able to provide conditions for differentiability of the process $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V}$ with respect to the parameters $\bar{x}$, as well as for the interchange of derivative and quantiles in this setting.

Theorem 2.5.3. *Let* $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ *be such that there are* $\eta : \mathcal{Y} \to \mathbb{R}$, $\phi : \mathcal{X} \to \mathbb{R}$ *and* $\psi : \mathcal{X} \to \mathbb{R}$ *with* $h(x, y) = \phi(x) + \psi(x)\eta(y)$, *for all* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$. *Assume that* $Y \in L^0(\Omega, \mathcal{F}, P)$, *so that its support is in* $\mathcal{Y}$, *and both* ψ *and* ϕ *are differentiable at* x.

1. *If* $\psi(\bar{x}) \geqslant 0$, *for all* $\bar{x} \in V \cap \mathcal{X}$ *in an open neighbourhood* V *of* x, *then the stochastic process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *admits a modification differentiable at* x, *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = \phi'(x) + \psi'(x) Q_\tau[\eta(Y)|\mathcal{G}].$$

Additionally, if $\psi'(x) \geqslant 0$, *then the stochastic process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *has a modification differentiable at* $x \in \mathcal{X}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

However, if $\psi'(x) < 0$, *then the stochastic process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *has a modification differentiable at* $x \in \mathcal{X}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

2. If $\psi(\bar{x}) \leqslant 0$, *for all* $\bar{x} \in V \cap \mathcal{X}$ *in an open neighbourhood* V *of* x, *then the stochastic* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap X}$ *admits a modification differentiable at* x, *such that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = \phi'(x) + \psi'(x) Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}].$$

Moreover, if $\psi'(x) \geqslant 0$, *then the process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *has a modification differentiable at* $x \in \mathcal{X}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\Big|\mathcal{G}\right], \text{ a.s..}$$

Nevertheless, if $\psi'(x) < 0$, *then the process* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *has a modification differentiable at* $x \in \mathcal{X}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\Big|\mathcal{G}\right], \text{ a.s..}$$

3. If $\psi(x) = 0$, $\bar{x} \in V \cap \mathcal{X} \mapsto \psi(\bar{x})$ *is either non-decreasing or non-increasing in a neighbourhood* $V \subset \mathcal{X}$ *of* x, *and* $\psi'(x) Q_\tau[\eta(Y)|\mathcal{G}] = \psi'(x) Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}]$ *a.s., then the stochastic* $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *admits a modification differentiable at* x, *such that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = \phi'(x) + \psi'(x) Q_\tau[\eta(Y)|\mathcal{G}], \text{ a.s..}$$

Beyond that, the process $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$ *has a modification differentiable at* $x \in \mathcal{X}$ *so that:*

$$\frac{d}{dx} Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\Big|\mathcal{G}\right], \text{ a.s..}$$

We conclude with two distinct examples, restricting ourselves to the unconditional framework. The first example shows explicitly when it is not possible to differentiate $Q_\tau[h(x, Z)]$.

Example 2.5.4. *Let* Y *be a random variable such that* $P[Y = 1] = P[Y = -1] = \frac{1}{2}$, *and* $h : \mathbb{R} \times \{-1, 1\} \to \mathbb{R}$: $h(x, y) = xy$. *Fixed* $\tau \in (\frac{1}{2}, 1)$, *then:*

$$\begin{aligned} Q_\tau[h(x, Y)] &= x Q_\tau[Y] \mathbb{1}_{x \geqslant 0} + x Q_{(1-\tau)+}[Y] \mathbb{1}_{x<0} \\ &= |x|, \end{aligned}$$

since $Q_\tau[Y] = 1$ *and* $Q_{(1-\tau)+}[Y] = -1$. *It is clear that* $x \mapsto Q_\tau[h(x, Y)]$ *is not differentiable at* 0.

Our second example in this section exhibits that we may have $Q_\tau[\frac{\partial h}{\partial x}(x, Y)] = \frac{d}{dx} Q_\tau[h(x, Y)]$, though $\frac{d}{dx} E[h(x, Y)] \neq E[\frac{\partial h}{\partial x}(x, Y)]$.

Example 2.5.5. *Let* Y *be a random variable whose support is* $\text{supp}(Y) = (0, 1)$ *and its law is given by:*

$$P[Y \in A] = \int_A \frac{y^{-\frac{1}{2}}}{2} dy, \text{for all } A \in \mathcal{B}((0, 1)).$$

Suppose $h : \mathbb{R}_+ \times (0, 1) \to \mathbb{R}$ *is the function* $h(x, y) = \ln(x + y) + 2\frac{y}{x+y}$. *Then* $E[h(x, Y)] = \ln(x + 1)$ *for all* $x \geqslant 0$. *In particular,* $\frac{d}{dx} E[h(x, Y)] = \frac{1}{x+1}$ *for all* $x \geqslant 0$ *and, consequently,* $\frac{d}{dx} E[h(0, Y)] = 1$.

Now observe that $\frac{\partial h}{\partial x}(0, Y) = \frac{-1}{Y}$*. This implies that:*

$$\mathsf{E}\Big[\frac{\partial h}{\partial x}(0, Y)\Big] = -\infty \neq 1 = \frac{d}{dx}\mathsf{E}[h(x, Y)]\Big|_{x=0}.$$

Therefore, it is not possible to differentiate under the integral sign. Nevertheless, for all τ*-quantile,* $\tau \in (0, 1)$*, one is able to do it.*

Notice that $\frac{\partial h}{\partial y}(x, y) = \frac{1}{x+y} + \frac{2x}{(x+y)^2} > 0$ *and, consequently,* $y \in (0, 1) \mapsto h(x, y)$ *is strictly increasing and continuous for all* $x \in \mathbb{R}_+$ *fixed. Secondly,* $x \mapsto h(x, y)$ *is differentiable everywhere. Finally,* $y \mapsto \frac{\partial h}{\partial x}(0, y)$ *is strictly increasing, because* $\frac{\partial^2 h}{\partial y \partial x}(0, y) = \frac{1}{y^2} > 0$*. Therefore, Corollary 2.5.2 assures that:*

$$\frac{d}{dx}\mathsf{Q}_\tau\left[h(x, Y)\right]\Big|_{x=0} = \mathsf{Q}_\tau\left[\frac{\partial h}{\partial x}(0, Y)\right], \text{ for all } \tau \in (0, 1).$$

2.6 Composition of conditional quantiles

Along this section, we investigate the behavior of the composition of conditional quantiles with respect to different σ-algebras. Firstly, Section 2.6.1 exhibits a general counterexample to the "law of iterated quantiles", which would be the analog of the "law of iterated expectations" to conditional quantiles.[8] Besides that, we describe the properties of the domains where the "law of iterated quantiles" holds, building an analogy to the projection approach to both expected values and quantiles. Lastly, Section 2.6.2 analyzes the problem of countable compositions of conditional quantiles along a filtration. As we will show, this problem is much more complex than its expected value counterpart, since the "law of iterated quantiles" is generally false. We also provide two distinct conditions for the existence of the limit of infinitely many compositions of conditional quantiles.

2.6.1 The law of iterated quantiles

Recall that given $\mathcal{F} \supset \mathcal{G} \supset \mathcal{H}$ σ-algebras and a random variable $X \in L^1(\Omega, \mathcal{F}, \mathsf{P})$, the law of iterated expectations is the following equality:

$$\mathsf{E}[\mathsf{E}[X|\mathcal{G}]|\mathcal{H}] = \mathsf{E}[X|\mathcal{H}] = \mathsf{E}[\mathsf{E}[X|\mathcal{H}]|\mathcal{G}], \text{ a.s.}$$

This equation can be interpreted as a commutative relation between the maps $\mathsf{E}[\cdot|\mathcal{H}] : L^1(\Omega, \mathcal{F}, \mathsf{P}) \to L^1(\Omega, \mathcal{H}, \mathsf{P})$ and $\mathsf{E}[\cdot|\mathcal{G}] : L^1(\Omega, \mathcal{F}, \mathsf{P}) \to L^1(\Omega, \mathcal{G}, \mathsf{P})$.

For the conditional quantiles, recall that $\mathsf{Q}_\tau[\mathsf{Q}_\tau[X|\mathcal{H}]|\mathcal{G}] = \mathsf{Q}_\tau[X|\mathcal{H}]$, due to Proposition 2.2.9 item 5 and $\mathsf{Q}_\tau[X|\mathcal{H}] \in L^0(\Omega, \mathcal{H}, \mathsf{P})$. However, as we pointed out and the following result shows, in general, the law of iterated quantiles, $\mathsf{Q}_\tau[\mathsf{Q}_\tau[X|\mathcal{G}]|\mathcal{H}] = \mathsf{Q}_\tau[X|\mathcal{G}]$, does not hold unrestrictedly in a wide class of probability spaces.

Proposition 2.6.1. *Suppose that* $(\Omega, \mathcal{F}, \mathsf{P})$ *is a probability space, such that* $\mathcal{F}$ *is a non-trivial* σ*-algebra on* Ω *and* $\tau \in (0, 1)$*. If there are disjoint sets* $\{A_i\}_{i=1}^3 \subset \mathcal{F}$*, with* $\mathsf{P}[A_i] = p_i \in (0, 1)$*,* $i = 1, 2$ *and* 3*, satisfying:*

1. $A_1 \cup A_2 \cup A_3 = \Omega$*;*

[8] The law of iterated expectations is also known as the law of total expectation or the tower property – Williams (1991).

2. $0 < p_1 < \tau$;

3. $\tau - p_1 \leqslant p_2 < \tau - \tau p_1$.

Then there are a random variable $X \in L^0(\Omega, \mathcal{F}, P)$ *and sub-*σ*-algebras* $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$*, so that:*

$$Q_\tau[Q_\tau[X|\mathcal{G}]|\mathcal{H}] \neq Q_\tau[X|\mathcal{H}]. \tag{2.8}$$

Although Proposition 2.6.1 shows that there is no hope for a general law of iterated quantiles, we are able to characterize the maximal set where it holds by analyzing the family of conditional quantile operators as a family of non-linear projections.

Recall that, when restricted to $L^2(\Omega, \mathcal{F}, P)$ random variables, the conditional expectation may be computed from a specific class of optimization problems. According to this formulation, for all sub-σ-algebra $\mathcal{S} \subset \mathcal{F}$, then:

$$E[X|\mathcal{S}] = \operatorname*{argmin}_{Y \in L^2(\Omega,\mathcal{S},P)} E[|X-Y|^2],$$

which is the minimization of the L^2 distance between a set and a point. In particular, $E[\cdot|\mathcal{S}]$ may be seen as a linear projections from $L^2(\Omega, \mathcal{F}, P)$ onto $L^2(\Omega, \mathcal{S}, P)$. Denoting by $\pi_{L^2(\Omega,\mathcal{S},P)} = E[\cdot|\mathcal{S}]$ and $\pi^2_{L^2(\Omega,\mathcal{S},P)}(X) = \pi_{L^2(\Omega,\mathcal{S},P)}(X) \circ \pi_{L^2(\Omega,\mathcal{S},P)}(X)$, this projection fulfills:

$$\begin{aligned}
\pi^2_{L^2(\Omega,\mathcal{S},P)}(X) &= \pi_{L^2(\Omega,\mathcal{S},P)}(X), \text{ for all } X \in L^2(\Omega, \mathcal{F}, P),\\
\pi_{L^2(\Omega,\mathcal{S},P)}(X) &= X, \text{ for all } X \in L^2(\Omega, \mathcal{S}, P).
\end{aligned}$$

Furthermore, restated in terms of projections, the law of iterated expectations is then just a commutative property enjoyed by projection operators,

$$\pi_{L^2(\Omega,\mathcal{H},P)} \circ \pi_{L^2(\Omega,\mathcal{G},P)}(X) = \pi_{L^2(\Omega,\mathcal{G},P)}(X) = \pi_{L^2(\Omega,\mathcal{G},P)} \circ \pi_{L^2(\Omega,\mathcal{H},P)}(X).$$

Similarly, the conditional median may also be seen as a non-linear projection when restricted to $L^1(\Omega, \mathcal{F}, P)$, projecting it onto $L^1(\Omega, \mathcal{S}, P)$. Indeed, by Proposition 2.2.7 the conditional median satisfies:

$$Q_{\frac{1}{2}}[X|\mathcal{S}] = \inf\left\{ Z \in \operatorname*{argmin}_{Y \in L^1(\Omega,\mathcal{S},P)} \frac{1}{2} E\big[|X-Y|\big] \right\}.$$

This characterization shows that conditional medians are the minimal random variables that minimize the L^1 distance between a closed convex subset, $L^1(\Omega, \mathcal{S}, P)$, and a point $X \in L^1(\Omega, \mathcal{F}, P)$. Moreover, defining $Q_{\frac{1}{2}}[\cdot|\mathcal{S}] = \pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{S},P)} : L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{S}, P)$, then $\pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{F},P)}$ is both an idempotent non-linear operator and invariant on $L^1(\Omega, \mathcal{F}, P)$ – see item 5 in Proposition 2.2.9.[9] Consequently,

$$\begin{aligned}
\pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{S},P)} \circ \pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{S},P)}(X) &= \pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{S},P)}(X), \text{ for all } X \in L^1(\Omega, \mathcal{F}, P)\\
\pi^{\frac{1}{2}}_{L^1(\Omega,\mathcal{S},P)}(X) &= X, \text{ for all } X \in L^1(\Omega, \mathcal{S}, P).
\end{aligned}$$

For the general case, i.e. $\tau \in (0,1)$, the conditional quantile operator may also be seen as the minimal minimizer of a quasimetric, which differs from a metric by not being symmetric. Indeed, fixed $\tau \in (0,1)$, define the quasimetric $d_\tau : L^1(\Omega, \mathcal{F}, P) \times L^1(\Omega, \mathcal{F}, P) \to \mathbb{R}_+$ by:

$$d_\tau(X, Y) = E[\rho_\tau(X - Y)].$$

It is immediate to see that d_τ is, indeed, a quasimetric, since it satisfies the following items 1 and 3, though not 2 – except when $\tau = \frac{1}{2}$:

[9] In fact, the invariance property, item 5 in Proposition 2.2.9, implies the idempotency property.

1. $d_\tau(X, Y) \geqslant 0$ and $d_\tau(X, Y) = 0$ if, and only if, $X = Y$ a.s.

2. $d_\tau(X, Y) = d_\tau(Y, X)$ for all $X, Y \in L^1$.

3. $d_\tau(X, Y) \leqslant d_\tau(X, Y) + d_\tau(Y, Z)$, for all $X, Y, Z \in L^1$.

Actually, item 2 above can be replaced by $d_\tau(X, Y) = d_{1-\tau}(Y, X)$, for the general case. Restating in this way, the conditional quantile is the minimal minimizer of the distance, measured by a quasimetric d_τ, between a point and a closed – in L^1-norm – convex set, $L^1(\Omega, \mathcal{S}, P)$:

$$Q_\tau[X|\mathcal{S}] = \inf \left\{ Z \in \underset{Y \in L^1(\Omega,\mathcal{S},P)}{\operatorname{argmin}} d_\tau(X, Y) \right\}.$$

Because the τ-conditional quantile is both an idempotent operator, invariant on its image – Proposition 2.2.9 item 5 – and a minimizer of a quasimetric between a convex closed set and a point, we conclude that it might also be seen as a nonlinear projection. Indeed, abusing the terminology from the L^2 framework, we will call any pair (H, π_H), or simply π_H, a projection onto H, if $H \subset L^1(\Omega, \mathcal{F}, P)$ is a closed convex set, in L^1-norm, and $\pi_H : L^1(\Omega, \mathcal{F}, P) \to H$ is an idempotent non-linear operator invariant on H. Thus, for all $\tau \in (0, 1)$ the pair $(L^1(\Omega, \mathcal{S}, P), \pi^\tau_{L^1(\Omega,\mathcal{S},P)})$ defined by $\pi^\tau_{L^1(\Omega,\mathcal{S},P)} = Q_\tau[\cdot|\mathcal{S}]$ is, due to Propositions 2.2.9 item 5 and Proposition 2.2.5 item 1, a (generalized) projection onto $L^1(\Omega, \mathcal{S}, P)$.

Using the terminology of commutative algebra, the commutator of two maps A, B is known as $[A, B] = A \circ B - B \circ A$. Thus, the set of random variables such that the law of iterated conditional quantiles holds can be defined by:

$$\mathcal{C}^\tau_{\mathcal{H},\mathcal{G}} = \left\{ X \in L^1(\Omega, \mathcal{F}, P) : \left[\pi^\tau_{L^1(\Omega,\mathcal{H},P)}, \pi^\tau_{L^1(\Omega,\mathcal{G},P)}\right] = 0 \right\}.$$

Fixed $\tau \in (0, 1)$, the formulation of conditional quantiles as projections allows us to characterize the domains of $L^1(\Omega, \mathcal{F}, P)$ where the law of iterated quantiles holds as the set where two projections commute with the following properties.

Proposition 2.6.2. *Fixed $\tau \in (0, 1)$ and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ σ-algebras, then:*

1. $L^1(\Omega, \mathcal{H}, P) \subset L^1(\Omega, \mathcal{G}, P) \subset \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$. *This last inclusion is proper provided that there is at least one variable independent of $\mathcal{G}$, i.e.* $L^1(\Omega, \mathcal{G}, P) \subsetneq \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$

2. *Given any $a \in L^1(\Omega, \mathcal{H}, P)$, $b \in L^\infty(\Omega, \mathcal{H}, P)$, with $b \geqslant 0$ a.s., and $X \in \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$, then $a + bX \in \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$. In particular, $\mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$ is a cone.*

2.6.2 Infinite composition of conditional quantiles

Instead of characterizing the domains where the conditional quantiles commute, $\mathcal{C}^\tau_{\mathcal{G},\mathcal{F}}$, in this section we investigate what happens when we compose the conditional quantiles infinitely many times. We show that this object has a richer structure than the infinite composition of conditional expectations, being able to provide a concrete example where the infinite composition of conditional quantiles may lead to a random variable which is infinity a.s. Besides that, we derive conditions for the existence of a finite and measurable random variable which is the limit of the infinite composition of conditional quantiles either when $E[X|\mathcal{F}_t] - E[X|\mathcal{F}_{t-1}]$ is independent of $\mathcal{F}_{t-1}$, for all $t \in \mathbb{N}$, or when $X \in L^\infty(\Omega, \mathcal{F}, P)$.

Suppose that we have a filtered probability space, $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}\cup\{0\}}, \mathsf{P}\big)$, with $\mathcal{F} = \mathcal{F}_\infty$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The law of iterated expectations is a powerful tool when we are considering the dynamics of conditional expectation. To see this, take any $\mathsf{X} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, then, for all $n < m \in \mathbb{N}$, we obtain

$$\mathsf{E}[\mathsf{E}[\dots \mathsf{E}[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathsf{E}[\mathsf{X}|\mathcal{F}_n]. \tag{2.9}$$

From this identity, we can apply the conditional expectation infinitely many times to obtain directly that $\mathsf{E}[\mathsf{X}|\mathcal{F}_n] = \lim_{m\uparrow+\infty} \mathsf{E}[\cdots \mathsf{E}[\mathsf{X}|\mathcal{F}_m] \cdots |\mathcal{F}_n]$.

For the conditional quantile, however, even in the finite case, an equation similar to (2.9) does not have to hold. Beyond that, as the following example show, there exist a random variable and filtration such that $\lim_{m\uparrow+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n]$ diverges a.s. except when $\tau = \frac{1}{2}$.

Example 2.6.3. *Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}\cup\{0\}}, \mathsf{P}\big)$ be such that $(\mathsf{B}_t)_{t\geqslant 0} : \Omega \to \mathbb{R}$ is a Brownian Motion, $\mathsf{T} > 0$ fixed, $t_0 = 0$, $t_n - t_{n-1} = \frac{6\mathsf{T}}{\pi^2 n^2}$, for $n \in \mathbb{N}$, and $\mathcal{F}_n = \sigma(\mathsf{B}_s : 0 \leqslant s \leqslant t_n)$. Define $\mathsf{X} = \mathsf{B}_\mathsf{T}$, then, fixed $n \geqslant 0$, for all $m > n$ – see the details in the Appendix A:*

$$\mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathsf{B}_{t_n} + \left(\frac{\sqrt{6\mathsf{T}}}{\pi} \sum_{j=1}^{m-n} \frac{1}{j} + \sqrt{\mathsf{T} - t_m}\right) \mathsf{Q}_\tau[N(0,1)].$$

Hence, because $\sum_{j=1}^{m-n} \frac{1}{j} \underset{m\to\infty}{\longrightarrow} +\infty$ and $\sqrt{\mathsf{T} - t_m} \underset{m\to\infty}{\longrightarrow} 0$, we conclude that:

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \begin{cases} -\infty, & \text{if } \tau < \frac{1}{2} \\ \mathsf{B}_{t_n}, & \text{if } \tau = \frac{1}{2} \\ +\infty, & \text{if } \tau > \frac{1}{2}. \end{cases}$$

The preceding computation on the example given above suggests that, for some particular cases, the series $\sum_{j\geqslant 1} \|\mathsf{E}[\mathsf{X}|\mathcal{F}_j] - \mathsf{E}[\mathsf{X}|\mathcal{F}_{j-1}]\|_{\mathsf{L}^1}$ may play an important role for the existence of

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n].$$

Indeed, our next results exhibit this connection.

Proposition 2.6.4. *Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{N}\cup\{0\}}, \mathsf{P}\big)$ be a filtered probability space, with $\mathcal{F}_\infty = \mathcal{F}$. Define $\mathsf{H} \subset \mathsf{L}^1(\Omega, \mathcal{F}, \mathsf{P})$, such that if $\mathsf{X} \in \mathsf{H}$, then:*

1. $\sum_{j\geqslant 1} \|\mathsf{E}[\mathsf{X}|\mathcal{F}_j] - \mathsf{E}[\mathsf{X}|\mathcal{F}_{j-1}]\|_{\mathsf{L}^1} < +\infty$.
2. $\mathsf{E}[\mathsf{X}|\mathcal{F}_j] - \mathsf{E}[\mathsf{X}|\mathcal{F}_{j-1}]$ *independent of* $\mathcal{F}_{j-1}$, $\forall j \geqslant 1$.

Then, for all $\mathsf{X} \in \mathsf{H}$ and $\tau \in (0,1)$, $\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n] \in \mathsf{L}^1(\Omega, \mathcal{F}_n, \mathsf{P})$, and:

$$\lim_{m\to+\infty} \mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n] = \mathsf{E}[\mathsf{X}|\mathcal{F}_n] + \sum_{j\geqslant 1} \mathsf{Q}_\tau[\mathsf{E}[\mathsf{X}|\mathcal{F}_{n+j}] - \mathsf{E}[\mathsf{X}|\mathcal{F}_{n+j-1}]], \text{ a.s..}$$

Although restrictive, the assumption of independent increments permits us to compute explicitly each iteration of the conditional quantile. Instead of explicitly calculating each iteration, we may bound them. In order to do so, we will restrict the random variables to $\mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$. In this domain, we are able to show that, for the sequence of compositions $(\mathsf{Q}_\tau[\mathsf{Q}_\tau[\dots \mathsf{Q}_\tau[\mathsf{X}|\mathcal{F}_m] \dots |\mathcal{F}_{n+1}]|\mathcal{F}_n])_{m>n}$, at least $\liminf$ and $\limsup$ exist and are finite a.s. This is a direct consequence of the fact that this sequence is bounded a.s. by $\pm\|\mathsf{X}\|_{+\infty}$, due to Proposition 2.2.5 item 2. Moreover, we are also able to provide a subset where both $\liminf$ and $\limsup$ agree, leaving the question whether they agree on all $\mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$.

Proposition 2.6.5. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}\cup\{0\}}, \mathsf{P})$ *be a filtered probability space, with* $\mathcal{F}_\infty = \mathcal{F}$. *Then, for each* $n \in \mathbb{N}\cup\{0\}$, *both operators* $\liminf_{m\in\mathbb{N}} \mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[\cdot|\mathcal{F}_m]\ldots|\mathcal{F}_n] : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{F}_n, \mathsf{P})$ *and* $\limsup_{m\in\mathbb{N}} \mathsf{Q}_\tau[\ldots \mathsf{Q}_\tau[\cdot|\mathcal{F}_m]\ldots|\mathcal{F}_n] : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{F}_n, \mathsf{P})$ *are well-defined non-linear operators. Moreover, they agree on* $\overline{\cup_{n\in\mathbb{N}\cup\{0\}}\mathsf{L}^\infty(\Omega, \mathcal{F}_n, \mathsf{P})}$, *where the closure is taken in* L^∞*-norm.*

2.7 Conclusion

This work investigates the properties of conditional quantiles viewed as nonlinear operators. The results are organized in parallel to the usual properties of the expectation operator. We generalize well-known properties of unconditional quantiles to the conditional case, such as translation invariance, comonotonicity, and equivariance to monotone transformations. Moreover, we provide a simple proof for Jensen's inequality for conditional quantiles.

Continuity and differentiability of the conditional expectation operator are widely used in practice. Therefore, we extend these concepts to conditional quantiles. We investigate continuity of conditional quantiles as operators with respect to different topologies. We obtain a novel Fatou's lemma for quantiles, provide conditions for continuity in L^p, and also weak continuity. Moreover, we also investigate the differentiability properties of quantiles. We show the validity of the Leibniz's rule for conditional quantiles for the cases of monotone, as well as separable functions.

Finally, we investigate the validity of the law of iterated expectations – also known as law of total expectation or tower property – to the quantile case. We show that the law of iterated quantiles or does not hold in general. Nevertheless, we characterize the maximum set of random variables for which this law holds, and investigate its consequences for the infinite composition of conditional quantiles.

The results are intended to shed new light and be useful for applications of quantiles, such a statistical applications as quantile regressions, risk management in finance, and decision theory in statistics and economics.

Chapter 3

Convex and Conditionally Law-Invariant Risk Measures

3.1 Introduction

The risk assessment of random outcomes for market investments has been subject of debate and research for mathematicians, statisticians, and economists. Due to its increasing usage by practitioners as well as regulatory agencies, a large amount of research has been devoted to explore and catalog the different properties of risk measures, which are, in general, unconditional. The practical importance of risk measures became influential by the Basel Accords directive requiring the usage of financial risk measures to quantify risk exposure. McNeil et al. (2005), and the references therein, have a comprehensive discussion about the historical developments of risk management, its central role in some recent regulatory accords, as well as its importance to financial institutions.

After the seminal article of Artzner et al. (1999), the systematization and axiomatic treatment of financial risks were intensely discussed in the literature. By assigning a set of desirable economic properties for a static risk measure, many authors provided a series of distinguished representations for such maps. Essentially, the most relevant regularity conditions ensure diversification, such as subadittivity and convexity, or allow to determine the risk from historical data, law-invariance. These different features and their corresponding characterizations were developed, for example, in Delbaen (2002), for the class of coherent risk measures, Föllmer and Schied (2002) and Frittelli and Rosazza Gianin (2005), for convex risk measures, and Kusuoka (2001), when the risk measure is law-invariant. Nevertheless, major drawbacks in these approaches are their inability of capturing the evolution and the availability of additional information, i.e. the risk measures are, often, static and unconditional.

In this chapter, we extend existing results for unconditional law-invariant convex risk measures to the conditional case, when additional information is available. We consider risk measurements conditioned on any σ-algebra, acting on L^∞-spaces, and assume the conditional risk measurement to be conditionally law-invariant, as coined by Dela Vega and Elliott (2021). Equipped with such machinery, we provide necessary and sufficient conditions for static and dynamic conditional convex risk measures, as well as their corresponding penalty functions, to be represented either as the integral of conditional quantiles, conditional average value-at-risk, random concave probability distortions or transition capacities. Subsequently, we delve into the dynamic representation of conditionally law-invariant risk measures, reconciling it with time-consistency and relevance through composition. We conclude by briefly connecting our dynamic representation results to limit theorems for iterated risk measures developed in Stadje (2010), which helps to explicitly identify the continuous-time risk measurement linked to a particular class of discrete-time risk measures.

In order to better understand the evolution of risk over time and accommodate intermediate payoffs, many authors have also been investigating and generalizing the risk measures' literature to the dynamic and/or conditional framework – e.g., but no limited to, Detlefsen and Scandolo (2005), Rosazza Gianin (2006), Stadje (2010), Madan et al. (2017), and Dela Vega and Elliott (2021). There are several theoretical and practical reasons for expanding the literature on conditional risk measures. First, an agent in the market often makes decisions based on a priori set of information. In this situation, he/she faces the problem of a static conditional risk measurement. Secondly, in a dynamic framework, the agent's flow of information about the market is modeled by a filtration such that, at each time, the risk measurement is updated conditioned to the current available information. Finally, conditional risk measures are also connected with Monetary Utility Functions (MUF) (see, e.g., Cheridito and Kupper (2011) and Klöppel and Schweizer (2007) for a detailed discussion), so that results concerning the former might be used to assess the risk attitudes of MUF's decision makers. As representative in this extensive literature, we refer to the dual representations of conditional convex risk measures obtained by Detlefsen and Scandolo (2005) and Dela Vega and Elliott (2021); the continuous-time risk measurement through g-expectations in Rosazza Gianin (2006), Stadje (2010), Madan et al. (2017); the investigation of acceptance sets on Föllmer and Penner (2006); and the conditional law-invariance properties and its implications explored in Weber (2006), Kupper and Schachermayer (2009) and Dela Vega and Elliott (2021).[1]

The structure of the present work is following. In Section 3.2, an axiomatic foundation for conditional risk measures is presented. Then, in Section 3.3, under a mild condition on the probability space, we describe sufficient and necessary conditions for a static conditional convex risk measure and its penalty function to be represented as an integral of conditional quantiles. In Section 3.4, we investigate the discrete-time dynamic risk assessment problem with an intraperiod law-invariance, describing some applications. We conclude the work in Section 3.5. The proofs of all results are collected in the appendices.

3.2 Preliminaries

In this section, we consider the basic definitions and recall the main properties of risk measures that are employed along the present work.

3.2.1 Conditonal Risk Measures

We start this section by providing a comprehensive introduction to conditional risk measures and their features. We adopt the axiomatic analysis of risk assessment in terms of capital requirements initiated by Artzner et al. (1999). Then, we analyze the definition of conditional law-invariance, coined by Dela Vega and Elliott (2021), showing that it is an equivalency relation in the space of almost surely finite random vectors. We discuss the implications of conditional law-invariance in the acceptance sets of conditional risk measures, and the relationships between conditional law-invariance and distribution-invariance.

[1] Equipped with the latter, Weber (2006) showed that, under appropriate assumption over the probability space, a representation theorem for distribution-invariant risk measures are derived in terms of static risk measures and expected shortfall risk. Dela Vega and Elliott (2021), on the other hand, extended the Kusuoka (2001)'s characterization for conditionally law-invariant coherent risk-measures. Nevertheless, law-invariance might pose serious restrictions on the dynamic behavior of risk measures. As demonstrated in Kupper and Schachermayer (2009), law-invariance might turn the risk measurement dynamically inconsistent, reducing the class of law-invariant and time-consistent risk measures to the entropic family.

In this work, we are concerned with the assessment of the risk of a bounded random outcome $X \in L^\infty(\Omega, \mathcal{F}, P)$ conditioned to the information available in $\mathcal{G}$. This intermediate σ-algebra, $\mathcal{G}$, stands for the information the agent has during the decision process, which can be either static or dynamic. To model this conditional risk assessment, we adopt the following definition:

Definition 3.2.1. *A map* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ *is a conditional risk measure if the following holds:*

1. *(Normalized)* $\rho(0) = 0$.

2. *(Conditional translational invariance) For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $Z \in L^\infty(\Omega, \mathcal{G}, P)$*, then:*
$$\rho(X + Z) = \rho(X) - Z.$$

3. *(Monotonicity) For any* $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$ *such that* $X \leqslant Y$ *a.s., then:*
$$\rho(X) \geqslant \rho(Y), \text{ a.s.}$$

Beyond the above standard features, ρ *may also have any of these additional properties:*

4. *(Conditional convexity) For any* $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $\Lambda \in L^0(\Omega, \mathcal{G}, P)$*, such that* $0 \leqslant \Lambda \leqslant 1$ *a.s., then:*
$$\rho\left(\Lambda X + (1 - \Lambda) Y\right) \leqslant \Lambda\rho(X) + (1 - \Lambda)\rho(Y), \text{ a.s.}$$

5. *(Conditional positive homogeneity) For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $\Lambda \in L^\infty(\Omega, \mathcal{G}, P)$*, such that* $\Lambda \geqslant 0$*, then:*
$$\rho(\Lambda X) = \Lambda\rho(X), \text{ a.s.}$$

6. *(Regularity) Given* $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$*, then for any* $A \in \mathcal{G}$ *such that* $X\mathbb{1}_A = Y\mathbb{1}_A$ *a.s.*
$$\rho(X)\mathbb{1}_A = \rho(Y)\mathbb{1}_A, \text{ a.s.}$$

7. *(Continuity from above) For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $(X_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, P)$*, so that* $X_n \downarrow X$ *a.s., then:*
$$\rho(X_n) \uparrow \rho(X), \text{ a.s.}$$

8. *(Continuity from below) For any* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $(X_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega, \mathcal{F}, P)$*, so that* $X_n \uparrow X$ *a.s., then:*
$$\rho(X_n) \downarrow \rho(X), \text{ a.s.}$$

We say that a conditional risk measure is conditionally coherent if it is conditionally convex and conditionally positively homogeneous. Moreover, it is continuous if it is both continuous from below and above. Every risk measure in this thesis conditional and, for this reason, we will sometimes omit this adjective to avoid unnecessary repetition and heavy notation.

The regularity condition was introduced in Detlefsen and Scandolo (2005) and it is useful for their representation theorem. Essentially, this condition says that if two variables are equal given some measurable event, then their risk in this event should be the same. Since every conditional risk measure in this thesis will be convex and normalized, regularity condition always holds. For a further discussion, see Detlefsen and Scandolo (2005).

A natural way to define a conditional risk measure is by specifying its acceptance set, $\mathcal{A} \subset \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ – see, for example, Föllmer and Penner (2006). Adopting this approach, a risk measurement of $\mathsf{X} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ will be the least $\mathsf{Y} \in \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P})$, measurable capital requirement, that, when added to X, makes their combination an acceptable position, i.e. $\mathsf{X}+\mathsf{Y} \in \mathcal{A}$. Thus, this translates into the following:

$$\rho_{\mathcal{A}}(\mathsf{X}) = \operatorname{essinf}\{\mathsf{Y} \in \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P}) : \ \mathsf{X}+\mathsf{Y} \in \mathcal{A}\},$$

Given a conditional risk measure as in Definition 3.2.1, we may also define its acceptance set as those positions whose underline risk is almost surely less or equal to zero, i.e. $\mathcal{A}_\rho = \{\mathsf{Y} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) : \ \rho(\mathsf{X}) \leqslant 0\}$. As it was pointed out by Detlefsen and Scandolo (2005), conditional translational invariance implies that $\rho_{\mathcal{A}_\rho} = \rho$, meaning that this acceptance set actually generates a set of minimal capital requirements for this risk measure. Moreover, the properties of ρ, as defined in Definition 3.2.1, are reflected in equivalent geometric and topological properties of its acceptance set, $\mathcal{A}_\rho$, such as convexity, positive homogeneity and continuity – see Proposition 2.14 in Detlefsen and Scandolo (2005).

In the unconditional framework, it is also natural to consider risk measurements that depend only on the distribution of the random variable, i.e. the risk measure is law-invariant. To analyze this property in the conditional setting, we first introduce the notion of conditional similarity. As the following definition shows, it describes a binary relation such that two variables are similar provided they have transition kernels that coincides a.s.

Definition 3.2.2 (Conditional similarity)**.** *Given* $\mathsf{X}, \mathsf{Y} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$*, we say that* $\mathsf{X} \sim_{\mathcal{G}} \mathsf{Y}$ *if there exists a set* $\Omega' \in \mathcal{G}$ *with full probability measure such that on it:*

$$\mathsf{P}[\mathsf{X} \in \mathsf{A}|\mathcal{G}] = \mathsf{P}[\mathsf{Y} \in \mathsf{A}|\mathcal{G}], \text{ for any } \mathsf{A} \in \mathcal{B}(\mathbb{R}^n).$$

This binary relation possesses the following useful properties.

Proposition 3.2.3. $\sim_{\mathcal{G}}$ *is an equivalence relation in* $\mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$*. Moreover, if* $\mathsf{X}, \mathsf{Y} \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P}; \mathbb{R}^n)$*, then* $\mathsf{X} \sim_{\mathcal{G}} \mathsf{Y}$ *if, and only if,* $\mathsf{X} = \mathsf{Y}$ *a.s. For any* $\mathsf{X} \in \mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P})$*, if* $\bar{\mathsf{X}} \sim_{\mathcal{G}} \mathsf{X}$*, then* $\bar{\mathsf{X}} \in \mathsf{L}^p(\Omega, \mathcal{F}, \mathsf{P})$*. Finally* $\bar{\mathsf{X}} \sim_{\mathcal{G}} \mathsf{X}$ *if, and only if,* $\bar{\mathsf{X}}+\mathsf{Y} \sim_{\mathcal{G}} \mathsf{X}+\mathsf{Y}$*, for any* $\mathsf{Y} \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$*.*

In order to extend law-invariance to dynamical setting, Weber (2006) proposes the concept of distribution-invariant conditional risk measure in terms of acceptance indicators. With this approach, he obtains appropriated representation of such measures. Nevertheless, we adopt the following natural concept of conditional law-invariance for a conditional risk measure based on conditional similarity, proposed initially by Dela Vega and Elliott (2021).

Definition 3.2.4 (Conditional law-invariance)**.** *A conditional risk measure,* $\rho : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P})$*, is conditionally law-invariant if, for any* $\mathsf{X}, \mathsf{Y} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ *such that* $\mathsf{X} \sim_{\mathcal{G}} \mathsf{Y}$*, then:*

$$\rho(\mathsf{X}) = \rho(\mathsf{Y}), \text{ a.s.}$$

It is important to notice that the above definition and the one proposed in Weber (2006) are different. Following his notation, when dividing the risk measure by the value of the zero-coupon bond, Weber (2006)'s definition of distribution-invariant risk measure imposes the following additional property.

Definition 3.2.5 (Certainty on Independent Variables)**.** *A conditional risk measure,* $\rho : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P})$*, is certain on independent variables if, for any* $\mathsf{X} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P})$ *independent of* $\mathcal{G}$*, then:*

$$\rho(\mathsf{X}) \text{ is constant a.s.}$$

In Proposition B.1.1 item 4, we show that the average value-at-risk conditioned to $\mathcal{G}$ with random parameters is not certain on independent variables. Therefore, an example of risk measure that is conditional law-invariant, as in Definition 3.2.4, though not distribution-invariant, as in Weber (2006), is readily available by multiplying it with the zero-coupon bond. Moreover, in the presence of conditional comonotonicity, as in Section 3.3.4, it is possible to prove that distribution-invariance, as in Weber (2006), forces the concave distortion functions to be deterministic, due to its certainty on independent variables. For example, Madan et al. (2017) also implicitly assumed the certainty on independent variables to obtain their representation of spectral risk measures in terms of non-random concave distortions.

When $\mathcal{G} = \{\emptyset, \Omega\}$, conditional law-invariance reduces to law-invariance, since $\mathcal{G}$-similarity is equality in law. Furthermore, conditional law-invariance simply says that the risk of a position in the market is completely determined by its law given the information available in $\mathcal{G}$.

Similarly to Proposition 2.14 in Detlefsen and Scandolo (2005), conditional law-invariance can also be characterized as a property of the acceptance set. Indeed, we say that a set $\mathcal{A} \subset \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$ is invariant under $\sim_{\mathcal{G}}$ if for every $\mathsf{X} \in \mathcal{A}$ and $\mathsf{Y} \in \mathsf{L}^0(\Omega, \mathcal{F}, \mathsf{P})$, such that $\mathsf{Y} \sim_{\mathcal{G}} \mathsf{X}$, then $\mathsf{Y} \in \mathcal{A}$. Thus, as the following result shows, conditional law-invariance of a risk measure is linked to $\sim_{\mathcal{G}}$ invariance of its acceptance set.

Proposition 3.2.6. *Let $\rho : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P})$ be a conditional risk measure. Then, ρ is conditionally law-invariant if, and only if, A_ρ is invariant under $\sim_{\mathcal{G}}$.*

As an example , we show a situation where Proposition 3.2.6 can be readily applied to characterize conditional law-invariance.

Example 3.2.7. *Let $u : \Omega \times \mathbb{R} \to \mathbb{R}$ be a non-decreasing concave function, such that $u(\cdot, x) \in \mathsf{L}^0(\Omega, \mathcal{G}, \mathsf{P})$, for any $x \in \mathbb{R}$. We define the conditional utility-based risk measure, $\rho_u : \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P})$, as:*

$$\rho_u(\mathsf{X}) = \operatorname{essinf}\{\mathsf{Y} \in \mathsf{L}^\infty(\Omega, \mathcal{G}, \mathsf{P}) : \ \mathsf{E}[u(\cdot, \mathsf{X}+\mathsf{Y})|\mathcal{G}] \geqslant u(\cdot, 0) \text{ a.s.}\}, \text{ for any } \mathsf{X} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}).$$

It is trivial to prove ρ_u is, indeed, a continuous from above conditional risk measure, whose acceptance set is $\mathcal{A}_{\rho_u} = \{\mathsf{X} \in \mathsf{L}^\infty(\Omega, \mathcal{F}, \mathsf{P}) : \ \mathsf{E}[u(\cdot, \mathsf{X})|\mathcal{G}] \geqslant u(\cdot, 0)\}$. Furthermore, because $\mathcal{A}_{\rho_u}$ is convex and $\sim_{\mathcal{G}}$-invariant, we conclude from Proposition 2.14 in Detlefsen and Scandolo (2005) and Proposition 3.2.6 that ρ_u is convex and conditionally law-invariant risk measure.

The previous example generates a whole class of conditional law-invariant and convex risk measures and will be studied in detail in Section 3.4.3. In fact, along the next sections, we will investigate how these two properties, convexity and conditional law-invariance, are helpful when characterizing conditional risk measures and their penalty functions, both in the static and dynamic cases.

3.3 Static Conditionally Law-Invariant Risk Measures

In this section, we demonstrate a series of representations for conditionally law-invariant and convex risk measures, as well as for their penalty functions. Firstly, a conditional quantile-based representation is derived in Section 3.3.1. Secondly, by determining a bijection between random mixtures and conditional quantiles of Radon-Nykodym derivatives, we establish a conditional average value-at-risk-based representation in Section 3.3.2. Then, Section 3.3.3 deals with a further equivalent representation in terms of random concave distortions. Finally, we conclude in Section 3.3.4 showing an alternative in terms of random Choquet's integrals.

3.3.1 A Quantile-based Representation of Conditionally Law-Invariant Risk Measures

Recall that conditionally convex and continuous from above risk measures admit the following robust representation – see Detlefsen and Scandolo (2005).

Theorem 3.3.1 (Detlefsen and Scandolo (2005))**.** *A conditionally convex risk measure,* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$*, is continuous from above if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{Q \in \mathcal{P}_\mathcal{G}} \left(E^Q[-X|\mathcal{G}] - \alpha_*(Q) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where

$$\alpha_*(Q) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(E^Q[-X|\mathcal{G}] - \rho(X) \right), \text{ for any } Q \in \mathcal{P}_\mathcal{G}.$$

Remark 3.3.2. *As it was shown in Detlefsen and Scandolo (2005), the penalization map,* $\alpha_* : \mathcal{P}_\mathcal{G} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}}_+)$*, satisfies the normalization condition:*

$$\inf_{Q \in \mathcal{P}_\mathcal{G}} \alpha_*(Q) = 0.$$

Along this chapter, Theorem 3.3.1 will be the starting point for the derivation of all subsequent representations. Additionally, we will also suppose the following:

Assumption 1. *There exist* $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $\mathcal{G}$*-measurable set,* Ω_X*, with full probability, such that* $x \in \mathbb{R} \mapsto P[X \leqslant x|\mathcal{G}]$ *is continuous in* Ω_X*.*

As discussed in Föllmer and Schied (2002) for the unconditional case, $\mathcal{G} = \{\emptyset, \Omega\}$, this assumption is equivalent to the existence of a uniform random variable in the interval $(0, 1)$, $U(0, 1)$, and the probability space being atomless. A similar situation holds in the conditional setting, since Assumption 1 also precludes the existence of a random variable, $U \in L^\infty(\Omega, \mathcal{F}, P)$, such that $P[U \in \cdot|\mathcal{G}] \overset{d}{=} U(0, 1)$ a.s. Nevertheless, it is not necessarily true that, even in an atomless space, there is a further random variable whose conditional c.d.f. is continuous. For instance, take $\mathcal{G} = \mathcal{F}$, and every conditional c.d.f. is discontinuous. For this reason, we assume 1.

Equipped with the previous assumption and employing the definition of conditional quantiles, Dela Vega and Elliott (2021) demonstrated the following generalization of Hardy-Littlewood type of identity to the conditional framework.

Lemma 3.3.3 (Dela Vega and Elliott (2021))**.** *If* $Y \in L^1(\Omega, \mathcal{F}, P)$ *and* $X \in L^\infty(\Omega, \mathcal{F}, P)$*, then:*

$$\operatorname*{esssup}_{\bar{X} \sim_\mathcal{G} X} E[\bar{X}Y|\mathcal{G}] = \int_0^1 Q_\tau[X|\mathcal{G}] Q_\tau[Y|\mathcal{G}] d\tau, \text{ a.s.}$$

Lemma 3.3.3 states that, as in the unconditional case, when we fix the conditional marginal of a given random variable, X, the maximum value for the conditional expected value of its product with a further fixed random variable, Y, is achieved by perfectly correlating the random variables when conditioned to $\mathcal{G}$.

After introducing and discussing the basic machinery needed, we now have enough tools to the demonstrate the main result of this section, which extends the representation theorem for law-invariant convex risk measures as integrals of quantiles, derived in Frittelli and Rosazza Gianin (2005).

Theorem 3.3.4. *A conditionally convex and continuous from above risk measure,* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$*, is conditionally law-invariant if, and only if,* ρ *admits the following representation:*

$$\rho(X) = \operatorname*{esssup}_{Q \in \mathcal{P}_\mathcal{G}} \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau - \alpha_*(Q) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where:

$$\alpha_*(Q) = \operatorname*{esssup}_{X \in L^\infty(\Omega, \mathcal{F}, P)} \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau - \rho(X) \right), \text{ for any } Q \in \mathcal{P}_\mathcal{G}$$

Moreover, $\alpha_* : \mathcal{P}_\mathcal{G} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}}_+)$ *is invariant under* $\sim_\mathcal{G}$*, that is if* $Q, \bar{Q} \in \mathcal{P}_\mathcal{G}$*, so that* $\frac{d\bar{Q}}{dP} \sim_\mathcal{G} \frac{dQ}{dP}$*, then* $\alpha_*(\bar{Q}) = \alpha_*(Q)$ *a.s.*

As an immediate corollary, we obtain the characterization of conditionally law-invariant coherent risk measures that are continuous from above, as in Dela Vega and Elliott (2021).

Corollary 3.3.5. *Let* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ *be a conditionally coherent risk measure continuous from above. Then, it is conditionally law-invariant if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{Q \in \mathcal{Q}} \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{Q} \subset \mathcal{P}_\mathcal{G}$*.*

3.3.2 Conditional Risk Measures as Mixtures of Conditional Average Value-at-Risk.

Theorem 3.3.4 above is the first step to generate a conditional version for the representation of convex law-invariant risk measures in terms of mixtures of average value-at-risk, derived by Frittelli and Rosazza Gianin (2005) and Kusuoka (2001). To accomplish this, we will need to define the following conditional risk measures:

Definition 3.3.6. *(Value-at-Risk conditional to* $\mathcal{G}$*) For any* $\tau \in (0, 1)$*, we denote by Value-at-Risk operator conditional to* $\mathcal{G}$*,* $V@R_\tau[\cdot|\mathcal{G}] : L^0(\Omega, \mathcal{F}, P) \to L^0(\Omega, \mathcal{G}, P)$*, the operator:*

$$V@R_\tau[X|\mathcal{G}] = Q_{1-\tau}[-X|\mathcal{G}].$$

Besides the aforementioned conditional risk measure, we can also define the Average Value-at-Risk conditional to $\mathcal{G}$ as:

Definition 3.3.7. *(Average Value-at-Risk conditional to* $\mathcal{G}$*) For any* $\tau \in (0, 1)$*, the* τ*-Average Value-at-Risk operator conditional to* $\mathcal{G}$*,* $AV@R_\tau : L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{G}, P)$*, is defined by*

$$AV@R_\tau[X|\mathcal{G}] = \frac{1}{\tau} \int_0^\tau V@R_s[X|\mathcal{G}] ds, \text{ for any } X \in L^1(\Omega, \mathcal{F}, P).$$

In Appendix B, we explore the properties enjoyed by $AV@R_\tau[\cdot|\mathcal{G}]$ as operator, such as continuity, invariance of domain, stability under basic operations and additivity under conditional comonotonicity. See Proposition B.1.1.

Differently from the unconditional setting, we allow the measures mixing the distinct average value-at-risk of the financial position to be random. Consequently, they should belong to the following set of transition kernels.

Definition 3.3.8. *The set of $\mathcal{G}$-measurable transition kernels in* $(0,1]$, $\mathcal{M}^{\mathcal{G}}_{(0,1]} = \{\mu : \Omega \times \mathcal{B}((0,1]) \to [0,1]\}$, *is composed by those* μ *satisfying:*

1. *For any* $A \in \mathcal{B}((0,1])$, *then* $\omega \in \Omega \mapsto \mu(\omega, A)$ *is* $\mathcal{G}$*-measurable.*

2. *For any* $\omega \in \Omega$, *then* $A \in \mathcal{B}((0,1]) \mapsto \mu(\omega, A)$ *is a probability measure on* $(0,1]$.

Although the elements of this space are understood as "random measures", we consciously omit the ω when writing $\mu(\omega, d\tau)$. This convention is done solely to avoid heavy notation.

By appropriately adapting the proof given in Föllmer and Schied (2002) to the conditional setting and employing the objects defined above, we are now able to generalize the representation theorems in Frittelli and Rosazza Gianin (2005) for risk measures that are conditionally convex, conditionally law-invariant and continuous from above in terms of random mixtures of average value-at-risk conditional to $\mathcal{G}$.

Theorem 3.3.9. *Let* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ *be conditionally convex and continuous from above. Then,* ρ *is conditionally law-invariant if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]\, d\mu(\tau) - \beta_*(\mu) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where

$$\beta_*(\mu) = \operatorname*{esssup}_{X \in L^\infty(\Omega, \mathcal{F}, P)} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]\, d\mu(\tau) - \rho(X) \right), \text{ for any } \mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}.$$

Moreover, if ρ is coherent and conditionally law-invariant, Dela Vega and Elliott (2021)'s result holds.

Corollary 3.3.10 (Dela Vega and Elliott (2021))**.** *Let* $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ *be a conditionally coherent and continuous from above risk measure. Then,* ρ *is conditionally law-invariant if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{\mu \in \mathcal{M}} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]\, d\mu(\tau) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{M} \subset \mathcal{M}^{\mathcal{G}}_{(0,1]}$.

3.3.3 Random Concave Distortions.

In this section, we describe an alternative representation for convex and conditionally law-invariant risk measures in terms of random concave distortions. This random distortions were proposed initially by Dela Vega and Elliott (2021) to describe conditions for a conditional risk measure to be represented by a unique distortion of the the conditional probability, whose definition is the following.

Definition 3.3.11. *The set of* $\mathcal{G}$*-measurable concave stochastic processes on* $\Omega \times [0,1]$, $\mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, *consist of maps* $\psi : \Omega \times [0,1] \to [0,1]$ *such that:*

1. *For any* $\omega \in \Omega$, $\tau \in [0,1] \mapsto \psi(\omega, \tau)$ *is concave, continuous, non-decreasing with* $\psi(\omega, 0) = 0$ *and* $\psi(\omega, 1) = 1$.

2. *For any* $\tau \in [0,1]$, $\omega \in \Omega \mapsto \psi(\omega, \tau)$ *is* $\mathcal{G}$*-measurable.*

It is trivial to show that, under these conditions, every $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ is $\mathcal{G} \otimes \mathcal{B}([0,1])$ measurable. This allows us to consider its composition with $\mathcal{F}$ and $\mathcal{G}$-measurable maps taking values in $[0,1]$.

In the unconditional framework, there exists a bijection between probability measures in the interval $(0,1]$ and concave distortions – see e.g. Föllmer and Schied (2002). This bijection extends similarly in the conditional setup as argued by Dela Vega and Elliott (2021).

Lemma 3.3.12. *The following map,* $\Phi : \mathcal{M}^{\mathcal{G}}_{(0,1]} \to \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$*, is a bijection:*

$$\Phi: \ \mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]} \mapsto \left(\tau \in (0,1) \mapsto \psi(\cdot,\tau) := 1 - \mu\left(\cdot,(\tau,1]\right) + \int_{(\tau,1]} \frac{\tau}{s}\mu(\cdot,ds)\right) \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1]),$$

with $\psi(\cdot,0) = 0$ *and* $\psi(\cdot,1) = 1$.

Equipped with the above map, the next result, Theorem 4.14 in Dela Vega and Elliott (2021), characterizes conditionally coherent, conditionally law-invariant and continuous risk measures, $\rho_\mu : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{G},P)$, of the form

$$\rho_\mu(X) = \int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau), \ \text{for any } X \in L^\infty(\Omega,\mathcal{F},P), \tag{3.1}$$

where $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, as the integral of a concave random distortion, ψ, of the conditional distribution of X given $\mathcal{G}$. From it, we derive an equivalent representation for convex and conditionally law-invariant risk measures.

Theorem 3.3.13. *For any* $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$*, there is a* $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ *so that the conditional risk measure in* (3.1) *satisfies, for every* $X \in L^\infty(\Omega,\mathcal{F},P)$,

$$\rho_\mu(X) = \int_0^{+\infty} \left(\psi\left(\cdot, P[X<x|\mathcal{G}]\right) - 1\right)dx + \int_{-\infty}^0 \psi\left(\cdot, P[X<x|\mathcal{G}]\right)dx, \ \text{a.s.}$$

As claimed before, we obtain that:

Theorem 3.3.14. *Let* $\rho : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{G},P)$ *be conditionally convex and continuous from above. Then,* ρ *is conditionally law-invariant if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{\psi \in \mathrm{Conc}(\Omega,\mathcal{G},[0,1])} \left(\int_{-\infty}^0 \psi\left(\cdot, P[X<x|\mathcal{G}]\right)dx + \int_0^{+\infty} \left(\psi\left(\cdot, P[X<x|\mathcal{G}]\right) - 1\right)dx - \gamma_*(\psi)\right),$$

for any $X \in L^\infty(\Omega,\mathcal{F},P)$*, where*

$$\gamma_*(\psi) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int_{-\infty}^0 \psi\left(\cdot, P[X<x|\mathcal{G}]\right)dx + \int_0^{+\infty} \left(\psi\left(\cdot, P[X<x|\mathcal{G}]\right) - 1\right)dx - \rho(X)\right),$$

for any $\psi \in \mathrm{Conc}(\Omega,\mathcal{G},[0,1])$.

When ρ is coherent, the above formula simplifies to:

Corollary 3.3.15. *Let* $\rho : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{G},P)$ *be a conditionally coherent and continuous from above risk measure. Then,* ρ *is conditionally law-invariant if, and only if,*

$$\rho(X) = \operatorname*{esssup}_{\psi \in \mathcal{C}} \left(\int_{-\infty}^0 \psi\left(\cdot, P[X<x|\mathcal{G}]\right)dx + \int_0^{+\infty} \left(\psi\left(\cdot, P[X<x|\mathcal{G}]\right) - 1\right)dx\right), \ \text{for any } X \in L^\infty(\Omega,\mathcal{F},P),$$

where $\mathcal{C} \subset \mathrm{Conc}(\Omega,\mathcal{G},[0,1])$.

Moreover, Theorem 3.3.13 and Theorem 3.3.4 allows us to identify the set of probability measures $\mathcal{Q}$, where the esssup is being taken in Theorem 3.3.1 for the robust representation of ρ_μ.

Corollary 3.3.16. *Let* $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ *and* $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ *be its corresponding concave distortion as in Theorem 3.3.13. The conditional risk measure (3.1) admits a robust representation given by:*

$$\rho_\mu(X) = \operatorname*{esssup}_{Q \in \mathcal{Q}} E^Q[-X|\mathcal{G}], \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{Q}$ *is characterized by:*

$$\mathcal{Q} = \left\{ Q \in \mathcal{P}_{\mathcal{G}} : \int_t^1 Q_\tau \left[\frac{dQ}{dP} \Big| \mathcal{G} \right] d\tau \leqslant \psi(\cdot, 1-t) \text{ a.s., for any } t \in (0,1). \right\}$$

As an example, we employ the identification of the core of probability measures where the esssup is being evaluated to prove that our definition for $AV@R_\Lambda$, as in Proposition B.1.1 item 4, is equivalent to Detlefsen and Scandolo (2005)'s definition.

Example 3.3.17. *For any* $\Lambda \in L^\infty(\Omega, \mathcal{G}, P)$*, so that* $0 < \Lambda \leqslant 1$ *a.s., then:*

$$AV@R_\Lambda[X|\mathcal{G}] = \operatorname*{esssup}_{Q \in \mathcal{Q}} E^Q[-X|\mathcal{G}], \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where

$$\mathcal{Q} = \left\{ Q \in \mathcal{P}_{\mathcal{G}} : \frac{dQ}{dP} \leqslant \frac{1}{\Lambda} \text{ a.s.} \right\}.$$

For a detailed proof of this statement, see the Appendix B.

3.3.4 Transition Capacities and Conditional Comonotonicity.

In this section, we describe random capacities and relate them to convex conditional risk measurements. As it was shown in Kusuoka (2001), for the unconditional case, and Dela Vega and Elliott (2021), for conditional settings, we demonstrate that conditional risk measures of the form of Theorem 3.3.13 are the ones additive under conditional comonotonicity. Besides that, this family of risk measures are also represented as the Choquet's integral of random capacities, similar to Madan et al. (2017). As a consequence, we derive a further equivalent characterization for conditionally convex risk measures and their penalty functions in terms of Choquet's integrals of transition capacities, based on a novel disintegration theorem for capacities.

The definition of transition capacity is not standard. For the purpose of the subsequent results, we will define them to be the following random set functions.

Definition 3.3.18. *A map* $c : \Omega \times \mathcal{B}(\mathbb{R}^d) \to [0,1]$ *will be a transition capacity if it satisfies the following:*

1. *For any* $\omega \in \Omega$, $A \in \mathcal{B}(\mathbb{R}^d) \mapsto c(\omega, A)$ *is a monotone set function, i.e. for any* $A, B \in \mathcal{B}(\mathbb{R}^d)$, c *satisfies:*

   $$c(\omega, A) \leqslant c(\omega, B), \text{ if } A \subset B.$$

   Moreover, $c(\omega, \emptyset) = 0$ *and* $c(\omega, \mathbb{R}^d) = 1$.

2. *For any* $A \in \mathcal{B}(\mathbb{R}^d)$, $\omega \in \Omega \mapsto c(\omega, A)$ *is* $\mathcal{G}$*-measurable.*

If, additionally, it also satisfies that:

$$c(\omega, A \cup B) + c(\omega, A \cap B) \leqslant c(\omega, A) + c(\omega, B), \text{ for any } A, B \in \mathcal{B}\left(\mathbb{R}^d\right) \text{ and } \omega \in \Omega,$$

then c *is said to be a submodular transition capacity. The transition capacity is compactly supported if there exists a compact* $K \subset \mathbb{R}^d$*, so that* $c(\cdot, K^c) = 0$ *a.s.*

Observe that Definition 3.3.18 coincides with the definition of a conditional probability if we demand that the set function in $\mathcal{B}(\mathbb{R}^d)$ is always a probability measure. Beyond that, we will show that, under some additional structure, a capacity on $(\Omega, \mathcal{F})$ might be disintegrated conditionally to $\mathcal{G}$ for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $A \in \mathcal{B}(\mathbb{R})$. This disintegrated capacity will be a transition capacity as defined previously. Therefore, this object is the natural generalization for a transition set function satisfying the desirable properties of a Choquet capacity.

One useful additional regularity property is its continuity as a set function.

Definition 3.3.19. *A map* $c : \Omega \times \mathcal{B}(\mathbb{R}^d) \to [0, 1]$ *will be a transition capacity continuous from above provided the following holds:*

$$c(\omega, A) = \lim_{n \in \mathbb{N}} c(\omega, A_n),$$

for any sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{B}(\mathbb{R}^d)$*, so that* $A_n \downarrow A \in \mathcal{B}(\mathbb{R}^d)$*, and* $\omega \in \Omega$*. It will be considered continuous from below if:*

$$c(\omega, A) = \lim_{n \in \mathbb{N}} c(\omega, A_n),$$

for any sequence $(A_n)_{n \in \mathbb{N}} \subset \mathcal{B}(\mathbb{R})$*, so that* $A_n \uparrow A \in \mathcal{B}(\mathbb{R}^d)$*, and* $\omega \in \Omega$*. Finally, it is continuous if it is both lower and upper continuous.*

For any compactly supported and continuous transition capacity, $c : \Omega \times \mathcal{B}(\mathbb{R}) \to [0, 1]$, the Choquet's integral of any $f : \mathbb{R} \to \mathbb{R}$ continuous is defined as:

$$\int f(x) c(\omega, dx) = \int_{-\infty}^{0} \left(c\left(\omega, \{y : f(y) > x\}\right) - 1\right) dx + \int_{0}^{+\infty} c\left(\omega, \{y : f(y) > x\}\right) dx.$$

It is trivial to show that this object is in $L^\infty(\Omega, \mathcal{G}, P)$. Observe first that, since c is compactly supported, continuous, monotone and f is continuous, then $x \in \mathbb{R} \mapsto c\left(\omega, \{y : f(y) > x\}\right) \in [0, 1]$ is monotone, bounded, compactly supported, right-continuous with left-limits and Borel measurable, for any $\omega \in \Omega$, assuring the finiteness of the integral. Moreover, it is immediate to check that this integral will be bounded by $2 \sup_{x \in K} |f(x)|$. Finally, monotonicity, right-continuity with left-limits, and uniform boundedness of the stochastic process $\left(c\left(\omega, \{y : f(y) > x\}\right)\right)_{x \in \mathbb{R}}$ guarantee that the integral is a $\mathcal{G}$-measurable map. In particular, the Choquet's integral of $-x$ with respect to this transition capacity is given by:

$$\int (-x)\, c(\omega, dx) = \int_{0}^{+\infty} \left(c(\omega, (-\infty, x)) - 1\right) dx + \int_{-\infty}^{0} c(\omega, (-\infty, x))\, dx.$$

Our objective is to show a correspondence between conditional risk measures, with some regularity conditions, and a family of transition capacities. These transition capacities are designed to represent bounds for the conditional probabilities in Theorem 3.3.4. Differently from the unconditional case and in order to avoid working on nice metric spaces equipped with their induced Borel σ-algebra, we need to define a large family of capacities satisfying some regular conditions. The reason for using such large family is that we do not want to force stringent assumptions on the probability space that guarantee the existence of a regular conditional probability map, $P[\cdot|\mathcal{G}] : \Omega \times \mathcal{F} \to [0, 1]$. Consequently, we need a family of transition capacities indexed by all possible finite, bounded and $\mathcal{F}$-measurable random vectors satisfying:

Definition 3.3.20. *Let* $\mathrm{C}(\mathcal{G}) = \{c_X : \Omega \times \mathcal{B}(\mathbb{R}^d) \to [0,1]\}_{\{X \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d), d \in \mathbb{N}\}}$ *be a family of transition capacities.*

1. *(Consistency)* $\mathrm{C}(\mathcal{G})$ *is consistent, if for any Borel measurable function* $\phi : \mathbb{R}^d \to \mathbb{R}^k$ *and* $X \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d)$ *there exists a set* $\Omega_{X,\phi} \in \mathcal{G}$*, with probability one, such that:*

$$c_{\phi(X)}(\omega, A) = c_X\left(\omega, \phi^{-1}(A)\right), \text{ for any } \omega \in \Omega_{X,\phi} \text{ and } A \in \mathcal{B}(\mathbb{R}^k).$$

2. *(Conditional Law-Invariance)* $\mathrm{C}(\mathcal{G})$ *is conditionally law-invariant, if for any* $X \sim_{\mathcal{G}} Y$ *there exists a set* $\Omega_{X,Y} \in \mathcal{G}$*, with probability one, so that* $c_X(\omega, A) = c_Y(\omega, A)$*, for any* $\omega \in \Omega_{X,Y}$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$*.*

3. *(*$\|\cdot\|_\infty$*-compactly supported)* $\mathrm{C}(\mathcal{G})$ *is* $\|\cdot\|_\infty$*-compactly supported, if for any* $X = (X_1, \ldots, X_d) \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d)$*,* c_X *is compactly supported in* $K = \prod_{n=1}^d [-\|X_i\|_\infty, \|X_i\|_\infty]$*.*

4. *(Conditional Translation Invariance)* $\mathrm{C}(\mathcal{G})$ *is conditionally translational invariant, if for any* $X \in L^\infty(\Omega,\mathcal{F},P)$ *and* $Y \in L^\infty(\Omega,\mathcal{G},P)$ *there is a set* $\Omega_{X,Y} \in \mathcal{G}$*, with probability one, such that:*

$$c_{X+Y}(\omega, A) = c_X\left(\omega, A - Y(\omega)\right), \text{ for any } A \in \mathcal{B}(\mathbb{R}) \text{ and } \omega \in \Omega_{X,Y}.$$

5. *(Spectral Family)*[2] $\mathrm{C}(\mathcal{G})$ *is a spectral family if all the conditions above hold and, for any* $X \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d)$*,* $d \in \mathbb{N}$*, then* c_X *is a continuous and submodular transition capacity.*

Similarly to Kusuoka (2001), the additivity of a conditional risk measure along conditional comonotonic vectors is a necessary condition for it to be representable as a Choquet's integral of capacities in a spectral family. Thus, to prove this claim we recall the definition of conditional comonotonicity.

Definition 3.3.21. *Let* $\mathcal{G} \subset \mathcal{F}$ *be any sub-*σ*-algebra of* $\mathcal{F}$*, and* $X \in L^0(\Omega,\mathcal{F},P;\mathbb{R}^n)$*.* X *is a* $\mathcal{G}$*-comonotonic random vector, if* $\operatorname{supp} P[X \in \cdot\,|\mathcal{G}](\omega)$ *is comonotonic almost surely on* Ω*.*

We refer to Chapter 2 and the references therein for a discussion of equivalent definitions of conditional comonotonicity, as well as its consequence for conditional quantiles. The additive condition with respect to conditional comonotonic random vector is, hence, defined as:

Definition 3.3.22. *A conditional risk measure,* $\rho : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{G},P)$*, is conditionally comonotonic if for any* $(X,Y) \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^2)$ $\mathcal{G}$*-comonotonic random vector:*

$$\rho(X+Y) = \rho(X) + \rho(Y), \text{ a.s.}$$

In Dela Vega and Elliott (2021), the additive condition above is required to hold for comonotonic variables. This requirement alone is stronger than Definition 3.3.22, as every comonotonic random vector is also conditional comonotonic, while the opposite is not generally true. Nevertheless, if a conditional risk measure is conditionally law-invariant, continuous from above and regular, then additivity along comonotonic vectors implies additivity along conditional comonotonic vectors. Thus, Dela Vega and Elliott (2021) and our conditions are equivalent.

As claimed before and equipped with all the machinery described above, we can now demonstrate the connection between coherence, conditional law-invariance, upper continuity and additivity in conditional comonotonic variables with a spectral family of transition capacities.

[2] We adopt this terminology since this family of set functions will uniquely determine non-linear operators acting on $L^\infty(\Omega,\mathcal{F},P)$ and taking values on $L^\infty(\Omega,\mathcal{G},P)$. In some sense, this connection between set functions and representation of operators resembles the Spectral Theorem and its spectral measures for self-adjoint operators – see Reed and Simon (1972).

Theorem 3.3.23. *A conditional risk measure, $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$, is conditionally coherent, conditionally law-invariant, continuous from above and conditionally comonotonic if, and only if, there exists a spectral family of transition capacities, $C(\mathcal{G})$, such that:*

$$\rho(X) = \int (-x) c_X(\cdot, dx), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Moreover, in the representation of Theorem 3.3.4, $Q \in \mathcal{Q}$ if, and only if, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$, there exists $\Omega_X \in \mathcal{G}$, with $P[\Omega_X] = 1$, so that:

$$Q[X < x|\mathcal{G}](\omega) \leqslant c_X(\omega, (-\infty, x)) \text{ for any } x \in \mathbb{R},\ \omega \in \Omega_X.$$

The result above permits us to demonstrate that spectral families are intimately connected to deterministic capacities. To show this, we first define what is meant by disintegrating a capacity with respect to a reference measure P and conditioned to a σ-algebra $\mathcal{G}$.

Definition 3.3.24. *Let $c : (\Omega, \mathcal{F}) \to [0, 1]$ be a submodular and continuous capacity. We say that c admits a disintegration with respect to P conditioned to $\mathcal{G}$, if there exists a spectral family of transition capacities, $C(\mathcal{G})$, such that for any $d \in \mathbb{N}$, $X \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^d)$, $A \in \mathcal{G}$ and $B \in \mathcal{B}(\mathbb{R}^d)$:*

$$E[c_X(B)\mathbb{1}_A] = c(X \in B, A),$$

where $c_X \in C(\mathcal{G})$ is the transition capacity of X in the spectral family, as in Definition 3.3.18.

It is clear from the definition that, if a capacity c admits a disintegration with respect to P conditioned to $\mathcal{G}$, then each element of its associated spectral family is uniquely determined up to a $\mathcal{G}$-measurable negligible set. Moreover, we can explicitly describe all the capacities that admits a disintegration.

Proposition 3.3.25 (Disintegration of Capacities)**.** *Let $c : (\Omega, \mathcal{F}) \to [0, 1]$ be a submodular and continuous capacity. Then, c admits a disintegration with respect to P conditioned to $\mathcal{G}$ if, and only if, the following conditions hold.*

1. *For any $A \in \mathcal{G}$, then $c(A) = P[A]$.*
2. *For any $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$, such that $X \sim_{\mathcal{G}} Y$, and $B \in \mathcal{B}(\mathbb{R})$, then $c(X \in B) = c(Y \in B)$.*
3. *For any fixed $\tau \in (0, 1)$, the set function $A \in \mathcal{G} \mapsto c(A, U \leqslant \tau)$ is a measure absolutely continuous with respect to P, where $U \in L^\infty(\Omega, \mathcal{F}, P)$, so that $P[U \in \cdot|\mathcal{G}] = U(0, 1)$ a.s.*

We will denote the set of all capacities described in Proposition 3.3.25 as C. For any $c \in \mathrm{C}$, we let $C(\mathcal{G}, c)$ be its associated and unique disintegration, and $c_X \in C(\mathcal{G}, c)$ stands for the transition capacity of X in this spectral family. As a consequence, we obtain the following equivalent characterization for conditionally law-invariant and convex risk measures.

Theorem 3.3.26. *Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ be a convex and continuous from above risk measure. Then, ρ is conditionally law-invariant if, and only if, ρ admits the following representation:*

$$\rho(X) = \operatorname*{esssup}_{c \in \mathrm{C}} \left(\int (-x) c_X(dx) - \delta_*(c) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\delta_* : \mathrm{C} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}})$ satisfies:

$$\delta_*(c) = \operatorname*{esssup}_{X \in L^\infty(\Omega, \mathcal{F}, P)} \left(\int (-x) c_X(dx) - \rho(X) \right),$$

for any $c \in \mathrm{C}$.

3.4 Dynamic One-Step Conditionally Law-Invariant Risk Measures

The results obtained in Section 3.3 completely characterizes risk measures that are convex, continuous from above and conditionally law-invariant in a static framework. In this section, however, we focus ourselves on one-step law-invariant and iterative dynamic risk measurements, as defined by Cheridito and Kupper (2011) and Madan et al. (2017). First, in Section 3.4.3, we discuss the basic set of properties a dynamic risk measurement is expected to have. Then, in Section 3.4.2, we characterize the family of one-step law-invariant dynamic risk measurements in a finite-time setting. As a result, we show that the one-step construction allows one to avoid the time inconsistencies pointed out in Kupper and Schachermayer (2009), whilst maintaining some degree of law-invariance. We extend this representation to convex, coherent and one-step comonotonic risk measurements. Finally, the chapter ends in Section 3.4.3 by applying the normalization proposed in Stadje (2010) and our dynamic representation result to explicitly characterize the driver function of the continuous-time limiting risk process, which, as a consequence, is computed as a g-expectation.

3.4.1 Iterated Conditional Risk Measurements

As shown in the seminal article of Kupper and Schachermayer (2009), conditional law-invariance affects the dynamic consistency of risk measurements. Nevertheless, as we demonstrate in this section, it is possible to weaken the conditional law-invariance requirement in the dynamic setting, maintaining time-consistency. In order to achieve this, one can consider a finite-time setting, with one-step conditional law-invariance of the risk measurements, as in Elliott et al. (2015), and employing iterate (or composed) risk measures, as proposed in Cheridito and Kupper (2011). When adopting this approach, we can establish and describe a large family of suitable, time-consistent and relevant dynamic risk measures.

For this reason, we adopt the following convention in this section. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\Pi}, \mathsf{P})$ be a finite-time filtered probability space, where $\Pi = \{0, t_1, \dots, t_{n-1}, T\}$ and $\mathcal{F}_T = \mathcal{F}$. Besides, we will call a sequence of conditional risk measures, $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to L^\infty(\Omega, \mathcal{F}_t, \mathsf{P})$, by a dynamic risk measurement. We denote by (strong) time-consistent dynamic risk measurement the following.

Definition 3.4.1. *A dynamic risk measurement* $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to L^\infty(\Omega, \mathcal{F}_t, \mathsf{P})$, *is strongly time-consistent if:*

$$\rho_s\left(-\rho_t(X)\right) = \rho_s(X), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \text{ and } s \leqslant t \in \Pi.$$

Beyond time-consistency, we will also require the risk measurements to be one-step conditionally law-invariant.

Definition 3.4.2. *A dynamic risk measurement* $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to L^\infty(\Omega, \mathcal{F}_t, \mathsf{P})$, *is one-step conditionally law-invariant if, for any* $t = t_k \in \{0, t_1, \dots, t_{n-1}\}$ *and* $X, Y \in \mathcal{F}_{t_{k+1}}$, *such that* $X \sim_{\mathcal{F}_t} Y$, *then:*

$$\rho_t(X) = \rho_t(Y), \text{ a.s.}$$

Notice that Definition 3.4.2 does not require from the conditional risk measures to be law-invariant with respect to those random variables that are beyond the next-step of the economy. In the finite-time setting, this is precisely the weakest possible notion of conditional law-invariance. Moreover, one-step law-invariance differs from Weber (2006)'s definition of distribution law-invariance, as, instead of imposing distribution-invariance on the conditional law of the final position, we only require its invariance with respect the next-period value conditioned to the present.

As discussed in Madan et al. (2017), Cheridito and Kupper (2011), Ruszczynski and Shapiro (2006) and Hardy and Wirch (2004), one possible way to transform a sequence of risk measurements into a strongly time-consistent family is by considering their iteration. This leads to the following.

Definition 3.4.3. *A dynamic risk measurement $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_t,P)$, is said to be an iterate dynamic risk measurement if there exists a family $(\bar\rho_t)_{t\in\Pi}$, $\bar\rho_t : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_t,P)$, with $\bar\rho_T(X) = -X$, of conditional risk measures, such that for any $X \in L^\infty(\Omega,\mathcal{F},P)$:*

$$\rho_t(X) = \begin{cases} -X, & \text{if } t = T, \\ \bar\rho_t\left(-\rho_{t_{k+1}}(X)\right), & \text{if } t = t_k \in \Pi \text{ and } t < T. \end{cases}$$

It is trivial to show that every iterate dynamic risk measurement is strongly time-consistent. Moreover, the converse is obviously true, since time-consistency will allow us to set $\bar\rho_t = \rho_t$, for any $t \in \Pi$. Hence, as pointed out by Elliott et al. (2015), when considering a dynamic risk measurement as in Definition 3.4.3, we are, in fact, analyzing the set of all strongly time-consistent family of risk measures. The one-step risk measures associated to a discrete-time and time-consistent risk measurement, $(\bar\rho_{t\in\Pi})$, are also known as the generators – see Cheridito and Kupper (2011).

In order characterize a dynamic risk measurement as in previous sections, the remainder property to translate into this setting is comonotonicity. In this sense, we will adopt the following additivity condition.

Definition 3.4.4. *A dynamic risk measurement $(\rho_t)_{t\in\Pi}$ is dynamically conditionally comonotonic if for any $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ and $X, Y \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$, such that (X,Y) is $\mathcal{F}_t$-comonotonic, then:*

$$\rho_t(X+Y) = \rho_t(X) + \rho_t(Y), \text{ a.s.}$$

3.4.2 Representation of Dynamic One-Step Conditionally Law-Invariant Risk Measures

The concepts and assumptions described above are sufficient to characterize a large family of strongly time-consistent and one-step law-invariant dynamic risk measurements.

Theorem 3.4.5. *Let $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_t,P)$, be a strongly time-consistent dynamic risk measurement. Then, the following statements are equivalent.*

1. *$(\rho_t)_{t\in\Pi}$ is one-step conditionally law-invariant, with ρ_t convex (or coherent) and continuous from above, for any $t \in \Pi$.*

2. *For any $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ there exists a map $\alpha_*^t : \mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}} \to L^0(\Omega,\mathcal{F}_t,P;\bar{\mathbb{R}}_+)$ such that:*

$$\alpha_*^t(Q) = \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(\int_0^1 Q_\tau[\rho_{t_{k+1}}(X)|\mathcal{F}_t] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] d\tau - \rho_t(X) \right), \text{ a.s., for any } Q \in \mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}},$$

where $\mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}} = \left\{Q \in \mathcal{P}_{\mathcal{F}_t} : \frac{dQ}{dP} \in L^0(\Omega,\mathcal{F}_{t_{k+1}},P)\right\}$, and

$$\rho_t(X) = \operatorname*{esssup}_{Q\in\mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}}} \left(\int_0^1 Q_\tau[\rho_{t_{k+1}}(X)|\mathcal{F}_t] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] d\tau - \alpha_*^t(Q) \right), \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F},P).$$

3. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ *there exists a map* $\beta_*^t : \mathcal{M}_{(0,1]}^{\mathcal{F}_t} \to L^0(\Omega, \mathcal{F}_t, P; \bar{\mathbb{R}}_+)$ *such that:*

$$\beta_*^t(\mu) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int_0^1 AV@R_\tau[-\rho_{t_{k+1}}(X)|\mathcal{F}_t] d\mu(\tau) - \rho_t(X) \right), \text{ a.s., for any } \mu \in \mathcal{M}_{(0,1]}^{\mathcal{F}_t},$$

and,

$$\rho_t(X) = \operatorname*{esssup}_{\mu \in \mathcal{M}_{(0,1]}^{\mathcal{F}_t}} \left(\int_0^1 AV@R_\tau[-\rho_{t_{k+1}}(X)|\mathcal{F}_t] d\mu(\tau) - \beta_*^t(\mu) \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

4. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ *there exists a map* $\gamma_*^t : \mathrm{Conc}(\Omega, \mathcal{F}_t, [0,1]) \to L^0(\Omega, \mathcal{F}_t, P; \bar{\mathbb{R}}_+)$ *such that:*

$$\gamma_*^t(\psi) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int_{\mathbb{R}} \left(\psi\left(\cdot, P[-\rho_{t_{k+1}}(X) < x|\mathcal{F}_t]\right) - \mathbb{1}_{[0,+\infty)}(x) \right) dx - \rho_t(X) \right),$$

for any $\psi \in \mathrm{Conc}(\Omega, \mathcal{F}_t, [0,1])$*, and*

$$\rho_t(X) = \operatorname*{esssup}_{\psi \in \mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])} \left(\int_{\mathbb{R}} \left(\psi\left(\cdot, P[-\rho_{t_{k+1}}(X) < x|\mathcal{F}_t]\right) - \mathbb{1}_{[0,+\infty)}(x) \right) dx - \gamma_*^t(\psi) \right),$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$*.*

5. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ *there exists a map* $\delta_*^t : C^t \to L^0(\Omega, \mathcal{F}_t, P; \bar{\mathbb{R}}_+)$ *such that:*

$$\delta_*^t(c) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int x c_{\rho_{t_{k+1}}(X)}(dx) - \rho_t(X) \right), \text{ a.s., for any } c \in C^t,$$

and,

$$\rho_t(X) = \operatorname*{esssup}_{c \in C^t} \left(\int x c_{\rho_{t_{k+1}}(X)}(dx) - \delta_*^t(c) \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where C^t *is the set of submodular and continuous capacities disintegrable with respect to* P *conditioned to* $\mathcal{F}_t$*.*

Furthermore, if ρ_t *is coherent in item 1, for any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$*, then items 2, 3, and 4 are:*

2'. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$*, there exists* $\mathcal{P}^t \subset \mathcal{P}_{\mathcal{F}_t, \mathcal{F}_{t_{k+1}}}$*, such that:*

$$\rho_t(X) = \operatorname*{esssup}_{Q \in \mathcal{P}^t} \left(\int_0^1 Q_\tau[\rho_{t_{k+1}}(X)|\mathcal{F}_t] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{F}_t \right] d\tau \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

3'. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ *there exists* $\mathcal{M}^t \subset \mathcal{M}_{(0,1]}^{\mathcal{F}_t}$ *such that:*

$$\rho_t(X) = \operatorname*{esssup}_{\mu \in \mathcal{M}^t} \left(\int_0^1 AV@R_\tau[-\rho_{t_{k+1}}(X)|\mathcal{F}_t] d\mu(\tau) \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

4'. *For any* $t = t_k \in \{0, t_1, \ldots, t_{n-1}\}$ *there exists* $\mathcal{C}^t \subset \mathrm{Conc}(\Omega, \mathcal{F}_t, [0,1])$ *such that:*

$$\rho_t(X) = \operatorname*{esssup}_{\psi \in \mathcal{C}^t} \left(\int_{\mathbb{R}} \left(\psi\left(\cdot, P[-\rho_{t_{k+1}}(X) < x|\mathcal{F}_t]\right) - \mathbb{1}_{[0,+\infty)}(x) \right) dx \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Beyond characterizing all strongly time-consistent, one-step conditionally law-invariant, convex (or coherent) dynamic measurements, the above results allow us to explicit the subset of these dynamic risk measurements that are dynamically conditionally comonotonic, relating them to the existence of dominant transition capacities and random concave distortions

Theorem 3.4.6. *Let $(\rho_t)_{t\in\Pi}$, $\rho_t : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_t,P)$, be a strongly time-consistent dynamic risk measurement. Then, the following is equivalent.*

1. *$(\rho_t)_{t\in\Pi}$ is one-step conditionally law-invariant, dynamically conditionally comonotonic, coherent and continuous from above.*
2. *For any $t = t_k \in \{0, t_1, \dots, t_{n-1}\}$, there exists $\mu_t \in \mathcal{M}^{\mathcal{F}_t}_{(0,1]}$ such that:*

$$\rho_t(X) = \int_0^1 AV@R_\tau[\rho_{t_{k+1}}(X)|\mathcal{F}_t]\mu_t(\tau), \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F},P).$$

3. *For any $t = t_k \in \{0, t_1, \dots, t_{n-1}\}$ there exists $\psi_t \in \mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])$ such that, for any $X \in L^\infty(\Omega,\mathcal{F},P)$:*

$$\rho_t(X) = \int_{-\infty}^0 \left(\psi_t\left(\cdot, P[-\rho_{t_{k+1}}(X) < x|\mathcal{F}_t]\right) - 1\right) dx + \int_0^{+\infty} \psi_t\left(\cdot, P[-\rho_{t_{k+1}}(X) < x|\mathcal{F}_t]\right) dx, \text{ a.s.,}$$

4. *For any $t = t_k \in \{0, t_1, \dots, t_{n-1}\}$ there exists a spectral family of transitional capacities, $C_t(\mathcal{F}_t)$, such that:*

$$\rho_t(X) = \int x c_{\rho_{t_{k+1}}(X)}(\cdot, dx), \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F},P).$$

Furthermore, ρ_t admits the following robust representation:

$$\rho_t(X) = \operatorname*{esssup}_{Q\in\mathcal{Q}^t} E^Q[\rho_{t_{k+1}}(X)|\mathcal{F}_t], \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F},P).$$

such that $Q \in \mathcal{Q}^t$ if, and only if, $Q \in \mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}}$ and, for any $X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$, there exists $\Omega_X \in \mathcal{F}_t$, with $P[\Omega_X] = 1$, satisfying

$$Q_\tau[X < x|\mathcal{F}_t](\omega) \leqslant c_X(\omega,(-\infty,x)), \text{ for any } x \in \mathbb{R} \text{ and } \omega \in \Omega_X.$$

We conclude this subsection by analyzing the relevance (or sensitivity) of the previous risk measurements. This property is essential, for example, for super-hedging, as no-arbitrage is equivalent to relevance of the risk measurement – see Föllmer and Schied (2002). Recall also that Kupper and Schachermayer (2009) demonstrated that law-invariance and relevance of dynamic risk measurements lead to time-inconsistencies, except for entropic risk measures. Nevertheless, in the following proposition, we are able to reconcile relevance and law-invariance for a dynamic risk measurement by weakening the latter. Indeed, we show that, assuming the one-step law-invariance, then convex and continuous from above dynamic risk measurements are relevant. To prove this claim, we first recall the definition of relevance.

Definition 3.4.7. *A conditional risk measure, $\rho : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{G},P)$, is relevant (or sensitive) if, for any $\epsilon > 0$ and $A \in \mathcal{F}$, with $P[A] > 0$, then:*

$$P[\rho(-\epsilon\mathbb{1}_A) > 0] > 0.$$

A dynamic risk measurement $(\rho_t)_{t\in\Pi}$ is relevant if, for any $t \in \Pi$, ρ_t is relevant.

Similarly to Corollary 4.59 in Föllmer and Schied (2002), we conclude proving the aforementioned claim.

Proposition 3.4.8. *Let $(\rho_t)_{t\in\Pi}$ be an one-step law-invariant, convex and continuous from above dynamic risk measurement. Then $(\rho_t)_{t\in\Pi}$ is relevant.*

3.4.3 Additional Characterizations and Applications

Some applications of the above dynamic results will be discussed along this section. In our first example, we introduce the class of dynamic iterate utility-based risk measurements, showing that they form an one-step law-invariant, convex and continuous from above class of strongly time-consistent and relevant iterated risk measurements. Consequently, they are a concrete representative of those risk measures that admit a representation as in Theorem 3.4.5.

Example 3.4.9. *For any $k \in \{0, \dots, n-1\}$, let $u_{t_{k+1}} : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy:*

1. *For any $\omega \in \Omega$, then $u_{t_{k+1}}(\omega, \cdot) : \mathbb{R} \to \mathbb{R}$ is non-decreasing and concave.*
2. *For any $x \in \mathbb{R}$, $u_{t_{k+1}}(\cdot, x) \in L^0(\Omega, \mathcal{F}_{t_k}, P)$.*

For any $X \in L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P)$, we interpret $E[u_{t_{k+1}}(\cdot, X)|\mathcal{F}_{t_k}]$ as the discounted expected utility of X from time t_{k+1} at t_k.

We define the one-step acceptance set at time t_k as:

$$\mathcal{A}_{t_k,t_{k+1}} = \{X \in L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P) : E[u_{t_{k+1}}(\cdot, X)|\mathcal{F}_t] \geqslant u(\cdot, 0) \text{ a.s.}\}.$$

This set describes the financial positions of the next step of the market that are acceptable based on their discounted expected utility.

Setting $\rho_{t_n} = \rho_T : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}, P)$ as $\rho_T(X) = -X$, the dynamic iterate utility-based risk measurement associated to $(u_{t_k})_{k\in\{1,\dots,n\}}$ are the following recursively defined risk measures :

$$\begin{aligned}\rho_{t_k}(X) &= \operatorname{essinf}\left\{Y \in L^\infty(\Omega, \mathcal{F}_{t_k}, P) :\ -\rho_{t_{k+1}}(X+Y) \in \mathcal{A}_{t_k,t_{k+1}}\right\},\\ &= \operatorname{essinf}\left\{Y \in L^\infty(\Omega, \mathcal{F}_{t_k}, P) :\ E\left[u_{t_{k+1}}\left(\cdot, -\rho_{t_{k+1}}(X+Y)\right)|\mathcal{F}_{t_k}\right] \geqslant u(\cdot, 0) \text{ a.s.}\right\},\end{aligned}$$

for any $k \in \{0, \dots, n-1\}$.

To interpret the above identity, suppose $X \in L^\infty(\Omega, \mathcal{F}_{t_{k+2}}, P) \cap \left(L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P)\right)^c$. Then, in order to accept position X in time t_{k+1}, we need to add to it, at this time, an amount of $\rho_{t_{k+1}}(X)$. Now, for the decision process in time t_k, we no longer accept X, but the "artificial" required debt created at time t_{k+1} when accepting it, i.e. $-\rho_{t_{k+1}}(X)$. Thus, we need to identify the minimum $\mathcal{F}_{t_k}$-measurable amount that, when added to this debt, $-\rho_{t_{k+1}}(X)$, makes their discounted expected utility above the threshold value, $u(\cdot, 0)$. For $2 \leqslant j \leqslant n-k$, this reasoning can also be replicated inductively, for any position $X \in L^\infty(\Omega, \mathcal{F}_{t_{k+j}}, P) \cap \left(L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P)\right)^c$.

By its definition, the risk measurement above is a strongly time-consistent iterative risk measurement, as in Definitions 3.4.3 and 3.4.1. Moreover, due to Example 3.2.7, $(\rho_t)_{t\in\Pi}$ is a convex, continuous from above, one-step conditionally law-invariant risk measurement. Therefore, it is representable as in Theorem 3.4.5.

Notice also that the above class contains the entropic risk measures, by taking $u_{t_k}(x) = 1 - e^{-\gamma x}$, for $\gamma \in (0, +\infty)$ and $k \in \{1, \dots, n\}$. In this particular case, it is trivial to show that, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$:

$$E\left[u_{t_{k+1}}\left(\cdot, -\rho_{t_{k+1}}(X)\right)|\mathcal{F}_{t_k}\right] = E\left[u_{t_{k+1}}\left(\cdot, X\right)\right)|\mathcal{F}_{t_k}\right].$$

Hence, the decision process at time t_k *used to construct* $\rho_{t_k}(X)$ *is the same if considering either* $-\rho_{t_{k+1}}(X)$ *or* X.

Moreover, in the case of a deterministic utility function, $u : \mathbb{R} \to \mathbb{R}$*, as for the entropic risk measures, it is trivial to show that the penalty function will be described in terms of the Fenchel-Legendre transform of the associated loss function, such as in Lemma 3.3 from Weber (2006), i.e., for any* $t = t_k$ *and* $Q \in \mathcal{P}_{\mathcal{F}_{t_k},\mathcal{F}_{t_{k+1}}}$*:*

$$\alpha^t_*(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\left(u(0) + E\left[l^*\left(\lambda\frac{dQ}{dP}\right)\Big|\mathcal{F}_t\right]\right), \text{ a.s.}$$

As a further prospective application of the above results is the understanding of the asymptotic behavior of strongly-time consistent, one-step law-invariant, coherent, continuous from above and dynamically comonotonic risk measures, when the time interval of the composition shrinks to zero. Assuming these properties and following the steps in Stadje (2010) and Madan et al. (2017), we apply Theorem 3.4.6 to characterize explicitly the driver function g of the limiting g-expectation to where a scaled iterated risk measurement is converging.

If not properly re-scaled, Stadje (2010) demonstrates that the limit of an iterative coherent risk measurement of an L^2 random variable may diverge when the interval of composition approaches to zero. He also establishes the appropriated scaling that guarantees convergence of the iterative risk measurement to a g-expectation. On the other hand, Madan et al. (2017) employ this procedure to provide a limit theorem for dynamic spectral risk measures. Essentially, a dynamic spectral measure satisfies the conditions of Theorem 3.4.6, plus certainty on independent variables.

In the following proposition we use their scaling to characterize the limiting process of an iterate risk measurement. For this reason, we will assume that $T > 0$ and $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by a standard Brownian Motion, $(B_t)_{t\in[0,T]}$. For any $n \in \mathbb{N}$, let $\Pi^{(n)} = \{0 = t^n_0, \ldots, t^n_{k_n} = T\}$ be a partition of the interval $[0, T]$, such that $\Pi^{(n)} \subset \Pi^{(n+1)}$, $\Pi = \cup_{n\in\mathbb{N}}\Pi^{(n)}$ and $\lim_{n\in\mathbb{N}} |\Pi^{(n)}| = 0$, where $|\Pi^{(n)}| = \sup_{i\in\{0,\ldots,k_n-1\}}(t_{i+1} - t_i)$.

Proposition 3.4.10. *For any* $n \in \mathbb{N}$*, let* $\rho^n_{t^n_i,t^n_{i+1}} : L^\infty(\Omega, \mathcal{F}_{t^n_{i+1}}, P) \to L^\infty(\Omega, \mathcal{F}_{t^n_i}, P)$ *be an one-step conditionally law-invariant, coherent, continuous from above and dynamically conditionally comonotonic risk measurement. For every* $i \in \{0, \ldots, k_n - 1\}$*, let* $\psi_{t^n_i} \in \mathrm{Conc}(\Omega, \mathcal{F}_{t^n_i,[0,1]})$ *be its associated random concave distortion.*

Let the iterative and re-scaled risk measurement, $(\rho^n_t)_{t\in\Pi^{(n)}}$*, be defined as* $\rho_T(X) = -X$ *and:*

$$\rho^n_{t_i}(X) = E[\rho^n_{t_{i+1}}(X)|\mathcal{F}_{t_i}] + (t_{i+1} - t_i)\rho^n_{t_i,t_{i+1}}\left(-\frac{\rho^n_{t_{i+1}}(X) - E[\rho^n_{t_{i+1}}(X)|\mathcal{F}_{t_i}]}{\sqrt{(t_{i+1} - t_i)}}\right), \tag{3.2}$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and* $t_i \in \Pi^{(n)}$*,* $t_i \neq T$.

Suppose there exists an extension of the random concave distortion process to any $t \in [0, T]$*,* $\psi_t \in \mathrm{Conc}(\Omega, \mathcal{F}_t, P)$*, satisfying.*

1.

$$\sup_{t\in[0,T]}\left\|\int_0^1 \frac{\psi_t(\cdot,\tau)}{\tau\sqrt{\tau}}d\tau\right\|_\infty < M.$$

2. *For any* $t \in [0, T]$*:*

$$\lim_{s\to t}\sup_{\tau\in[0,1]}\frac{|\psi_t(\omega,\tau) - \psi_s(\omega,\tau)|}{\tau} = 0, \text{ for any } \omega \in \Omega.$$

Then, there exists a progressively measureable and finite stochastic process, $(\mu_t)_{t\in[0,T]}$*, such that for every* $X \in L^\infty(\Omega, \mathcal{F}, P)$*, the following continuous-time extension of* ρ^n*,*

$$\rho_t^n(X) = \sum_{i=1}^{k_n-1} \rho_{t_i^n}^n(X)\mathbb{1}_{[t_i^n, t_{i+1}^n)}(t), \tag{3.3}$$

converges to $\sup_{t\in[0,T]} E\left[|\rho_t^n(X) - \rho_t^g(X)|^2\right] \to 0$*, where* $\rho_t^g(X) = Y_t^g$ *solves*

$$\begin{cases} dY_t^g = -g(t, Z_t)dt + Z_t dB_t, \\ Y_T^g = -X, \end{cases} \tag{3.4}$$

with $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ *equals to* $g(\omega, t, z) = \mu_t(\omega)|z|$*. Moreover,* $(\mu_t)_{t\in[0,T]}$ *is given by:*

$$\mu_t(\omega) = \sqrt{2\pi}|z| \left(\int_0^{\frac{1}{2}} \psi_t(\omega, \tau) e^{\frac{Q_\tau[N(0,1)]^2}{2}} d\tau + \int_{\frac{1}{2}}^1 (\psi_t(\omega, \tau) - 1)\, e^{\frac{Q_\tau[N(0,1)]^2}{2}} d\tau \right).$$

For example, the proposition above allows us to describe the following continuous-time version of average value-at-risk with random paramerters.

Example 3.4.11. *Let* $(\lambda_t)_{t\in[0,1]}$ *be a continuous stochastic process adapted to* $(\mathcal{F}_t)_{t\in[0,1]}$ *and uniformly bounded by:*

$$0 < \lambda \leqslant \lambda_t(\omega) \leqslant 1, \text{ for any } \omega \in \Omega.$$

Define $(\psi_t)_{t\in[0,T]}$ *such that* $\psi_t \in \mathrm{Conc}(\Omega, \mathcal{F}_t, P)$*, for any* $t \in [0, T]$*, as the following concave distortion:*

$$\psi_t(\omega, \tau) = \frac{\tau \wedge \lambda_t}{\lambda_t}.$$

As we saw in Example 3.3.17, this concave distortion is associated with the risk measure $AV@R_{\lambda_t}[\cdot|\mathcal{F}_t] : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$*.*

Finally, let $(\Pi^{(n)})_{n\in\mathbb{N}}$ *be an increasing sequence of partitions of* $[0, T]$*, as in Example 3.4.10, and* $\rho_{t_i^n, t_{i+1}^n}^n : L^\infty(\Omega, \mathcal{F}_{t_{i+1}^n}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ *be the* $\lambda_{t_i^n}$*-average value at risk conditional to* $\mathcal{F}_{t_i^n}$

$$\rho_{t_i^n, t_{i+1}^n}^n(X) = AV@R_{\lambda_{t_i^n}}[X|\mathcal{F}_{t_i^n}], \text{ for any } X \in L^\infty(\Omega, \mathcal{F}_{t_{i+1}^n}, P).$$

Thus, the dynamic risk measurement defined by (3.2) and (3.3) satisfies:

$$\sup_{t\in[0,T]} E\left[|\rho_t^n(X) - \rho_t^g(X)|^2\right] \to 0, \text{ as } n \to +\infty.$$

where $\rho_t^g(X) = Y_t^g$*, for any* $t \in [0, T]$*, is the solution of:*

$$\begin{cases} dY_t^g = -g(t, Z_t)dt + Z_t dB_t, \\ Y_T^g = -X, \end{cases}$$

with $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$*, given by* $g(\omega, t, z) = \frac{|z|}{\sqrt{2\pi}\lambda_t} e^{-\frac{Q_{\lambda_t}[N(0,1)]^2}{2}}$*.*

3.5 Conclusion

The difficult task of identifying the dynamical behavior of the risk associated to a random outcome in time might be elucidated by applying the previous results and techniques, as they addressed and connected static, finite-time and continuous-time dynamic risk measurements. In this direction, a series of distinct representation theorems for conditional convex risk measures are obtained under the assumption of conditional law-invariance. We described convex risk measures and their respective penalty functions in terms of integrals of conditional quantiles, integrals of conditional average value-at-risk, random concave distortions and transition capacities. Subsequently, these representations were pushed forward to the dynamic and discrete-time setting, reconciling law-invariance, time-consistency and relevance of convex risk measures through iteration and by weaking law-invariance.

To exemplify the class of risk measures described in this thesis, we provided a large and intuitive class of convex and dynamic risk measurements based on utility functions. We showed that this class admits all the different representations derived along the previous sections, as well as one in terms of Fenchel-Legendre transforms.

The discrete-time characterization proposed might also shed a light into continuous-time risk measurements. For a specific class of conditional risk measures, we proved that it is possible to explicitly determine the associated continuous-time limiting dynamic risk measurement. This is achieved by controlling the behavior of the one-step risk measurement and taking advantage of its distinct characterizations obtained along this chapter. This procedure also generates a way of solving numerically a particular class of backward stochastic differential equations through the computation of iterative coherent risk measurements.

Appendix A

Appendix of Chapter 2

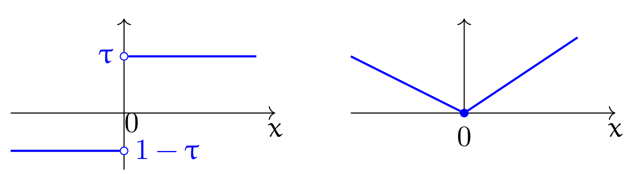
This is the appendix of Chapter 2. It contains a section devoted to examples examples, Section A.1 and the explicit proofs of each result stated in the aforementioned chapter. The present appendix also follows the organization and notation of the thesis, with exception to Subsection 2.2.2, as the proof of Proposition 2.2.5 depends on results derived in Proposition 2.2.9. We also enriched Section 2.6.2 with some results appropriated to the unconditional framework.

A.1 Examples

In this section, we offer some examples of our definitions of conditional quantiles. We begin by illustrating the check function $\rho_\tau : \mathbb{R} \to \mathbb{R}$, its derivative and integral: see Figure A.1.

Example A.1.1. *Let $X, Y, S \in L^0(\Omega, \mathcal{F}, P)$ be random variables such that X and Y are independent and uniformly distributed over the interval $[0,1]$ and $S = X + Y$. Set $\mathcal{G} = \sigma(Y)$. Then, the τ-conditional quantile random set of S conditioned to $\mathcal{G}$ is:*

$$\begin{aligned}
\Gamma_\tau[S|\mathcal{G}](\omega) &= \operatorname*{argmin}_{t\in\mathbb{R}} \int_{\mathbb{R}} \big(\rho_\tau(s-t) - \rho_\tau(s)\big) P[S \in ds|\mathcal{G}](\omega) \\
&= \operatorname*{argmin}_{t\in\mathbb{R}} \int_{[0,1]} \big[\rho_\tau(x + Y(\omega) - t) - \rho_\tau(x + Y(\omega))\big]\, dx \\
&= \operatorname*{argmin}_{t\in\mathbb{R}} \int_{[0,1]} \big[\rho_\tau(x + Y(\omega) - t)\big]\, dx.
\end{aligned}$$



(a) Graph of $x \mapsto \rho'_\tau(x)$ (b) Graph of $x \mapsto \rho_\tau(x)$

Figure A.1: Check function ρ_τ and its derivative.

Substituting $r = t - Y(\omega)$, *the above is*[1]

$$\begin{aligned}
\Gamma_\tau[S|\mathcal{G}](\omega) &= Y(\omega) + \operatorname*{argmin}_{r\in\mathbb{R}} \int_{[0,1]} \rho_\tau(x-r)dx \\
&= Y(\omega) + \operatorname*{argmin}_{r\in\mathbb{R}} \int_{[0,1]} \left[(\tau-1)(x-r)\mathbb{1}_{[x<r]} + \tau(x-r)\mathbb{1}_{[x\geqslant r]}\right]dx \\
&= Y(\omega) + \operatorname*{argmin}_{r\in\mathbb{R}} \left\{\int_{[0,r)} (\tau-1)(x-r)dx + \int_{[r,1]} \tau(x-r)dx\right\} \\
&= Y(\omega) + \operatorname*{argmin}_{r\in\mathbb{R}} \left\{(1-\tau)\frac{r^2}{2} + \tau\frac{(1-r)^2}{2}\right\}
\end{aligned}$$

Taking the derivative of the above objective function, we obtain the first order condition $(1-\tau)r - \tau(1-r) = r - \tau = 0$, *which implies* $r = \tau$. *Therefore,*

$$\Gamma_\tau[S|\mathcal{G}](\omega) = \{\tau + Y(\omega)\}.$$

Consequently,

$$Q_\tau[U|\mathcal{G}](\omega) = Q_{\tau+}[U|\mathcal{G}] = \tau + Y(\omega).$$

Example A.1.2. *Let* $U \in L^0(\Omega, \mathcal{F}, P)$ *be a random variable uniformly distributed over the interval* $[-1, 2]$, *and set* $\mathcal{G} = \sigma(U^2)$. *Then, the* τ*-conditional quantile random set of* U *conditioned to* $\mathcal{G}$ *is:*

$$\Gamma_\tau[U|\mathcal{G}] = \begin{cases} \{|U|\}, & \textit{if } |U| > 1 \textit{ or } |U| \leqslant 1 \textit{ and } \tau \in \left(\frac{1}{2}, 1\right) \\ \{-|U|\}, & \textit{if } |U| \leqslant 1 \textit{ and } \tau \in \left(0, \frac{1}{2}\right) \\ \left[-|U|, |U|\right], & \textit{if } |U| \leqslant 1 \textit{ and } \tau = \frac{1}{2} \end{cases}$$

Consequently, the left conditional quantiles is:

$$Q_\tau[U|\mathcal{G}] = \begin{cases} \{|U|\}, & \textit{if } |U| > 1 \textit{ or } |U| \leqslant 1 \textit{ and } \tau \in \left(\frac{1}{2}, 1\right) \\ \{-|U|\}, & \textit{if } |U| \leqslant 1 \textit{ and } \tau \in \left(0, \frac{1}{2}\right] \end{cases}$$

and the right conditional quantiles is:

$$Q_{\tau+}[U|\mathcal{G}] = \begin{cases} |U|, & \textit{if } |U| > 1 \textit{ or } |U| \leqslant 1 \textit{ and } \tau \in \left[\frac{1}{2}, 1\right) \\ -|U|, & \textit{if } |U| \leqslant 1 \textit{ and } \tau \in \left(0, \frac{1}{2}\right) \end{cases}$$

Notice that, as a stochastic process, $(Q_\tau[U|\mathcal{G}])_{\tau\in(0,1)}$ *is left-continuous with right-limits, whereas* $(Q_{\tau+}[U|\mathcal{G}])_{\tau\in(0,1)}$ *is right-continuous with left-limits. Moreover, since the conditional mean equals* $E[U|\mathcal{G}] = |U|\mathbb{1}_{[U>1]}$, *we stress that, when* $0 < |U| \leqslant 1$, *then both* $Q_\tau[U|\mathcal{G}] \neq E[U|\mathcal{G}]$ *and* $Q_{\tau+}[U|\mathcal{G}] \neq E[U|\mathcal{G}]$. *Finally, this extremely simple situation shows that the right and left conditional medians might not be the same,* $Q_{\frac{1}{2}}[U|\mathcal{G}] \neq Q_{\frac{1}{2}+}[U|\mathcal{G}]$.

Example A.1.3. *Let* $A \in \mathcal{G}$ *be such that* $P[A] \in (0, 1)$, $S : \Omega \to \{0, 1\}$, $S \sim \mathsf{Ber}\left(\frac{1}{2}\right)$ *be a Bernoulli random variable independent of* $\mathcal{G}$, $x_1 < x_2$, $y_1 < y_2$ *and* $x_2 - x_1 < y_2 - y_1$. *Set* $X, Y \in L^0(\Omega, \mathcal{F}, P)$ *as:*

$$X(\omega) = \begin{cases} x_1, & \textit{if } \omega \in A \cup \left(A^c \cap S^{-1}(0)\right) \\ x_2, & \textit{if } \omega \in A^c \cap S^{-1}(1) \end{cases} \quad \textit{and} \quad Y(\omega) = \begin{cases} y_1, & \textit{if } \omega \in A^c \cup \left(A \cap S^{-1}(0)\right) \\ y_2, & \textit{if } \omega \in A \cap S^{-1}(1) \end{cases}$$

[1]Here, we adopt the convention that for a number $a \in \mathbb{R}$ and a set $A \subset \mathbb{R}$, $a + A$ denotes the set $\{y \in \mathbb{R} : y = a + x$, for some $x \in A\}$.

Since $A \in \mathcal{G}$, conditional on $\mathcal{G}$, we know whether A or A^c. Then, the quantiles of X, Y conditioned to $\mathcal{G}$ are given by:

$$Q_\tau[X|\mathcal{G}](\omega) = \begin{cases} x_1, & \text{if } \omega \in A^c \text{ and } \tau \in \left(0, \frac{1}{2}\right], \text{ or } \omega \in A \\ x_2, & \text{if } \omega \in A^c \text{ and } \tau \in \left(\frac{1}{2}, 1\right) \end{cases}$$

and

$$Q_\tau[Y|\mathcal{G}](\omega) = \begin{cases} y_1, & \text{if } \omega \in A \text{ and } \tau \in \left(0, \frac{1}{2}\right], \text{ or } \omega \in A^c \\ y_2, & \text{if } \omega \in A \text{ and } \tau \in \left(\frac{1}{2}, 1\right) \end{cases}$$

Observe that

$$(X+Y)(\omega) = \begin{cases} x_1 + y_1, & \text{if } \omega \in S^{-1}(0) \\ x_2 + y_1, & \text{if } \omega \in A^c \cap S^{-1}(1) \\ x_1 + y_2, & \text{if } \omega \in A \cap S^{-1}(1) \end{cases} \text{ and } Q_\tau[X+Y|\mathcal{G}](\omega) = \begin{cases} x_1 + y_1, & \text{if } \tau \leqslant \frac{1}{2} \\ x_2 + y_1, & \text{if } \tau > \frac{1}{2} \text{ and } \omega \in A^c \\ x_1 + y_2, & \text{if } \tau > \frac{1}{2} \text{ and } \omega \in A \end{cases}$$

Therefore, for all $\tau \in (0,1)$, $Q_\tau[X+Y|\mathcal{G}] = Q_\tau[X|\mathcal{G}] + Q_\tau[Y|\mathcal{G}]$. This example will be further discussed in Section 2.3.1.

A.2 Proofs of Section 2.2

A.2.1 Proofs of Subsection 2.2.1

Proof of Proposition 2.2.2. Fix $\omega \in \Omega$, let $F_\omega : \mathbb{R} \to \mathbb{R}$ be $F_\omega(x) = P[X \leqslant x|\mathcal{G}](\omega)$. Then F_ω is a c.d.f. and, consequently, $y \in \Gamma_\tau[X|\mathcal{G}](\omega)$ if, and only if, $\lim_{x \uparrow y} F_\omega(x) \leqslant \tau \leqslant F_\omega(y)$ – see (Valadier, 2014, Theorem 1) for a explicit proof. Therefore, $\Gamma_\tau[X|\mathcal{G}](\omega)$ is a well-defined non-empty compact set, since F_ω is càd-làg, non-decreasing with $\lim_{x \downarrow -\infty} F_\omega(x) = 0$ and $\lim_{x \uparrow +\infty} F_\omega(x) = 1$.

Let $(U_n)_{n \in \mathbb{N} \cup \{0\}}$ be a basis of the topology in $\mathbb{R}$ composed by convex open sets such that $U_0 = \mathbb{R}$. Define, for each n:

$$m_n(\omega) = \inf_{y \in U_n} \left\{ \int_{\mathbb{R}} \big(\rho_\tau(x-y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega) \right\}.$$

Notice that $m_n : \Omega \to \mathbb{R}$ is well-defined, since it is the minimum of a continuous convex problem on a convex domain. Moreover, since U_n admits a countable dense subset D_n and the objective function is continuous with respect to y,

$$m_n(\omega) = \inf_{y \in D_n} \left\{ \int_{\mathbb{R}} \big(\rho_\tau(x-y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega) \right\}.$$

Fixed $y \in D_n$, $\omega \in \Omega \mapsto \int_{\mathbb{R}} \big(\rho_\tau(x-y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega)$ is $\mathcal{G}$-measurable, since $x \in \mathbb{R} \mapsto \rho_\tau(x-y) - \rho_\tau(x)$ is continuous and bounded. As a consequence, m_n is $\mathcal{G}$-measurable.

Given an open set V, from the assumptions we obtain $V = \cup_{U_n \subset V} U_n$. Therefore,

$$\begin{aligned} \{\omega \in \Omega : \Gamma_\tau[X|\mathcal{G}](\omega) \cap V \neq \emptyset\} &= \bigcup_{U_n \subset V} \{\omega \in \Omega : \Gamma_\tau[X|\mathcal{G}](\omega) \cap U_n \neq \emptyset\} \\ &= \bigcup_{U_n \subset V} \{\omega \in \Omega : m_n(\omega) = m_0(\omega)\}. \end{aligned}$$

Since $\{m_n = m_0\} \in \mathcal{G}$, we obtain the result. $\square$

Proof of Proposition 2.2.4. Consider the maps $\inf : (\mathcal{K}, \mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$ and $\sup : (\mathcal{K}, \mathcal{B}(\mathcal{K})) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$, which compute the infimum and supremum of any compact set of the real line. Then, it is trivial to show that these maps are measurable with respect to $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathbb{R})$. Consequently, since $Q_\tau[X|\mathcal{G}] = \inf \Gamma_\tau[X|\mathcal{G}]$ and $Q_{\tau+}[X|\mathcal{G}] = \sup \Gamma_\tau[X|\mathcal{G}]$ are composition of measurable maps and $\Gamma_\tau[X|\mathcal{G}](\omega)$ is not empty for all $\omega \in \Omega$, we obtain that they are measurable, i.e. $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau+}[X|\mathcal{G}] \in L^0(\Omega, \mathcal{G}, P)$. □

A.2.2 Proofs of Subsection 2.2.3

Proof of Theorem 2.2.6. 1. Fixed $\omega \in \Omega$, (Valadier, 2014, Theorem 1) demonstrates that $y \in \Gamma_\tau[X|\mathcal{G}](\omega)$ if, and only if, it satisfies:

$$P[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau \geqslant \lim_{x \uparrow y} P[X \leqslant x|\mathcal{G}](\omega)$$

Since $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau+}[X|\mathcal{G}] \in \Gamma_\tau[X|\mathcal{G}]$, we have that $P\left[X \leqslant Q_\tau[X|\mathcal{G}](\omega)|\mathcal{G}\right](\omega) \geqslant \tau$. Then, $Q_\tau[X|\mathcal{G}] \in \{Y \in L^0(\Omega, \mathcal{G}, P) : P[X \leqslant Y|\mathcal{G}] \geqslant \tau\}$.

Take $Y \in L^0(\Omega, \mathcal{G}, P)$ satisfying $P[X \leqslant Y(\omega)|\mathcal{G}](\omega) \geqslant \tau$, for all $\omega \in \Omega$. Fix $\omega \in \Omega$ and denote by $F_\omega(x) = P[X \leqslant x|\mathcal{G}](\omega)$, for all $x \in \mathbb{R}$. If $y < Q_\tau[X|\mathcal{G}](\omega)$, then $F_\omega(y) < \tau$, otherwise:

$$\lim_{x \uparrow y} F_\omega(x) \leqslant \lim_{x \uparrow Q_\tau[X|\mathcal{G}](\omega)} F_\omega(x) \leqslant \tau \leqslant F_\omega(y) \leqslant F_\omega(Q_\tau[X|\mathcal{G}](\omega)),$$

and $y \in \Gamma_\tau[X|\mathcal{G}](\omega)$, an absurd.

Thus, $F_\omega(Y(\omega)) = P[X \leqslant Y|\mathcal{G}](\omega) \geqslant \tau$ implies that $Q_\tau[X|\mathcal{G}](\omega) \leqslant Y(\omega)$. From the fact that $\omega \in \Omega$ was arbitrarily chosen we have that $Y \geqslant Q_\tau[X|\mathcal{G}]$, and the result is proved.

2. Let $D \subset \mathbb{R}$ be countable and dense. Given $y, y' \in \mathbb{R}$, for all $x \in \mathbb{R}$:

$$\Big|\big(\rho_\tau(x-y) - \rho_\tau(x)\big) - \big(\rho_\tau(x-y') - \rho_\tau(x)\big)\Big| \leqslant (\frac{1}{2} + |\tau - \frac{1}{2}|)|y - y'|.$$

Define $\Omega' = \bigcap_{y \in D}\{E[\rho_\tau(X-y) - \rho_\tau(X)|\mathcal{G}] = \int_{\mathbb{R}} \big(\rho_\tau(x-y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}]\}$. Therefore $\Omega' \in \mathcal{G}$ and $P[\Omega'] = 1$. We claim that if $\omega \in \Omega'$, then $E[\rho_\tau(X-y) - \rho_\tau(X)] = \int_{\mathbb{R}} \big(\rho_\tau(x - y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}]$ holds for all $y \in \mathbb{R}$. To see this, let $D = \{y_n : n \in \mathbb{N}\}$. Fix $\omega \in \Omega'$, take any $y \in \mathbb{R}$ and let $y_{n_k} \xrightarrow{k} y$. Then, since both sample paths are continuous:

$$\int_{\mathbb{R}} \Big(\rho_\tau(x - y_{n_k}) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega) \to \int_{\mathbb{R}} \Big(\rho_\tau(x-y) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega)$$
$$E\Big[\rho_\tau(X - y_{n_k}) - \rho_\tau(X)\Big|\mathcal{G}\Big](\omega) \to E\Big[\rho_\tau(X-y) - \rho_\tau(X)\Big|\mathcal{G}\Big](\omega).$$

However, realize that $\int_{\mathbb{R}} \big(\rho_\tau(x-y_{n_k}) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega) = E\Big[\rho_\tau(X-y_{n_k}) - \rho_\tau(X)\Big|\mathcal{G}\Big](\omega)$. Consequently, $E\Big[\rho_\tau(X-y) - \rho_\tau(X)\Big|\mathcal{G}\Big](\omega) = \int_{\mathbb{R}} \big(\rho_\tau(x - y) - \rho_\tau(x)\big) P[X \in dx|\mathcal{G}](\omega)$, for all $y \in \mathbb{R}$ and $\omega \in \Omega'$.

Hence, on Ω', the objective function in the optimization problem, equation (2.1), is the same as in item 2, and we obtain the result.

3. Fix $\omega \in \Omega$ and denote by $F_\omega(x) = P[X \leqslant x|\mathcal{G}](\omega)$. F_ω is a c.d.f and the solution of equation (2.2) at ω implies that $Q_\tau[X|\mathcal{G}](\omega) = Q_\tau[F_\omega]$. Since $Q_\tau[F_\omega] = \inf\{y \in \mathbb{R} : P[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau\}$, we have that $Q_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : P[X \leqslant y|\mathcal{G}](\omega) \geqslant \tau\}$. From the fact that $\omega \in \Omega$ was arbitrarily chosen, the result is proved.

$\square$

Proof of Proposition 2.2.7. Fix any $p \in [1, +\infty]$. Firstly, notice that for all $Y \in L^p(\Omega, \mathcal{G}, P)$:

$$\begin{aligned} E[\rho_\tau(X - Y) - \rho_\tau(X)|\mathcal{G}] &= \int_{\mathbb{R}} \Big(\rho_\tau(x - Y(\omega)) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega), \text{ a.s.} \\ &\geqslant \int_{\mathbb{R}} \Big(\rho_\tau(x - Q_\tau[X|\mathcal{G}](\omega)) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega) \\ &= \int_{\mathbb{R}} \Big(\rho_\tau(x - Q_{\tau+}[X|\mathcal{G}](\omega)) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega) \end{aligned}$$

Since $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau+}[X|\mathcal{G}] \in \Gamma_\tau[X|\mathcal{G}]$. Then,

$$E[\rho_\tau(X - Y) - \rho_\tau(X)] \geqslant E[\rho_\tau(X - Q_\tau[X|\mathcal{G}]) - \rho_\tau(X)]$$

Moreover, since $E[\rho_\tau(X - Y) - \rho_\tau(X)|\mathcal{G}] \geqslant \int_{\mathbb{R}} \Big(\rho_\tau(x - Q_\tau[X|\mathcal{G}](\omega)) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega)$ a.s. for all $Y \in L^p(\Omega, \mathcal{G}, P)$, we have that $E[\rho_\tau(X - Y) - \rho_\tau(X)] = E[\rho_\tau(X - Q_\tau[X|\mathcal{G}]) - \rho_\tau(X)]$ if, and only if, $E[\rho_\tau(X - Y) - \rho_\tau(X)|\mathcal{G}] = \int_{\mathbb{R}} \Big(\rho_\tau(x - Q_\tau[X|\mathcal{G}](\omega)) - \rho_\tau(x)\Big) P[X \in dx|\mathcal{G}](\omega)$ a.s. However, this implies that $Y \in \Gamma_\tau[X|\mathcal{G}]$ a.s. and, consequently, $Q_\tau[X|\mathcal{G}] \leqslant Y \leqslant Q_{\tau+}[X|\mathcal{G}]$ a.s. Hence,

$$\begin{aligned} Q_\tau[X|\mathcal{G}] &= \inf\{Y \in L^p(\Omega, \mathcal{G}, P) : Y \in \operatorname*{argmin}_{Z \in L^p(\Omega,\mathcal{G},P)} E[\rho_\tau(X - Z) - \rho_\tau(X)]\} \\ Q_{\tau+}[X|\mathcal{G}] &= \sup\{Y \in L^p(\Omega, \mathcal{G}, P) : Y \in \operatorname*{argmin}_{Z \in L^p(\Omega,\mathcal{G},P)} E[\rho_\tau(X - Z) - \rho_\tau(X)]\} \end{aligned}$$

Where the infimum and supremum are taken in the essential sense, as in Peskir and Shiryaev (2006). $\square$

Proof of Proposition 2.2.9.

1. Fix $\omega \in \Omega$ and let $F_\omega : \mathbb{R} \to \mathbb{R}$ be the function $F_\omega(x) = P[X \leqslant x|\mathcal{G}](\omega)$. F_ω is a c.d.f. and, by Theorem 2.2.6 item 3, $Q_\tau[X|\mathcal{G}](\omega) = \inf\{y \in \mathbb{R} : F_\omega(y) \geqslant \tau\}$, for all $\tau \in (0, 1)$. Therefore, since it is the quantile of a c.d.f, $\tau \in (0, 1) \mapsto Q_\tau[X|\mathcal{G}](\omega)$ is left-continuous with right-limits – see van der Vaart (1998).

   Moreover, its right-limit at τ is $\lim_{s\downarrow\tau} Q_s[X|\mathcal{G}](\omega) = z_\tau = \sup\{y \in \mathbb{R} : F_\omega(y) \leqslant \tau\}$. We claim that $Q_{\tau+}[X|\mathcal{G}](\omega) = z_\tau$. Notice first that $F_\omega(z_\tau) \geqslant \tau$, since F_ω is right-continuous. Furthermore, if $y < z_\tau$, then $F_\omega(y) \leqslant \tau$, otherwise $F_\omega(z_\tau) > \tau$, an absurd. Therefore, $\lim_{y\uparrow z_\tau} F_\omega(y) \leqslant \tau \leqslant F_\tau(z_\tau)$, and $z_\tau \in \Gamma_\tau[X|\mathcal{G}](\omega)$.

   If $z_\tau < Q_{\tau+}[X|\mathcal{G}](\omega)$, then there is a $1 > \tau' > \tau$, sufficiently close to τ, so that for a $y \in (z_\tau, Q_{\tau+}[X|\mathcal{G}](\omega))$ we would obtain $F_\omega(y) \geqslant \tau' > \tau$. Nevertheless, this would imply that $F_\omega(Q_{\tau+}[X|\mathcal{G}](\omega)) \geqslant \lim_{x\uparrow Q_{\tau+}[X|\mathcal{G}](\omega)} F_\omega(x) \geqslant \tau' > \tau$ and, consequently, $Q_{\tau+}[X|\mathcal{G}](\omega) \notin \Gamma_\tau[X|\mathcal{G}](\omega)$, an absurd.

   Hence, $z_\tau = Q_{\tau+}[X|\mathcal{G}](\omega)$ and the claim is proved.

2. We denote by suppX the set:

$$\text{supp}X = \Big(\bigcup_{\substack{B \text{ open} \\ P[X\in B]=0}} B\Big)^c.$$

Since $\mathbb{R}$ is separable, there exist a countable set $\{x_n\}_{n\in\mathbb{N}} \subset \text{supp}X$ and $\{\epsilon_n\}_{n\in\mathbb{N}} \in (0,\infty)$ so that:

$$\big(\text{supp}X\big)^c = \bigcup_{n\in\mathbb{N}} B(x_n;\epsilon_n),$$

where $B(x_n;\epsilon_n) = (x_n - \epsilon_n, x_n + \epsilon_n)$. We claim that for all $n \in \mathbb{N}$ there is a set $\Omega_n \subset \Omega$, with full measure, so that on Ω_n:

$$Q_\tau[X|\mathcal{G}](\omega) \notin B(x_n;\epsilon_n), \text{ for every } \tau \in (0,1) \text{ and } \omega \in \Omega_n.$$

Indeed, fixed $n \in \mathbb{N}$ realize that there is a set $\Omega_n \subset \Omega$, with full measure, so that $P[X \in B(x_n;\epsilon_n)|\mathcal{G}](\omega) = 0$, for every $\omega \in \Omega_n$, since $P[X \in B(x_n;\epsilon_n)] = 0$ and $P[X \in B(x_n;\epsilon_n)|\mathcal{G}] = E[\mathbb{1}_{X\in B(x_n;\epsilon_n)}|\mathcal{G}] = 0$ a.s. Fixed $\omega \in \Omega_n$, define $\tau_\omega = P[X \leqslant x_n|\mathcal{G}](\omega)$. If $\tau_\omega \in (0,1]$, let $\tau \leqslant \tau_\omega$, then $P[X \leqslant x_n - \epsilon_n|\mathcal{G}](\omega) = P[X \leqslant x_n|\mathcal{G}](\omega) = \tau_\omega$. Therefore, $Q_\tau[X|\mathcal{G}](\omega) \leqslant x_n - \epsilon$, by Theorem 2.2.6 item 3. On the other hand, if $\tau_\omega \in [0,1)$, then for all $\tau > \tau_\omega$ observe that $P[X < x_n + \epsilon|\mathcal{G}](\omega) = P[X \leqslant x_n|\mathcal{G}](\omega) = \tau_\omega$, which implies $Q_\tau[X|\mathcal{G}](\omega) \geqslant x_n + \epsilon_n$, and the claim is proved.

Now, take $\Omega' = \bigcap_{n\in\mathbb{N}} \Omega_n$, then Ω' has full measure and on this set:

$$Q_\tau[X|\mathcal{G}](\omega) \notin \bigcup_{n\in\mathbb{N}} B(x_n;\epsilon_n) = \big(\text{supp}X\big)^c, \text{ for every } \tau \in (0,1).$$

Thus, for every $\tau \in (0,1)$ $Q_\tau[X|\mathcal{G}] \in \text{supp}X$ a.s.

3. Suppose that $Y \geqslant X$ a.s. Then, for all $q \in \mathbb{Q}$, because $\mathbb{1}_{[Y\leqslant q]} \leqslant \mathbb{1}_{[X\leqslant q]}$ a.s., there is a set $\Omega_q \in \mathcal{G}$, with full probability measure, such that for $\omega \in \Omega_q$:

$$\begin{aligned} P[Y \leqslant q|\mathcal{G}](\omega) &= E[\mathbb{1}_{[Y\leqslant q]}|\mathcal{G}](\omega) \\ &\leqslant E[\mathbb{1}_{[X\leqslant q]}|\mathcal{G}](\omega) \\ &= P[X \leqslant q|\mathcal{G}](\omega) \end{aligned}$$

Defining $\Omega' = \cap_{q\in\mathbb{Q}}\Omega_q$, then $\Omega' \in \mathcal{G}$, with full measure, and $P[Y \leqslant x|\mathcal{G}](\omega) \leqslant P[X \leqslant x|\mathcal{G}](\omega)$, for all $x \in \mathbb{R}$ and $\omega \in \Omega'$, since it holds for all $x \in \mathbb{Q}$ and transition kernels are measures.

Thus, by Theorem 2.2.6 item 1 and Proposition 2.2.9 item 1, on Ω', we have that for all $\tau \in (0,1)$:

$$\begin{aligned} Q_\tau[X|\mathcal{G}](\omega) &= \inf\{x \in \mathbb{R} : P[X \leqslant x|\mathcal{G}](\omega) \geqslant \tau\} \\ &\leqslant \inf\{x \in \mathbb{R} : P[Y \leqslant x|\mathcal{G}](\omega) \geqslant \tau\} \\ &= Q_\tau[Y|\mathcal{G}](\omega), \end{aligned}$$

and

$$\begin{aligned} Q_{\tau+}[X|\mathcal{G}](\omega) &= \sup\{x \in \mathbb{R} : P[X \leqslant x|\mathcal{G}](\omega) \leqslant \tau\} \\ &\leqslant \sup\{x \in \mathbb{R} : P[Y \leqslant x|\mathcal{G}](\omega) \leqslant \tau\} \\ &= Q_{\tau+}[Y|\mathcal{G}](\omega). \end{aligned}$$

4. Suppose that Y is independent of $\mathcal{G}$. Using item 2 in Theorem 2.2.6, we obtain a.s.:

$$\begin{aligned} Q_\tau[Y|\mathcal{G}] &= \inf \operatorname*{argmin}_{y\in\mathbb{R}} E[\rho_\tau(Y-y)-\rho_\tau(Y)|\mathcal{G}] \\ &= \inf \operatorname*{argmin}_{y\in\mathbb{R}} E[\rho_\tau(Y-y)-\rho_\tau(Y)] \\ &= Q_\tau[Y]. \end{aligned}$$

5. If X is $\mathcal{G}$-measurable, then a.s.:

$$\begin{aligned} Q_\tau[X|\mathcal{G}] &= \inf \operatorname*{argmin}_{y\in\mathbb{R}} E[\rho_\tau(X-y)-\rho_\tau(X)|\mathcal{G}] \\ &= \inf \operatorname*{argmin}_{y\in\mathbb{R}} \big(\rho_\tau(X-y)-\rho_\tau(X)\big) \\ &= X. \end{aligned}$$

6. Let $\Omega' \subset \Omega$ be such that $P[\Omega'] = 1$, $E[g(X)|\mathcal{G}](\omega) = \int_{\mathbb{R}} g(x)P[X \in dx|\mathcal{G}](\omega)$ and $|E[g(X)|\mathcal{G}](\omega)| < +\infty$ for all $\omega \in \Omega'$. Fix $\omega \in \Omega'$, define the c.d.f. $F_\omega : \mathbb{R} \to \mathbb{R}$, as $F_\omega(x) = P[X \leqslant x|\mathcal{G}](\omega)$. Item 3 in Theorem 2.2.6 guarantees that $Q_\tau[X|\mathcal{G}](\omega)$ is its τ-quantile, for all $\tau \in (0,1)$. Moreover, we know that $E[g(X)|\mathcal{G}](\omega) = \int_{\mathbb{R}} g(x)dF_\omega(x)$, since $\omega \in \Omega'$. Therefore, Lemma A.19 on Föllmer and Schied (2002) implies $\int_{\mathbb{R}} g(x)dF_\omega(x) = \int_0^1 g(Q_\tau[F_\omega])d\tau$. In other words, we have that:

$$E[g(X)|\mathcal{G}](\omega) = \int_0^1 g(Q_\tau[X|\mathcal{G}](\omega))d\tau, \text{ a.s..}$$

□

A.2.3 Proof of Subsection 2.2.2

Proof of Proposition 2.2.5. 1. ($\Leftarrow$) Suppose that $Q_\tau[X|\mathcal{G}] \in L^p(\Omega, \mathcal{G}, P)$, for all $\tau \in (0,1)$, $s \mapsto E[|Q_s[X|\mathcal{G}]|^p]$ is left-continuous with right-limits and:

$$\int_0^1 E[|Q_\tau[X|\mathcal{G}]|^p]d\tau < +\infty.$$

Then, for each $n \in \mathbb{N}$ the function $g_n : \mathbb{R} \to \mathbb{R}$, defined by $g_n(x) = |x|^p \wedge n$, is in $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$. Hence, by item 6 of Proposition 2.2.9 we obtain:

$$\begin{aligned} E[|X|^p \wedge n] &=_0^1 E[|Q_\tau[X|\mathcal{G}]|^p \wedge n]d\tau \\ &\leqslant \int_0^1 E[|Q_\tau[X|\mathcal{G}]|^p]d\tau < +\infty. \end{aligned}$$

Since this holds for all $n \in \mathbb{N}$, we obtain $X \in L^p$.

($\Rightarrow$) If $X \in L^p$, item 6 of Proposition 2.2.9 assures that $E[|X|^p|\mathcal{G}] = \int_0^1 |Q_\tau[X|\mathcal{G}]|^p d\tau$ a.s. Integrating both sides and applying Fubini, we have that $\int_0^1 E[|Q_\tau[X|\mathcal{G}]|^p]d\tau < +\infty$. Therefore, a.s. on $(0,1)$ we have that $E[|Q_\tau[X|\mathcal{G}]|^p] < +\infty$.

Suppose, for the sake of contradiction, that $E[|Q_\tau[X|\mathcal{G}]|^p] = +\infty$ for some $\tau \in (0,1)$. Take any $0 < \tau' < \tau < \tau'' < 1$ so that $E[|Q_{\tau'}[X|\mathcal{G}]|^p]$ and $E[|Q_{\tau''}[X|\mathcal{G}]|^p]$ are finite. Notice that item 1 in Proposition 2.2.9 implies $Q_{\tau'}[X|\mathcal{G}] \leqslant Q_\tau[X|\mathcal{G}] \leqslant Q_{\tau''}[X|\mathcal{G}]$. Hence, $|Q_\tau[X]|^p \leqslant 2^p(|Q_{\tau'}[X|\mathcal{G}]|^p + |Q_{\tau''}[X|\mathcal{G}]|^p)$. Consequently, $E[|Q_\tau[X|\mathcal{G}]|^p] < +\infty$, an absurd.

If $\tau_n \uparrow \tau \in (0,1)$, then $Q_{\tau_n}[X|\mathcal{G}] \uparrow Q_\tau[X|\mathcal{G}]$. However, $|Q_{\tau_n}[X|\mathcal{G}]| \leqslant |Q_{\tau_1}[X|\mathcal{G}]| + |Q_\tau[X|\mathcal{G}]|$, and both $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau_1}[X|\mathcal{G}]$ belong to L^p. Therefore, dominated convergence theorem implies $E[|Q_{\tau_n}[X|\mathcal{G}]|^p] \uparrow E[|Q_\tau[X|\mathcal{G}]|^p]$.

If $\tau_n \downarrow \tau \in (0,1)$, then $Q_{\tau_n}[X|\mathcal{G}] \downarrow Q_{\tau+}[X|\mathcal{G}]$, which exists and is finite since $(Q_{\tau_n}[X|\mathcal{G}])_{n\in\mathbb{N}}$ is decreasing and bounded below by $Q_\tau[X|\mathcal{G}]$. Moreover,

$$Q_\tau[X|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}] \leqslant Q_{\tau_n}[X|\mathcal{G}] \leqslant Q_{\tau_1}[X|\mathcal{G}].$$

Since both $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau_1}[X|\mathcal{G}]$ are in L^p, dominated convergence theorem implies that $E[|Q_{\tau_n}[X|\mathcal{G}]|^p] \downarrow E[|Q_{\tau+}[X|\mathcal{G}]|^p]$.

2. ($\Rightarrow$) Suppose that $X \in L^\infty(\Omega, \mathcal{F}, P)$. Let $+\infty > C > 0$ be such that $-C \leqslant X \leqslant C$ a.s. Therefore, $-C \leqslant Q_\tau[X|\mathcal{G}] \leqslant C$ a.s., for all $\tau \in (0,1)$ due to item 3 in Proposition 2.2.9. Consequently, $Q_\tau[X|\mathcal{G}] \in L^\infty(\Omega, \mathcal{F}, P)$ for all $\tau \in (0,1)$, and:

$$\sup_{\tau\in(0,1)} \|Q_\tau[X|\mathcal{G}]\|_\infty \leqslant C.$$

($\Leftarrow$) Conversely, suppose that $Q_\tau[X|\mathcal{G}] \in L^\infty(\Omega, \mathcal{F}, P)$, for all $\tau \in (0,1)$, and:

$$\sup_{\tau\in(0,1)} \|Q_\tau[X|\mathcal{G}]\|_\infty \leqslant C < +\infty.$$

Then, for all $p \in [1, +\infty)$ and $n \in \mathbb{N}$, we obtain:

$$\begin{aligned} E[|X|^p \wedge n] &= E\Big[E\big[|X|^p \wedge n \big| \mathcal{G}\big]\Big] \\ &= E\left[\int_0^1 (|Q_\tau[X|\mathcal{G}]|^p \wedge n)\, d\tau\right] \\ &\leqslant E\left[\int_0^1 |Q_\tau[X|\mathcal{G}]|^p\, d\tau\right] \\ &\leqslant C^p. \end{aligned}$$

Therefore, we obtain $|E[|X|^p]|^{\frac{1}{p}} \leqslant C$, for all $p \in [1, +\infty)$. Consequently, $\|X\|_\infty \leqslant C < +\infty$, and we obtain the result.

3. Suppose that $X \in L^p$. If $\tau_n \uparrow \tau \in (0,1)$, then $Q_{\tau_n}[X|\mathcal{G}] \uparrow Q_\tau[X|\mathcal{G}]$. Furthermore, $|Q_{\tau_n}[X|\mathcal{G}]| \leqslant |Q_{\tau_1}[X|\mathcal{G}]| + |Q_\tau[X|\mathcal{G}]|$, and both $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau_1}[X|\mathcal{G}]$ belong to L^p. Therefore, dominated convergence theorem implies $Q_{\tau_n}[X|\mathcal{G}] \uparrow Q_\tau[X|\mathcal{G}]$ in L^p.

   If $\tau_n \downarrow \tau \in (0,1)$, then $Q_{\tau_n}[X|\mathcal{G}] \downarrow Q_{\tau+}[X|\mathcal{G}]$. Moreover,

$$Q_\tau[X|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}] \leqslant Q_{\tau_n}[X|\mathcal{G}] \leqslant Q_{\tau_1}[X|\mathcal{G}].$$

   Since both $Q_\tau[X|\mathcal{G}]$ and $Q_{\tau_1}[X|\mathcal{G}]$ are in L^p, dominated convergence theorem implies that $Q_{\tau_n}[X|\mathcal{G}] \downarrow Q_{\tau+}[X|\mathcal{G}]$ in L^p.

□

A.3 Proofs of Section 2.3

A.3.1 Proofs of Subsection 2.3.1

Proof of Lemma 2.3.3. This proof is essentially the same as in Jouini and Napp (2004), except that we modify the equations to hold a.s. instead of pointwise, as they do. We present the proof here for the sake of completeness.

1. $\Rightarrow$ 2. Let $\Omega' \subset \Omega$, with $\mathsf{P}[\Omega'] = 1$ and $\Omega' \in \mathcal{G}$, be such that supp $\mathsf{P}[X \in \cdot|\mathcal{G}](\omega)$ is comonotonic and $\mathsf{P}\big[X \in \mathbb{R}^{i-1} \times A \times \mathbb{R}^{n-i}\big|\mathcal{G}\big](\omega) = \mathsf{P}[X_i \in A|\mathcal{G}](\omega)$ for every $\omega \in \Omega'$, $A \in \mathcal{B}(\mathbb{R})$ and $i \in \{1, \dots, n\}$. Fix $\omega \in \Omega'$ and take any $x \in \mathbb{R}^n$. Define $A_i = \{y \in \text{supp } \mathsf{P}[X \in \cdot|\mathcal{G}](\omega) : y_i \leqslant x_i\}$ for each $i = 1, \dots, n$, then $A_i = \bigcap_{1 \leqslant j \leqslant n} A_j$ for some $i \in \{1, \dots, n\}$, due to the comonotonicity of supp $\mathsf{P}[X \in \cdot|\mathcal{G}](\omega)$. Thus,

$$\begin{aligned}
\mathsf{P}[X \leqslant x|\mathcal{G}](\omega) &= \mathsf{P}\left[X \in \bigcap_{1 \leqslant j \leqslant n} A_j \middle| \mathcal{G}\right](\omega) \\
&= \mathsf{P}[X \in A_i|\mathcal{G}](\omega) \\
&= \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega) \\
&= \min_{1 \leqslant i \leqslant n} \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega).
\end{aligned}$$

The last identity hold for all $x \in \mathbb{R}^n$ and $\omega \in \Omega'$, hence, 1 $\Rightarrow$ 2.

2. $\Rightarrow$ 3. Let $U \in L^0(\Omega, \mathcal{F}, \mathsf{P})$ be such that $U \sim U(0,1)$. Take $\Omega' \in \mathcal{G}$ such that $\mathsf{P}[\Omega'] = 1$ and $\mathsf{P}\big[X \in \mathbb{R}^{i-1} \times A \times \mathbb{R}^{n-i}\big|\mathcal{G}\big](\omega) = \mathsf{P}[X_i \in A|\mathcal{G}](\omega)$ for every $\omega \in \Omega$, $A \in \mathcal{B}(\mathbb{R})$, and $i \in \{1, \dots, n\}$. Moreover, assume that item 2 holds on Ω'.

   By Proposition 2.2.9 item 1, $(Q_\tau[X_i|\mathcal{G}])_{\tau \in (0,1)}$ càg-làd everywhere on Ω, and for each $i \in \{1, \dots, n\}$. Thus, fixed $\omega' \in \Omega'$, $(Q_U[X_1|\mathcal{G}](\omega'), \dots, Q_U[X_n|\mathcal{G}](\omega'))$ is a well-defined random vector, which is the composition of $\tau \in (0,1) \mapsto (Q_\tau[X_1|\mathcal{G}](\omega'), \dots, Q_\tau[X_n|\mathcal{G}](\omega'))$ and U. We also obtain that, for all $\omega \in \Omega'$ and $x \in \mathbb{R}^n$:

$$\begin{aligned}
&\big[\omega \in \Omega : \big(Q_{U(\omega)}[X_1|\mathcal{G}](\omega'), \dots, Q_{U(\omega)}[X_n|\mathcal{G}](\omega')\big) \leqslant x\big] = \\
&= \left[\omega \in \Omega : U(\omega) \leqslant \min_{i \in \{1,\dots,n\}} \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega')\right]
\end{aligned}$$

   Therefore, for all $x \in \mathbb{Q}^n$:

$$\begin{aligned}
\mathsf{P}[(Q_U[X_1|\mathcal{G}](\omega'), \dots, Q_U[X_n|\mathcal{G}](\omega')) \leqslant x] &= \mathsf{P}\left[U \leqslant \min_{i \in \{1,\dots,n\}} \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega')\right] \\
&= \min_{1 \leqslant i \leqslant n} \mathsf{P}[X_i \leqslant x_i|\mathcal{G}](\omega') \\
&= \mathsf{P}[X \leqslant x|\mathcal{G}](\omega')
\end{aligned}$$

   This holds for all $x \in \mathbb{R}^n$, and $\omega' \in \Omega'$. Consequently, since both sides are probability measures for each $\omega \in \Omega'$, we have that the above equality also holds for all $A \in \mathcal{B}(\mathbb{R}^n)$, and the proof is done.

3. $\Rightarrow$ 1. Let $\Omega' \in \mathcal{G}$, with $\mathsf{P}[\Omega'] = 1$, be such that, for all $\omega \in \Omega'$:

$$\mathsf{P}[(Q_U[X_1|\mathcal{G}](\omega), \dots, Q_U[X_n|\mathcal{G}](\omega)) \leqslant x] = \mathsf{P}[X \leqslant x|\mathcal{G}](\omega), \text{ for all } x \in \mathbb{R}^n.$$

Thus, supp $\mathsf{P}[X \in \cdot|\mathcal{G}](\omega) = $ supp $\mathsf{P}[(Q_U[X_1|\mathcal{G}](\omega), \ldots, Q_U[X_n|\mathcal{G}](\omega)) \in \cdot]$, for all $\omega \in \Omega'$. However,

$$\begin{aligned}&\text{supp } \mathsf{P}\left[(Q_U[X_1|\mathcal{G}](\omega), \ldots, Q_U[X_n|\mathcal{G}](\omega)) \in \cdot|\mathcal{G}\right] \\ &= \{(Q_\tau[X_1|\mathcal{G}](\omega), \ldots, Q_\tau[X_n|\mathcal{G}](\omega)) : \tau \in (0,1)\},\end{aligned}$$

which is a comonotonic set. Therefore, $3 \Rightarrow 1$.

□

Proof of Theorem 2.3.4. By item 3 in Lemma 2.3.3, there exists an uniform random variable U and a set $\Omega' \in \mathcal{G}$, with full measure, such that for every $\omega \in \Omega'$:

$$\begin{aligned}\mathsf{P}[Y \in A|\mathcal{G}](\omega) &= \mathsf{P}[(Q_U[Y_1|\mathcal{G}](\omega), \ldots, Q_U[Y_n|\mathcal{G}](\omega)) \in A] \\ \mathsf{P}[\psi(X,Y) \in A|\mathcal{G}](\omega) &= \mathsf{P}[Y \in \psi^{-1}(X(\omega), A)|\mathcal{G}](\omega), \text{ for all } A \in \mathcal{B}(\mathbb{R}^n).\end{aligned}$$

Thus, by item 3 in Theorem 2.2.6 and the above characterization we obtain that, for all $\omega \in \Omega'$:

$$\begin{aligned}Q_\tau[\psi(X,Y)|\mathcal{G}](\omega) &= \inf\{y \in \mathbb{R}|\mathsf{P}[\psi(X,Y) \leqslant y|\mathcal{G}](\omega) \geqslant \tau\} \\ &= \inf\left\{y \in \mathbb{R}|\mathsf{P}\left[Y \in \psi^{-1}\left(X(\omega), (-\infty, y]\right)\big|\mathcal{G}\right](\omega) \geqslant \tau\right\} \\ &= \inf\left\{y \in \mathbb{R}|\mathsf{P}\left[(Q_U[Y_1|\mathcal{G}](\omega), \ldots, Q_U[Y_n|\mathcal{G}](\omega)) \in \psi^{-1}\left(X(\omega), (-\infty, y]\right)\right] \geqslant \tau\right\} \\ &= \inf\{y \in \mathbb{R}|\mathsf{P}\left[\psi(X(\omega), Q_U[Y_1|\mathcal{G}](\omega), \ldots, Q_U[Y_n|\mathcal{G}](\omega)) \leqslant y\right] \geqslant \tau\} \\ &= Q_\tau[\phi_\omega(U)],\end{aligned}$$

where $\phi_\omega : (0,1) \to \mathbb{R}$ is the function $\phi_\omega(\tau) = \psi(X(\omega), Q_\tau[Y_1|\mathcal{G}](\omega), \ldots, Q_\tau[Y_n|\mathcal{G}](\omega))$. Observe that ϕ_ω is non-decreasing, left-continuous with right-limits by Proposition 2.2.9 item 1 and the hypothesis on ψ. Hence, we can apply Proposition 2.3.7 to obtain $Q_\tau[\phi_\omega(U)] = \phi_\omega(Q_\tau[U]) = \phi_\omega(\tau)$.

Therefore, we conclude that for all $\omega \in \Omega'$ we have

$$Q_\tau[\psi(X,Y)|\mathcal{G}](\omega) = \psi(X(\omega), Q_\tau[Y_1|\mathcal{G}](\omega), \ldots, Q_\tau[Y_n|\mathcal{G}](\omega))\ ,$$

and the result is proved. □

Proof of Corollary 2.3.5. Realize that X and $\psi_i : \mathcal{X}_i \subset \mathbb{R}^n \to \mathbb{R}$, for $i = 1, 2$, given by $\psi_1(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$ or $\psi_2(x_1, \ldots, x_n) = \prod_{i=1}^n x_i$, with $\mathcal{X}_1 = \mathbb{R}^n$ and $\mathcal{X}_2 = \mathbb{R}^n_+$, satisfy the conditions of Theorem 2.3.4 and, consequently, the result follows. □

Proof of Theorem 2.3.6. Fix $\tau \in (0,1)$, we demonstrated in the proof of Theorem 2.2.6 item 2 that there is $\Omega' \in \mathcal{G}$, $\mathsf{P}[\Omega'] = 1$, such that for all $\omega \in \Omega'$:

$$\begin{aligned}&\int \Big(\rho_\tau(a(\omega) + b(\omega)x - y) - \rho_\tau(a(\omega) + b(\omega)x)\Big)\mathsf{P}[X \in dx|\mathcal{G}](\omega) = \\ &= \mathsf{E}[\rho_\tau(a + bX - y) - \rho_\tau(a + bX)|\mathcal{G}](\omega),\end{aligned}$$

for all $y \in \mathbb{R}$. Additionally, for all $\omega \in \Omega'$:

$$\begin{aligned}Q_\tau[a + bX|\mathcal{G}](\omega) &= \inf \operatorname*{argmin}_{y \in \mathbb{R}} \mathsf{E}[\rho_\tau(a + bX - y) - \rho_\tau(a + bX)|\mathcal{G}](\omega) \\ &= \inf \operatorname*{argmin}_{y \in \mathbb{R}} \int_{\mathbb{R}} \Big(\rho_\tau(a(\omega) + b(\omega)x - y) - \rho_\tau(a(\omega) + b(\omega)x)\Big)\mathsf{P}[X \in dx|\mathcal{G}](\omega).\end{aligned}$$

Fix $\omega \in \Omega' \cap \{b \geqslant 0\} \cap \{|a| < +\infty\}$. Then, by using the positive homogeneity of ρ_τ, we have that:

$$
\begin{aligned}
Q_\tau[a+bX|\mathcal{G}](\omega) &= \inf \operatorname*{argmin}_{y\in\mathbb{R}} \int_{\mathbb{R}} \big(\rho_\tau(a(\omega)+b(\omega)x-y) - \rho_\tau(a(\omega)+b(\omega)x)\big) P[X\in dx|\mathcal{G}](\omega) \\
&= \inf \operatorname*{argmin}_{a(\omega)+b(\omega)y'\in\mathbb{R}} \left\{ \int_{\mathbb{R}} \Bigg(\rho_\tau\big(b(\omega)(x-y')\big) - \rho_\tau\big(b(\omega)x\big) \right. \\
&\left. + \rho_\tau\big(b(\omega)x\big) - \rho_\tau\big(a(\omega)+b(\omega)x\big) \Bigg) P[X\in dx|\mathcal{G}](\omega) \right\} \\
&= a(\omega) + b(\omega) \inf \operatorname*{argmin}_{y'\in\mathbb{R}} \left\{ \int_{\mathbb{R}} \big(\rho_\tau(b(\omega)(x-y')) - \rho_\tau(b(\omega)x)\big) P[X\in dx|\mathcal{G}](\omega) \right. \\
&\left. + \int_{\mathbb{R}} \Big(\rho_\tau\big(b(\omega)x\big) - \rho_\tau\big(a(\omega)+b(\omega)x\big)\Big) P[X\in dx|\mathcal{G}](\omega) \right\} \\
&= a(\omega) + b(\omega) \inf \operatorname*{argmin}_{y'\in\mathbb{R}} \left\{ \int_{\mathbb{R}} \Big(\rho_\tau\big(b(\omega)(x-y')\big) - \rho_\tau\big(b(\omega)x\big)\Big) P[X\in dx|\mathcal{G}](\omega) \right\} \\
&= a(\omega) + b(\omega) \inf \operatorname*{argmin}_{y'\in\mathbb{R}} \left\{ b(\omega) \int_{\mathbb{R}} \Big(\rho_\tau(x-y') - \rho_\tau(x)\Big) P[X\in dx|\mathcal{G}](\omega) \right\} \\
&= a(\omega) + b(\omega) Q_\tau[X|\mathcal{G}](\omega).
\end{aligned}
$$

Now, if $\omega \in \Omega' \cap \{b < 0\} \cap \{|a| < +\infty\}$, just realize that $\rho_\tau(b(\omega)x) = -b(\omega)\rho_{1-\tau}(x)$, for all $x \in \mathbb{R}$. Repeating the same computation and taking this observation into account, we obtain $Q_\tau[a+bX|\mathcal{G}](\omega) = a(\omega) + b(\omega)Q_{(1-\tau)+}[X|\mathcal{G}](\omega)$. Thus, the result is proved.

□

A.3.2 Proofs of Subsection 2.3.2

Proof of Proposition 2.3.7. Fix $\tau \in (0,1)$ and $x \in \mathbb{R}$ such that $y \in \mathbb{R} \mapsto g(x,y)$ is non-decreasing and left-continuous. Then, the following hold a.s.:

$$P[g(x,Y) \in A|\mathcal{G}] = P[Y \in g^{-1}(x,\cdot)(A)|\mathcal{G}], \text{ for all } A \in \mathcal{B}(\mathbb{R}),$$

where $g^{-1}(x,\cdot)(A) = \{y \in \mathbb{R} | g(x,y) \in A\}$. Moreover, we also know that, by Theorem 2.2.6 item 3,

$$
\begin{aligned}
Q_\tau[g(x,Y)|\mathcal{G}] &= \inf\{z \in \mathbb{R} : P[g(x,Y) \leqslant z|\mathcal{G}] \geqslant \tau\} \\
Q_\tau[Y|\mathcal{G}] &= \inf\{y \in \mathbb{R} : P[Y \leqslant y|\mathcal{G}] \geqslant \tau\}.
\end{aligned}
$$

Denoting by $\Omega' = \{\omega \in \Omega : g(x, Q_\tau[Y|\mathcal{G}](\omega)) < Q_\tau[g(x,Y)|\mathcal{G}](\omega)\}$ and $\Omega'' = \{\omega \in \Omega : g(x, Q_\tau[Y|\mathcal{G}](\omega)) > Q_\tau[g(x,Y)|\mathcal{G}](\omega)\}$, we claim that both Ω' and Ω'' have zero probability. Indeed, suppose that $P[\Omega'] > 0$. It is immediate to see that $\Omega' = \cup_{q\in\mathbb{Q}}\{g(x, Q_\tau[Y|\mathcal{G}]) < q < Q_\tau[g(x,Y)|\mathcal{G}]\} = \cup_{q\in\mathbb{Q}} A_q$. Thus, the last assumption implies $P[A_q] > 0$ for some $q \in \mathbb{Q}$. Notice also that:

$$P[g(x,Y) \leqslant q|\mathcal{G}]\mathbb{1}_{A_q} < \tau \mathbb{1}_{A_q} \text{ a.s.}$$

On the other hand, since $g(x,\cdot)$ is non-decreasing:

$$\begin{aligned}
\tau\mathbb{1}_{A_q} \leqslant P\big[Y \leqslant Q_\tau[Y|\mathcal{G}]\big|\mathcal{G}\big]\mathbb{1}_{A_q} &= E\big[\mathbb{1}_{\left[Y\leqslant Q_\tau[Y|\mathcal{G}]\right]}\big|\mathcal{G}\big]\mathbb{1}_{A_q} \\
&\leqslant E\big[\mathbb{1}_{\left[g(x,Y)\leqslant g(x,Q_\tau[Y|\mathcal{G}])\right]}\mathbb{1}_{A_q}\big|\mathcal{G}\big] \\
&\leqslant E[\mathbb{1}_{[g(x,Y)\leqslant q]}\mathbb{1}_{A_q}|\mathcal{G}] \\
&= P[g(x,Y)\leqslant q|\mathcal{G}]\mathbb{1}_{A_q} \\
&< \tau\mathbb{1}_{A_q} \text{ a.s.,}
\end{aligned}$$

which is an absurd. Hence, $P[\Omega'] = 0$.

Conversely, assume that $P[\Omega''] > 0$. Observe again that $\Omega'' = \cup_{q\in\mathbb{Q}}\{g(x, Q_\tau[Y|\mathcal{G}]) > q > Q_\tau[g(x,Y)|\mathcal{G}]\} = \cup_{q\in\mathbb{Q}}B_q$ and, at least for some $q \in \mathbb{Q}$, we have $P[B_q] > 0$. Therefore,

$$P[g(x,Y)\leqslant q|\mathcal{G}]\mathbb{1}_{B_q} \geqslant \tau\mathbb{1}_{B_q} \text{ a.s.}$$

Denoting by $\beta_q = \sup\{y \in \mathbb{R} : g(x,y) \leqslant q\}$, we observe that $\beta_q\mathbb{1}_{B_q} < Q_\tau[Y|\mathcal{G}]\mathbb{1}_{B_q}$, since $g(x,\cdot)$ is non-decreasing and left-continuous. Furthermore, $g(x,\beta_q) \leqslant q$ and a.s.:

$$\begin{aligned}
P[Y\leqslant \beta_q|\mathcal{G}]\mathbb{1}_{B_q} &= E[\mathbb{1}_{[Y\leqslant\beta_q]}|\mathcal{G}]\mathbb{1}_{B_q} \\
&= E[\mathbb{1}_{[g(x,Y)\leqslant q]}|\mathcal{G}]\mathbb{1}_{B_q} \\
&\geqslant \tau\mathbb{1}_{B_q}\text{a.s.}
\end{aligned}$$

Consequently, $Q_\tau[Y|\mathcal{G}]\mathbb{1}_{B_q} > \beta_q\mathbb{1}_{B_q} \geqslant Q_\tau[Y|\mathcal{G}]\mathbb{1}_{B_q}$ a.s., which is an absurd. Thus, $P[\Omega''] = 0$ and:

$$Q_\tau[g(x,Y)|\mathcal{G}] = g\left(x, Q_\tau[Y|\mathcal{G}]\right)\text{.a.s.} \tag{A.1}$$

Observe also that fixed $\tau \in (0,1)$ and taking $\tau_n \downarrow \tau$, then, by Proposition 2.2.9 item 1 and left-continuity of $g(x,\cdot)$, we obtain the following:

$$\begin{gathered}
Q_{\tau_n}[Y|\mathcal{G}] \downarrow Q_{\tau+}[Y|\mathcal{G}] \\
Q_{\tau_n}[g(x,Y)|\mathcal{G}] \downarrow Q_{\tau+}[g(x,Y)|\mathcal{G}] \\
g(x, Q_{\tau_n}[Y|\mathcal{G}]) \downarrow g(x, Q_{\tau+}[X|\mathcal{G}]).
\end{gathered}$$

However, because $Q_{\tau_n}[g(x,Y)|\mathcal{G}] = g(x, Q_{\tau_n}[Y|\mathcal{G}])$ a.s., we conclude that:

$$Q_{\tau+}[g(x,Y)|\mathcal{G}] = g(x, Q_{\tau+}[Y|\mathcal{G}]),\ \text{a.s.}$$

If $y \in \mathbb{R} \mapsto g(x,y)$ is non-increasing, then define $h = -g$, apply the previous result to obtain $Q_{\tau+}[h(x,Y)|\mathcal{G}] = h(x, Q_{\tau+}[Y|\mathcal{G}])$. Then, use Theorem 2.3.6 to conclude.

□

Proof of Proposition 2.3.8. Define the function $\beta : \mathbb{R}\times\mathbb{R} \mapsto \bar{\mathbb{R}}$ by:

$$\beta(x,z) = \begin{cases} \sup\{y\in\mathbb{R} : g(x,y)\leqslant z\}, & \text{if } \{y\in\mathbb{R} : g(x,y)\leqslant z\} \neq \emptyset \\ -\infty, & \text{otherwise,} \end{cases}$$

which is well-defined by the first assumption on g. Furthermore, it is trivial to show that:

1. For all $z \in \mathbb{R}$, then $\beta(\cdot, z)$ is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\bar{\mathbb{R}}))$-measurable.

2. For all $x \in \mathbb{R}$, then $\beta(x, \cdot)$ is non-decreasing and left-continuous.

Fixed any $z \in \mathbb{Q}$ arbitrary, then $\beta(X, z)$ is a well-defined $\mathcal{G}$-measurable map, not necessarily finite a.s. Besides, $[Y \leqslant \beta(X, z)] = [g(X, Y) \leqslant z]$. Thus, there is a set $\Omega_z \in \mathcal{G}$, with $P[\Omega_z] = 1$, so that:

$$P[Y \leqslant \beta(X(\omega), z)|\mathcal{G}](\omega) = P[g(X, Y) \leqslant z|\mathcal{G}](\omega), \text{ for all } \omega \in \Omega_z.$$

Let $\Omega' = \cap_{z \in \mathbb{Q}} \Omega_z$, then $P[\Omega'] = 1$ and:

$$P[Y \leqslant \beta(X(\omega), z)|\mathcal{G}](\omega) = P[g(X, Y) \leqslant z|\mathcal{G}](\omega), \text{ for all } \omega \in \Omega' \text{ and } z \in \mathbb{Q}$$

Moreover, if $z \in \mathbb{R}$ is arbitrary, take $z_n \downarrow z$, $(z_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$, then $\beta(X(\omega), z_n) \downarrow \beta(X(\omega), z)$ for all $\omega \in \Omega'$, by left-continuity. Hence, using the fact that $P[Y \leqslant \cdot|\mathcal{G}]$ and $P[g(X, Y) \leqslant \cdot|\mathcal{G}]$ are right-continuous for all $\omega \in \Omega'$, we have that:

$$P[Y \leqslant \beta(X(\omega), z)|\mathcal{G}](\omega) = P[g(X, Y) \leqslant z|\mathcal{G}](\omega), \text{ for all } \omega \in \Omega' \text{ and } z \in \mathbb{R}$$

Using the characterization obtained in Theorem 2.2.6 item 3, and Proposition 2.2.9 item 1, it is possible to show that, on Ω',

$$\begin{aligned}
Q_\tau[g(X, Y)|\mathcal{G}](\omega) &= \inf\{z \in \mathbb{R}|P[g(X, Y) \leqslant z|\mathcal{G}](\omega) \geqslant \tau\} \\
&= \inf\{z \in \mathbb{R}|P[Y \leqslant \beta(X(\omega), z)|\mathcal{G}](\omega) \geqslant \tau\} \\
&= \inf\{z \in \mathbb{R}|\beta(X(\omega), z) \geqslant Q_\tau[Y|\mathcal{G}](\omega)\} \\
&= \inf\{z \in \mathbb{R}|z \geqslant g\left(X(\omega), Q_\tau[Y|\mathcal{G}](\omega)\right)\} \\
&= g\left(X(\omega), Q_\tau[Y|\mathcal{G}](\omega)\right),
\end{aligned}$$

and

$$\begin{aligned}
Q_{\tau+}[g(X, Y)|\mathcal{G}](\omega) &= \sup\{z \in \mathbb{R}|P[g(X, Y) \leqslant z|\mathcal{G}](\omega) \leqslant \tau\} \\
&= \sup\{z \in \mathbb{R}|P[Y \leqslant \beta(X(\omega), z)|\mathcal{G}](\omega) \leqslant \tau\} \\
&= \sup\{z \in \mathbb{R}|\beta(X(\omega), z) \leqslant Q_{\tau+}[Y|\mathcal{G}](\omega)\} \\
&= \sup\{z \in \mathbb{R}|z < g(X(\omega), Q_\tau[Y|\mathcal{G}](\omega))\} \\
&= g(X(\omega), Q_{\tau+}[Y|\mathcal{G}](\omega)).
\end{aligned}$$

Now if in the first assumption of the proposition we assume that $y \in \mathbb{R} \mapsto g(x, y)$ is non-increasing for all $x \in \mathbb{R}$, then we may define $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $h(x, y) = -g(x, y)$, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Consequently, we can apply the above result to $h(X, Y)$, obtaining a.s. that:

$$\begin{aligned}
Q_\tau[h(X, Y)|\mathcal{G}] &= h(X, Q_\tau[Y|\mathcal{G}]) \\
Q_{\tau+}[h(X, Y)|\mathcal{G}] &= h(X, Q_{\tau+}[Y|\mathcal{G}])
\end{aligned}$$

Now, due to Theorem 2.3.6, we can exchange the minus sign on $Q_\tau[\cdot|\mathcal{G}]$ and $Q_{\tau+}[\cdot|\mathcal{G}]$ in the following way:

$$\begin{aligned}
Q_\tau[g(X, Y)|\mathcal{G}] &= Q_\tau[-h(X, Y)|\mathcal{G}] \\
&= -Q_{(1-\tau)+}[h(X, Y)|\mathcal{G}] \\
&= -h(X, Q_{(1-\tau)+}[Y|\mathcal{G}]) \\
&= g(X, Q_{(1-\tau)+}[Y|\mathcal{G}]), \text{ a.s.,}
\end{aligned}$$

and

$$\begin{aligned} Q_{\tau+}[g(X,Y)|\mathcal{G}] &= Q_{\tau+}[-h(X,Y)|\mathcal{G}] \\ &= -Q_{(1-\tau)}[h(X,Y)|\mathcal{G}] \\ &= -h(X, Q_{(1-\tau)}[Y|\mathcal{G}]) \\ &= g(X, Q_{(1-\tau)}[Y|\mathcal{G}]), \text{ a.s.,} \end{aligned}$$

concluding the argument. Note that the same set Ω' holds the equalities for all $\tau \in (0,1)$ □

A.4 Proofs of Subsection 2.3.3

Proof of Theorem 2.3.9. 1. Take u concave and let $u'_+ : \mathbb{R} \to \mathbb{R}$ be the right derivative of u, which exists and it is well-defined since u is concave. Then, for all $x, x' \in \mathbb{R}$:

$$u(x') \leqslant u(x) + u'_+(x)(x'-x).$$

Therefore,

$$u(X) \leqslant u(Q_\tau[X|\mathcal{G}]) + u'_+\big(Q_\tau[X|\mathcal{G}]\big)(X - Q_\tau[X|\mathcal{G}]).$$

The function $x \mapsto u'_+(x)$ is right-continuous, non-increasing and, consequently, $\mathcal{B}(\mathbb{R})$-measurable. Thus, $u'_+(Q_\tau[X|\mathcal{G}]))$ is $\mathcal{G}$-measurable. Taking the τ-conditional quantile of $u(X)$, using the monotonicity of quantiles and Theorem 2.3.6:

$$\begin{aligned} Q_\tau[u(X)|\mathcal{G}] &\leqslant Q_\tau\Big[u(Q_\tau[X|\mathcal{G}]) + u'_+(Q_\tau[X|\mathcal{G}])(X - Q_\tau[X|\mathcal{G}])\Big|\mathcal{G}\Big] \\ &= \begin{cases} u(Q_\tau[X|\mathcal{G}]), & \text{a.s. if } u'_+(Q_\tau[X|\mathcal{G}]) \geqslant 0 \\ u(Q_\tau[X|\mathcal{G}]) + u'_+(Q_\tau[X|\mathcal{G}])(Q_{(1-\tau)+}[X|\mathcal{G}] - Q_\tau[X|\mathcal{G}]), & \text{a.s. if } u'_+(Q_\tau[X|\mathcal{G}]) < 0. \end{cases} \end{aligned}$$

Notice that, since $\tau \in (0, \frac{1}{2}]$, then $Q_\tau[X|\mathcal{G}] \leqslant Q_{(1-\tau)+}[X|\mathcal{G}]$, therefore, $u'_+\big(Q_\tau[X|\mathcal{G}]\big)(Q_{(1-\tau)+}[X|\mathcal{G}] - Q_\tau[X|\mathcal{G}]) \leqslant 0$, when $u'_+\big(Q_\tau[X|\mathcal{G}]\big) < 0$. Thus, we obtain $Q_\tau[u(x)|\mathcal{G}] \leqslant u\big(Q_\tau[x|\mathcal{G}]\big)$ a.s.

2. Now suppose that u is convex and let $u'_+ : \mathbb{R} \to \mathbb{R}$ be the right derivative of u. Then, for all $x, x' \in \mathbb{R}$:

$$u(x') \geqslant u(x) + u'_+(x)(x'-x).$$

Therefore,

$$u(X) \geqslant u(Q_\tau[X|\mathcal{G}]) + u'_+\big(Q_\tau[X|\mathcal{G}]\big)(X - Q_\tau[X|\mathcal{G}]).$$

The function $x \mapsto u'_+(x)$ is right-continuous, non-decreasing and, consequently, $\mathcal{B}(\mathbb{R})$-measurable. Thus, $u'_+(Q_\tau[X|\mathcal{G}]))$ is $\mathcal{G}$-measurable. Taking the τ-conditional quantile of $u(X)$, using the monotonicity of quantiles and Theorem 2.3.6:

$$\begin{aligned} Q_\tau[u(X)|\mathcal{G}] &\geqslant Q_\tau\Big[u(Q_\tau[X|\mathcal{G}]) + u'_+(Q_\tau[X|\mathcal{G}])(X - Q_\tau[X|\mathcal{G}])\Big|\mathcal{G}\Big] \\ &= \begin{cases} u(Q_\tau[X|\mathcal{G}]), & \text{a.s. if } u'_+(Q_\tau[X|\mathcal{G}]) \geqslant 0 \\ u(Q_\tau[X|\mathcal{G}]) + u'_+(Q_\tau[X|\mathcal{G}])(Q_{(1-\tau)+}[X|\mathcal{G}] - Q_\tau[X|\mathcal{G}]), & \text{a.s. if } u'_+(Q_\tau[X|\mathcal{G}]) < 0. \end{cases} \end{aligned}$$

Since $\tau \in (\frac{1}{2}, 1)$, then $Q_\tau[X|\mathcal{G}] \geqslant Q_{(1-\tau)+}[X|\mathcal{G}]$, therefore, $u'_+\big(Q_\tau[X|\mathcal{G}]\big)(Q_{1-\tau}[X|\mathcal{G}] - Q_\tau[X|\mathcal{G}]) \geqslant 0$, when $u'_+\big(Q_\tau[X|\mathcal{G}]\big) < 0$. Thus, we obtain $u\big(Q_\tau[x|\mathcal{G}]\big) \leqslant Q_\tau[u(X)|\mathcal{G}]$ a.s.

3. If u is convex and $Q_{\frac{1}{2}}[X|\mathcal{G}] = Q_{\frac{1}{2}+}[X|\mathcal{G}]$ a.s., then, by the computations made previously in item 2, $Q_{\frac{1}{2}}[u(X)] \geqslant u(Q_{\frac{1}{2}}[X|\mathcal{G}])$ a.s.

   If $Q_{\frac{1}{2}}[X|\mathcal{G}] \neq Q_{\frac{1}{2}+}[X|\mathcal{G}]$ on a set $\Omega' \in \mathcal{G}$ so that $P[\Omega'] > 0$, then take $u(x) = -x$. This function is convex and, by Theorem 2.3.6, $Q_{\frac{1}{2}}[u(X)|\mathcal{G}] = -Q_{\frac{1}{2}+}[X|\mathcal{G}]$. However, $-Q_{\frac{1}{2}+}[X|\mathcal{G}] < -Q_{\frac{1}{2}}[X|\mathcal{G}] = u(Q_{\frac{1}{2}}[X|\mathcal{G}])$ on Ω', forcing $u(Q_{\frac{1}{2}}[X|\mathcal{G}]) > Q_{\frac{1}{2}}[u(X)|\mathcal{G}]$ with positive probability.

□

Proof of Corollary 2.3.10. By items 1,2 and 3, for all pair (x, y) and (x, y') in $\mathbb{R} \times \mathbb{R}$, we have that $u(x, y) \leqslant u(x, y') + u'_{2,+}(x, y')(y - y')$ and $\omega \in \Omega \mapsto u(X(\omega), Q_\tau[Y|\mathcal{G}](\omega))$ is $\mathcal{G}$-measurable. Furthermore, for all $\omega \in \Omega$:

$$u(X(\omega), Y(\omega)) \leqslant u(X(\omega), Q_\tau[Y|\mathcal{G}](\omega)) + u'_{2,+}(X(\omega), Q_\tau[Y|\mathcal{G}](\omega))(Y(\omega) - Q_\tau[Y|\mathcal{G}](\omega))$$

Taking the $Q_\tau[\cdot|\mathcal{G}]$ on both sides, using monotonicity of conditional quantiles and the invariance properties, we obtain:

$$\begin{aligned} Q_\tau[u(X,Y)|\mathcal{G}] &\leqslant Q_\tau\Big[u(X, Q_\tau[Y|\mathcal{G}]) + u'_+(X, Q_\tau[Y|\mathcal{G}])(Y - Q_\tau[Y|\mathcal{G}])\Big|\mathcal{G}\Big], \text{ a.s.} \\ &= \begin{cases} u(X, Q_\tau[Y|\mathcal{G}]), & \text{a.s. if } u'_+(X, Q_\tau[Y|\mathcal{G}]) \geqslant 0 \\ u(X, Q_\tau[Y|\mathcal{G}]) + u'_+(X, Q_\tau[Y|\mathcal{G}])(Q_{(1-\tau)+}[Y|\mathcal{G}] - Q_\tau[Y|\mathcal{G}]), & \text{a.s. if } u'_+(X, Q_\tau[Y|\mathcal{G}]) < 0. \end{cases} \end{aligned}$$

Now, since $\tau \in (0, \frac{1}{2}]$ and $Q_{(1-\tau)+}[Y|\mathcal{G}] \geqslant Q_\tau[Y|\mathcal{G}]$ a.s., we have that $Q_\tau[u(X,Y)|\mathcal{G}] \leqslant u(X, Q_\tau[Y|\mathcal{G}])$ a.s.

If for all $x \in \mathbb{R}$, $y \in \mathbb{R} \mapsto u(x, y)$ is convex, then the argument stays the same except that we change the directions of the inequalities. Indeed, convexity implies that for all pair (x, y) and $(x, y') \in \mathbb{R} \times \mathbb{R}$, then $u(x, y) \geqslant u(x, y') + u'_{2,+}(x, y')(y - y')$. Thus,

$$\begin{aligned} Q_\tau[u(X,Y)|\mathcal{G}] &\geqslant Q_\tau\Big[u(X, Q_\tau[Y|\mathcal{G}]) + u'_+(X, Q_\tau[Y|\mathcal{G}])(Y - Q_\tau[Y|\mathcal{G}])\Big|\mathcal{G}\Big], \text{ a.s.} \\ &= \begin{cases} u(X, Q_\tau[Y|\mathcal{G}]), & \text{a.s. if } u'_+(X, Q_\tau[Y|\mathcal{G}]) \geqslant 0 \\ u(X, Q_\tau[Y|\mathcal{G}]) + u'_+(X, Q_\tau[Y|\mathcal{G}])(Q_{(1-\tau)+}[Y|\mathcal{G}] - Q_\tau[Y|\mathcal{G}]), & \text{a.s. if } u'_+(X, Q_\tau[Y|\mathcal{G}]) < 0. \end{cases} \end{aligned}$$

If $\tau \in (\frac{1}{2}, 1)$, then $Q_\tau[Y|\mathcal{G}] \geqslant Q_{(1-\tau)+}[Y|\mathcal{G}]$ a.s. and we obtain the desired result. Moreover, if $\tau = \frac{1}{2}$ and $Q_{\frac{1}{2}+}[Y|\mathcal{G}] = Q_{\frac{1}{2}}[Y|\mathcal{G}]$ a.s., then $u(X, Q_{\frac{1}{2}}[Y|\mathcal{G}]) \leqslant Q_{\frac{1}{2}}[u(X,Y)|\mathcal{G}]$ a.s. □

A.5 Proofs of Section 2.4

A.5.1 Proofs of Subsection 2.4.1

Proof of Theorem 2.4.1. Denote by $Y_n = \inf_{m \geqslant n} X_m$, which is in $L^0(\Omega, \mathcal{F}, P)$ since $\inf_{n \in \mathbb{N}} X_n \in L^0(\Omega, \mathcal{F}, P)$. Moreover, $Y_n \uparrow \liminf_{n \in \mathbb{N}} X_n \in L^0(\Omega, \mathcal{F}, P)$, and both $Q_\tau[Y_n|\mathcal{G}] \leqslant \inf_{m \geqslant n} Q_\tau[X_m|\mathcal{G}]$ and $Q_{\tau+}[Y_n|\mathcal{G}] \leqslant \inf_{m \geqslant n} Q_{\tau+}[X_m|\mathcal{G}]$ a.s., by Proposition 2.2.9 item 3. In particular, $\lim_{n \in \mathbb{N}} Q_\tau[Y_n|\mathcal{G}] \leqslant \liminf_{m \in \mathbb{N}} Q_\tau[X_m|\mathcal{G}]$ a.s.

Let $\Omega' = \{\omega \in \Omega : Q_\tau[\liminf_{m \in \mathbb{N}} X_m|\mathcal{G}](\omega) > \lim_{n \in \mathbb{N}} Q_\tau[Y_n|\mathcal{G}](\omega)\}$. It is easy to see that $\Omega' = \cup_{q \in \mathbb{Q}} \{Q_\tau[\liminf_{m \in \mathbb{N}} X_m|\mathcal{G}](\omega) > q > \lim_{n \in \mathbb{N}} Q_\tau[Y_n|\mathcal{G}](\omega)\} = \cup_{q \in \mathbb{Q}} A_q$. For the sake of contradiction, assume that $P[\Omega'] > 0$. Thus, there is at least one $q \in \mathbb{Q}$ such that $P[A_q] > 0$. Fixed such $q \in \mathbb{Q}$, then $\mathbb{1}_{[Y_n \leqslant q]} \downarrow \mathbb{1}_{[\liminf X_m \leqslant q]}$. Moreover, $E[\mathbb{1}_{[Y_n \leqslant q]}|\mathcal{G}] \downarrow E[\mathbb{1}_{[\liminf X_m \leqslant q]}|\mathcal{G}]$ a.s., by the dominated convergence theorem for conditional expectation. We also know that a.s. on A_q:

$$P[Y_n \leqslant q|\mathcal{G}] = E[\mathbb{1}_{[Y_n \leqslant q]}|\mathcal{G}] \downarrow E[\mathbb{1}_{[\liminf_{m \in \mathbb{N}} X_m \leqslant q]}|\mathcal{G}] = P[\liminf X_m \leqslant q|\mathcal{G}].$$

However, the following also holds a.s.:

$$\begin{aligned}\mathsf{E}\big[\mathbb{1}_{[Y_n\leqslant q]}\big|\mathcal{G}\big]\mathbb{1}_{A_q} &= \mathsf{E}\big[\mathbb{1}_{[Y_n\leqslant q]}\mathbb{1}_{A_q}\big|\mathcal{G}\big]\\ &\geqslant \mathsf{E}\big[\mathbb{1}_{\left[Y_n\leqslant Q_\tau[Y_n|\mathcal{G}]\right]}\mathbb{1}_{A_q}\big|\mathcal{G}\big]\\ &= \mathsf{E}\big[\mathbb{1}_{\left[Y_n\leqslant Q_\tau[Y_n|\mathcal{G}]\right]}\big|\mathcal{G}\big]\mathbb{1}_{A_q}\\ &= \mathsf{P}\big[Y_n \leqslant Q_\tau[Y_n|\mathcal{G}]\big|\mathcal{G}\big]\mathbb{1}_{A_q}\\ &\geqslant \tau\mathbb{1}_{A_q}.\end{aligned}$$

Consequently, $\mathsf{P}[\liminf_{m\in\mathbb{N}} X_m \leqslant q|\mathcal{G}]\mathbb{1}_{A_q} \geqslant \tau\mathbb{1}_{A_q}$ a.s. and $Q_\tau[\liminf_{m\in\mathbb{N}} fX_m|\mathcal{G}] \leqslant q$ a.s. on A_q, an absurd. Therefore, $\mathsf{P}[\Omega'] = 0$ and we have that:

$$Q_\tau[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \text{ a.s.}$$

To prove the other inequality, $\limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n|\mathcal{G}] \leqslant Q_{\tau+}[\limsup_{n\in\mathbb{N}}|\mathcal{G}]$, just apply the already proved inequality to $Z_n = -X_n$ at $1-\tau$, using Theorem 2.3.6. □

Proof of Corollary 2.4.3. If $\tau \in (0,1)$ is such that $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$, then, for all sequence $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega,\mathcal{F},\mathsf{P})$ converging to X, we obtain by Fatou's lemma for conditional quantiles, Theorem 2.4.1, that:

$$Q_\tau[X|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}] = Q_\tau[X|\mathcal{G}], \text{ a.s.}$$

Hence, there exists $\lim_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]$ and it equals $Q_\tau[X|\mathcal{G}]$ a.s. □

A.5.2 Proofs of Subsection 2.4.2

In order to prove the proposition, we will need the following lemmas.

Lemma A.5.1. *Let* $X, Y \in L^1(\Omega,\mathcal{F},\mathsf{P})$, *and* $u:\mathbb{R}\to\mathbb{R}$ *convex, such that* $\mathsf{E}[|u(X-Y)|] < +\infty$. *Then,*

$$\mathsf{E}[u(X-Y)|\mathcal{G}] \geqslant \int_0^1 u(Q_\tau[X|\mathcal{G}] - Q_\tau[Y|\mathcal{G}])d\tau, \text{ a.s.},$$

with equality a.s. if (X,Y) *is* $\mathcal{G}$*-comonotonic.*

Lemma A.5.1 is just a conditional version of Theorem 8.1 in Major (1978).

Proof of Lemma 4.1. Let $\Omega' \in \mathcal{G}$ be such that $\mathsf{P}[\Omega'] = 1$, and for all $\omega \in \Omega'$:

$$\begin{aligned}\mathsf{P}[(X,Y) \in A\times\mathbb{R}|\mathcal{G}](\omega) &= \mathsf{P}[X\in A|\mathcal{G}](\omega), \text{ for all } A\in\mathcal{B}(\mathbb{R}),\\ \mathsf{P}[(X,Y) \in \mathbb{R}\times B|\mathcal{G}](\omega) &= \mathsf{P}[Y\in B|\mathcal{G}](\omega), \text{ for all } B\in\mathcal{B}(\mathbb{R}),\\ \mathsf{E}[u(X-Y)|\mathcal{G}](\omega) &= \int_{\mathbb{R}^2} u(x-y)\mathsf{P}[(X,Y)\in dx\otimes dy|\mathcal{G}](\omega),\end{aligned}$$

By Theorem 2.2.6 item 1, for all $\omega\in\Omega'$, we know that:

$$\begin{aligned}Q_\tau[X|\mathcal{G}](\omega) &= \inf\{x\in\mathbb{R}|\mathsf{P}[X\leqslant x|\mathcal{G}](\omega)\geqslant\tau\} = \inf\{x\in\mathbb{R}|\mathsf{P}[(X,Y)\in(-\infty,x]\times\mathbb{R}|\mathcal{G}](\omega)\geqslant\tau\}\\ Q_\tau[Y|\mathcal{G}](\omega) &= \inf\{y\in\mathbb{R}|\mathsf{P}[Y\leqslant y|\mathcal{G}](\omega)\geqslant\tau\} = \inf\{y\in\mathbb{R}|\mathsf{P}[(X,Y)\in\mathbb{R}\times(-\infty,y]|\mathcal{G}](\omega)\geqslant\tau\}\end{aligned}$$

Then, Theorem 8.1. in Major (1978) implies that:

$$\int_{\mathbb{R}^2} u(x-y)P[(X,Y)\in dx\otimes dy|\mathcal{G}](\omega) \geqslant \int_0^1 u(Q_\tau[X|\mathcal{G}](\omega)-Q_\tau[Y|\mathcal{G}](\omega))d\tau, \text{ for all } \omega\in\Omega'$$

Thus, we conclude that $E[u(X-Y)|\mathcal{G}] \geqslant \int_0^1 u(Q_\tau[X|\mathcal{G}]-Q_\tau[Y|\mathcal{G}])d\tau$ a.s. Moreover, if (X,Y) is a $\mathcal{G}$-comonotonic vector, then item 3 in Lemma 2.3.3 forces that:

$$\int_{\mathbb{R}^2} u(x-y)P[(X,Y)\in dx\otimes dy|\mathcal{G}](\omega) = \int_0^1 u(Q_\tau[X|\mathcal{G}](\omega)-Q_\tau[Y|\mathcal{G}](\omega))d\tau, \text{ a.s.}$$

Hence, $E[u(X-Y)|\mathcal{G}] = \int_0^1 u(Q_\tau[X|\mathcal{G}]-Q_\tau[Y|\mathcal{G}])d\tau$ a.s., and we conclude the proof. □

Lemma A.5.2. *Let $X\in L^1(\Omega,\mathcal{F},P)$. Then, $s\mapsto E[Q_s[X|\mathcal{G}]]$ is continuous at $\tau\in(0,1)$ if, and only if, $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ a.s.*

Lemma A.5.2 characterizes and links the jumps of the sample paths of $(Q_\tau[X|\mathcal{G}])_{\tau\in(0,1)}$ to the jumps of $\tau\in(0,1)\mapsto E[Q_\tau[X|\mathcal{G}]]$.

Proof of Lemma 4.2. Fixed $X\in L^1(\Omega,\mathcal{F},P)$, observe that $s\in(0,1)\mapsto E[Q_s[X|\mathcal{G}]]$ is well-defined by Proposition 2.2.5 item 1. Furthermore, $\tau\in(0,1)$ is continuity point if, and only if,

$$E[Q_\tau[X|\mathcal{G}]] = \lim_{s\uparrow\tau} E[Q_s[X|\mathcal{G}] = \lim_{s\downarrow\tau} E[Q_s[X|\mathcal{G}]] = E[Q_{\tau+}[X|\mathcal{G}]].$$

Where the first and the later identities derived from item 1 in Proposition 2.2.9. Since $Q_\tau[X|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}]$, by item 1 in Proposition 2.2.9, then $E[Q_\tau[X|\mathcal{G}]] = E[Q_{\tau+}[X|\mathcal{G}]]$ if, and only if, $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ a.s., concluding the proof. □

Proof of Proposition 2.4.4. Let $X\in L^p(\Omega,\mathcal{F},P)$ and $(X_n)_{n\in\mathbb{N}}\subset L^p(\Omega,\mathcal{F},P)$, such that $X_n \xrightarrow[n\to+\infty]{L^p} X$. Then, by Lemma A.5.2, we have that $E[|X-X_n|^p] \geqslant \int_0^1 E[|Q_\tau[X|\mathcal{G}]-Q_\tau[X_n|\mathcal{G}]|^p]d\tau$, for every $n\in\mathbb{N}$. Since $E[|X-X_n|^p] \xrightarrow[n\to\infty]{} 0$, we obtain $E[|Q_\tau[X|\mathcal{G}]-Q_\tau[X_n|\mathcal{G}]|^p] \xrightarrow[n\to\infty]{} 0$ a.s. on $(0,1)$.

We claim that for all $\tau\in(0,1)$ where $s\mapsto E[Q_s[X|\mathcal{G}]]$ is continuous, then $Q_\tau[X_n|\mathcal{G}] \xrightarrow[n\to+\infty]{L^p} Q_\tau[X|\mathcal{G}]$. Indeed, fix such $\tau\in(0,1)$, $\epsilon>0$, and let $0<\tau'<\tau<\tau''<1$ be such that τ' and τ'' satisfy:

$$Q_{\tau'}[X_n|\mathcal{G}] \xrightarrow[n\to+\infty]{L^p} Q_{\tau'}[X|\mathcal{G}]$$
$$Q_{\tau''}[X_n|\mathcal{G}] \xrightarrow[n\to+\infty]{L^p} Q_{\tau''}[X|\mathcal{G}].$$

Because $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ a.s., by Lemma A.5.2, $\lim_{s\downarrow\tau} Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$ and $\lim_{s\uparrow\tau} Q_\tau[X|\mathcal{G}] = Q_\tau[X|\mathcal{G}]$, by Proposition 2.2.9, then, Proposition 2.2.5 item 3 implies that we can take τ' and τ'' also satisfying that $E[|Q_{\tau'}[X|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p] \leqslant \frac{\epsilon}{2^{1+2p}}$ and $E[|Q_{\tau''}[X|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p] \leqslant \frac{\epsilon}{2^{1+2p}}$. Consequently,

$$\begin{aligned}E[|Q_{\tau'}[X_n|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p] &\leqslant 2^p\left(E[|Q_{\tau'}[X_n|\mathcal{G}]-Q_{\tau'}[X|\mathcal{G}]|^p]+E[|Q_{\tau'}[X|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p]\right)\\ &\leqslant 2^pE[|Q_{\tau'}[X_n|\mathcal{G}]-Q_{\tau'}[X|\mathcal{G}]|^p]+\frac{\epsilon}{2^{p+1}}.\end{aligned}$$

And also:

$$\begin{aligned}E[|Q_{\tau''}[X_n|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p] &\leqslant 2^p\left(E[|Q_{\tau''}[X_n|\mathcal{G}]-Q_{\tau''}[X|\mathcal{G}]|^p]+E[|Q_{\tau''}[X|\mathcal{G}]-Q_\tau[X|\mathcal{G}]|^p]\right)\\ &\leqslant 2^pE[|Q_{\tau''}[X_n|\mathcal{G}]-Q_{\tau''}[X|\mathcal{G}]|^p]+\frac{\epsilon}{2^{p+1}}.\end{aligned}$$

Moreover, because $s \mapsto Q_s[X_n|\mathcal{G}]$ is non-decreasing, for all $n \in \mathbb{N}$, we have that $|Q_\tau[X|\mathcal{G}] - Q_\tau[X_n|\mathcal{G}]|^p \leqslant 2^p\left(|Q_{\tau'}[X_n|\mathcal{G}] - Q_\tau[X|\mathcal{G}]|^p + |Q_{\tau''}[X_n|\mathcal{G}] - Q_\tau[X|\mathcal{G}]|^p\right)$. Hence,

$$\begin{aligned}\mathsf{E}[|Q_\tau[X|\mathcal{G}] - Q_\tau[X_n|\mathcal{G}]|^p] &\leqslant 2^p\left(\mathsf{E}[|Q_{\tau'}[X_n|\mathcal{G}] - Q_\tau[X|\mathcal{G}]|^p] + \mathsf{E}[|Q_{\tau''}[X_n|\mathcal{G}] - Q_\tau[X_n|\mathcal{G}]|^p]\right)\\ &\leqslant 4^p\left(\mathsf{E}[|Q_{\tau'}[X_n|\mathcal{G}] - Q_{\tau'}[X|\mathcal{G}]|^p] + \mathsf{E}[|Q_{\tau''}[X_n|\mathcal{G}] - Q_{\tau''}[X|\mathcal{G}]|^p]\right) + \epsilon.\end{aligned}$$

Therefore, for all $\epsilon > 0$ we showed that $\limsup_{n\in\mathbb{N}} \mathsf{E}[|Q_\tau[X|\mathcal{G}] - Q_\tau[X_n|\mathcal{G}]|^p] \leqslant \epsilon$, which proves the claim.

Now, fix any $\tau \in (0,1)$, not necessarily a continuity point of $s \mapsto \mathsf{E}[Q_s[X|\mathcal{G}]]$. Observe that the random variables $\liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]$ and $\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]$ are well-defined a.s. and belong to L^p. Indeed, there are $0 < \tau' < \tau < \tau'' < 1$, continuity points of $s \mapsto \mathsf{E}[Q_s[X|\mathcal{G}]]$ such that:

$$\begin{aligned}\mathsf{E}[|\liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]|^p] &\leqslant 2^p\left(\mathsf{E}[|\liminf_{n\in\mathbb{N}} Q_{\tau'}[X_n|\mathcal{G}]|^p] + \mathsf{E}[|\liminf_{n\in\mathbb{N}} Q_{\tau''}[X_n|\mathcal{G}]|^p]\right)\\ &\leqslant 2^p\left(\mathsf{E}[|Q_{\tau'}[X|\mathcal{G}]|^p] + \mathsf{E}[|Q_{\tau''}[X|\mathcal{G}]|^p]\right) < +\infty.\end{aligned}$$

And,

$$\begin{aligned}\mathsf{E}[|\limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]|^p] &\leqslant 2^p\left(\mathsf{E}[|\limsup_{n\in\mathbb{N}} Q_{\tau'}[X_n|\mathcal{G}]|^p] + \mathsf{E}[|\limsup_{n\in\mathbb{N}} Q_{\tau''}[X_n|\mathcal{G}]|^p]\right)\\ &\leqslant 2^p\left(\mathsf{E}[|Q_{\tau'}[X|\mathcal{G}]|^p] + \mathsf{E}[|Q_{\tau''}[X|\mathcal{G}]|^p]\right) < +\infty.\end{aligned}$$

Furthermore, for all $\delta \in (0, \min 1-\tau, \tau)$, we may take τ' and τ'' also satisfying $0 < \tau - \delta < \tau' < \tau < \tau'' < \tau + \delta < 1$. By non-decreasingness for the sample paths of conditional quantiles and the fact that $\liminf_{n\in\mathbb{N}} Q_s[X_n|\mathcal{G}] = \limsup_{n\in\mathbb{N}} Q_s[X_n|\mathcal{G}] = Q_s[X|\mathcal{G}]$ in L^p for $s = \tau', \tau''$, then a.s.:

$$\begin{aligned}Q_{\tau'}[X|\mathcal{G}] = \liminf_{n\in\mathbb{N}} Q_{\tau'}[X_n|\mathcal{G}] &\leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}]\\ &\leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau''}[X_n|\mathcal{G}] = Q_{\tau''}[X|\mathcal{G}]\end{aligned}$$

Taking $\delta \downarrow 0$, we obtain:

$$Q_\tau[X|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}], \text{ a.s.}$$

and the proof is completed. □

Proof of Proposition 2.4.5. Fixed $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$, then $X \leqslant Y + \|X-Y\|_\infty$ and $Y \leqslant X + \|X-Y\|_\infty$. Thus, monotonicity (Proposition 2.2.9 item 3) and translational invariance (Theorem 2.3.6) imply:

$$\begin{aligned}Q_\tau[X|\mathcal{G}] &\leqslant Q_\tau[Y + \|X-Y\|_\infty|\mathcal{G}] = Q_\tau[Y|\mathcal{G}] + \|X-Y\|_\infty, \text{ a.s.}\\ Q_\tau[Y|\mathcal{G}] &\leqslant Q_\tau[X + \|X-Y\|_\infty|\mathcal{G}] = Q_\tau[X|\mathcal{G}] + \|X-Y\|_\infty, \text{ a.s.}\end{aligned}$$

Thus, $|Q_\tau[X|\mathcal{G}] - Q_\tau[Y|\mathcal{G}]| \leqslant \|X-Y\|_\infty$ a.s. □

A.5.3 Proofs of Subsection 2.4.3

Proof of Theorem 2.4.8. Firstly, we will prove the result for the trivial σ-algebra, $\mathcal{G} = \{\emptyset, \Omega\}$. As claimed in the text, in this case, $\mathcal{G}$-weakly a.s. is equivalent to weak convergence. After establishing the result for the unconditional case, we show how to derive the complete result from the unconditional.

Denote, as usual, by F_n the c.d.f. of X_n and by F the c.d.f. of X.

($\Rightarrow$) Fixed $\tau \in (0,1)$, we claim that $\sup_{n\in\mathbb{N}} |Q_\tau[X_n]| < +\infty$. For the sake of contradiction, suppose that it is not the case, i.e. $|Q_\tau[X_{n_k}]| \underset{k\to\infty}{\longrightarrow} +\infty$ for some subsequence. Then, for all $x \in \mathbb{R}$ continuity point of F, there is $k_0 \in \mathbb{N}$ such that either $+\infty > Q_\tau[X_{n_k}] > x$ for infinitely many $k \geqslant k_0$, or $-\infty < Q_\tau[X_{n_k}] < x$ for infinitely many $k \geqslant k_0$. Consequently either $F(x) \leqslant \tau$ or $F(x) \geqslant \tau$. Since the set of $x \in \mathbb{R}$ continuity point of F is dense in $\mathbb{R}$, either $\sup_{x\in\mathbb{R}} F(x) \leqslant \tau$ or $\inf_{x\in\mathbb{R}} F(x) \geqslant \tau$, which is a contradiction to the fact that F is a c.d.f.

Therefore, for all $\tau \in (0,1)$, there exists a convergent subsequence, $Q_\tau[X_{n_k}] \underset{k\to\infty}{\longrightarrow} x \in \mathbb{R}$, where $x = \liminf_{n\in\mathbb{N}} Q_\tau[X_n]$. Furthermore, $|x| < +\infty$. If $x < Q_\tau[X]$, then there exists $y \in (x, Q_\tau[X])$ such that F is continuous at y. Hence, there is a $k_0 \in \mathbb{N}$ such that, for all $k \geqslant k_0$, $Q_\tau[X_{n_k}] \leqslant y$, leading to:

$$\tau \leqslant F_{n_k}(Q_\tau[X_{n_k}]) \leqslant F_{n_k}(y)$$

However, we know that $F_{n_k}(y) \underset{k\to\infty}{\longrightarrow} F(y)$, which implies $F(y) \geqslant \tau$ and, consequently, $Q_\tau[X] \leqslant y$, an absurd. Therefore, $Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n]$.

By the same reason, there exists a subsequence, $(X_{n_k})_{k\in\mathbb{N}}$ such that $y = \lim_{k\in\mathbb{N}} Q_{\tau+}[X_{n_k}] = \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n]$. Suppose, for the sake of contradiction, that $y > Q_{\tau+}[X]$. Then there is $z \in (Q_{\tau+}[X], y)$ continuity point of F and $k_0 \in \mathbb{N}$ so that, for all $k \geqslant k_0$, $Q_{\tau+}[X_{n_k}] > z$. Hence, $F_{n_k}(z) \leqslant \tau$, for all $k \geqslant k_0$. However, the continuity of F at z implies that $F_{n_k}(z) \underset{k\to\infty}{\longrightarrow} F(z) \leqslant \tau$, forcing $z \leqslant Q_{\tau+}[X]$, an absurd, since $z > Q_{\tau+}[X]$.

Then, we proved that:

$$Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau+}[X], \text{ for all } \tau \in (0,1).$$

($\Leftarrow$) Conversely, suppose that, for all $\tau \in (0,1)$, we have the following:

$$Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau+}[X].$$

Fix $x \in \mathbb{R}$ continuity point of F, so that $F(x) \in (0,1)$. Then, for all $0 < \tau < F(x) < \tau' < 1$, we have $Q_\tau[X] < x < Q_{\tau'}[X]$, due to item 3 in Theorem 2.2.6. Given $\epsilon \in (0, \min\{F(x), 1 - F(x)\})$ there are $\tau, \tau' \in (F(x) - \epsilon, F(x) + \epsilon)$, with $\tau < \tau'$, such that $s \mapsto Q_s[X]$ is continuous at τ and τ'. Hence, there is $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $Q_\tau[X_n] < x < Q_{\tau'}[X_n]$. Consequently, $\tau \leqslant F_n(x) < \tau'$ and we obtain $|F_n(x) - F(x)| \leqslant \epsilon$.

If $F(x) = 1$, and $x \in \mathbb{R}$ a continuity point of F, then, for all $0 < \tau < 1$, we have $Q_\tau[X] < x$, due to item 3 in 2.6. Given $\epsilon \in (0,1)$, take $\tau \in (1-\epsilon, 1)$, such that $s \mapsto Q_s[X]$ is continuous at τ. Thus, there is $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $Q_\tau[X_n] < x$. Consequently, $\tau \leqslant F_n(x) \leqslant 1$ and we have $|F_n(x) - F(x)| \leqslant \epsilon$.

Finally, if $F(x) = 0$, and $x \in \mathbb{R}$ a continuity point of F, then, for all $0 < \tau < 1$, we have $Q_\tau[X] > x$. Given $\epsilon \in (0,1)$, take $\tau \in (0, \epsilon)$, such that $s \mapsto Q_s[X]$ is continuous at τ. Hence, there is $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$, $Q_{\tau+}[X_n] > x$. Consequently, $\tau \geqslant F_n(x) \geqslant 0$ and we obtain $|F_n(x) - F(x)| \leqslant \epsilon$.

In either case, we got that, for all $x \in \mathbb{R}$ continuity point of F, then $F_n(x) \to F(x)$, concluding the proof.

Assume now that $\mathcal{G}$ is any sub-σ-algebra of $\mathcal{F}$. As in the definition, a sequence $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega,\mathcal{F},P)$ converges $\mathcal{G}$-weakly a.s. if, and only if, there exists a set $\Omega' \in \mathcal{G}$, with full probability measure, so that for all $\omega \in \Omega'$:

$$P[X_n \in \cdot|\mathcal{G}](\omega) \Rightarrow P[X \in \cdot|\mathcal{G}](\omega).$$

If we denote by $F_{n,\omega} : \mathbb{R} \to [0,1]$, $F_{n,\omega}(x) = P[X_n \leqslant x|\mathcal{G}](\omega)$, then $Q_\tau[F_{n,\omega}] = Q_\tau[X_n|\mathcal{G}](\omega)$ according to Theorem 2.2.6 item 3. Therefore, due to the unconditional result above, $\mathcal{G}$-weak convergence a.s. is equivalent to:

$$\begin{aligned} Q_\tau[X|\mathcal{G}](\omega) \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}](\omega) &\leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}](\omega) \\ &\leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n|\mathcal{G}](\omega) \leqslant Q_{\tau+}[X|\mathcal{G}](\omega), \end{aligned}$$

for all $\tau \in (0,1)$ and $\omega \in \Omega'$, which concludes the proof.

□

Proof of Proposition 2.4.10. Given a first order monotone sequence of random variables, $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega,\mathcal{F},P)$, it is easy to see that there always exists a function, $F : \Omega\times\mathbb{R} \to [0,1]$, such that:

- For all $\omega \in \Omega$, $x \in \mathbb{R} \mapsto F(\omega,x)$ is right-continuous with left-limits, non-decreasing. (Helly's Selection Theorem)
- There exists a set $\Omega' \in \mathcal{G}$, with probability one, so that, for all $\omega \in \Omega'$, $F_n(\omega,x) \to F(\omega,x)$ in every continuity point of F.
- From the right-continuity with left-limits, we can also assume that F is $\mathcal{G}\otimes\mathcal{B}(\mathbb{R})$-measurable.

Unfortunately, F does not necessarily come from a conditional probability measure. To guarantee that a monotone sequence of random variables converges $\mathcal{G}$-weakly a.s. to some random variable $X \in L^0(\Omega,\mathcal{F},P)$, whose conditional c.d.f is F, is necessary and sufficient for the sequence to exist a $\Omega' \in \mathcal{G}$, with full probability, and, for all $\epsilon > 0$, a $m(\epsilon,\omega) > 0$ such that:

$$\sup_{\substack{X\in\Pi \\ \tau\in(\epsilon,1-\epsilon]}} |Q_\tau[X|\mathcal{G}](\omega)| \leqslant m(\epsilon,\omega) \quad \text{and} \quad \sup_{\substack{X\in\Pi \\ \tau\in[\epsilon,1-\epsilon)}} |Q_{\tau+}[X|\mathcal{G}](\omega)| \leqslant m(\epsilon,\epsilon),$$

for all $\omega \in \Omega'$.

To see this, first observe that the above equation is equivalent to the existence of a full measure set, $\Omega' \in \mathcal{G}$, so that on it:

$$\lim_{m\to+\infty} \inf_{n\in\mathbb{N}} \big(P[X_n \leqslant m|\mathcal{G}](\omega) - P[X_n \leqslant -m|\mathcal{G}](\omega)\big) = 1.$$

The above equation, on the other hand, is equivalent to:

$$\lim_{m\to+\infty} F(\cdot,m) = 1 \quad \text{and} \quad \lim_{m\to+\infty} F(\cdot,-m) = 0 \text{ a.s.},$$

which implies that F is, indeed, a transition kernel.

This, and the existence of a uniform random variable independent of $\mathcal{G}$ assures that there exists $X \in L^0(\Omega,\mathcal{F},P)$, so that $X_n \underset{\mathcal{G}}{\Rightarrow} X$ a.s.

Furthermore, under this condition we have that there is a $X \in L^0(\Omega, \mathcal{F}, P)$, so that, by Theorem 2.4.8 and the monotonicity of $(X_n)_{n\in\mathbb{N}}$,

$$Q_\tau[X|\mathcal{G}] \leqslant \lim_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] \leqslant Q_{\tau+}[X|\mathcal{G}], \text{ a.s. and for all } \tau \in (0,1),$$

since $(Q_\tau[X_n|\mathcal{G}])_{n\in\mathbb{N}}$ is also monotone. Finally, if $\tau \in (0,1)$ is such that $Q_\tau[X|\mathcal{G}] = Q_{\tau+}[X|\mathcal{G}]$, then $\lim_{n\in\mathbb{N}} Q_\tau[X_n|\mathcal{G}] = Q_\tau[X|\mathcal{G}]$, concluding the proof. □

We conclude this section with some results in the unconditional framework. Firstly, we show how Theorem 2.4.8 also implies Lemma 21.2 in van der Vaart (1998). Then, we restate Prohorov's Theorem in terms of quantiles.

Corollary A.5.3. *Any sequence $(X_n)_{n\in\mathbb{N}} \subset L^0(\Omega, \mathcal{F}, P)$ converges weakly to $X \in L^0(\Omega, \mathcal{F}, P)$ if, and only if, $Q_\tau[X_n] \to Q_\tau[X]$ for all $\tau \in (0,1)$ so that $s \in (0,1) \mapsto Q_s[X]$ is continuous.*

Proof of Corollary 5.3. Let's show that $Q_\tau[X_n] \to Q_\tau[X]$ for all $\tau \in (0,1)$ so that $s \in (0,1) \mapsto Q_s[X]$ is continuous if, and only if,

$$Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau+}[X], \text{ for all } \tau \in (0,1).$$

($\Rightarrow$) Assume that $Q_\tau[X_n] \to Q_\tau[X]$ for all $\tau \in (0,1)$ so that $s \in (0,1) \mapsto Q_s[X]$ is continuous. Thus, for all $\tau \in (0,1)$ and $\epsilon \in (0, \min\{\tau, 1-\tau\})$, there are $\tau - \epsilon < \tau' < \tau < \tau'' < \tau + \epsilon$ continuity points of $s \in (0,1) \mapsto Q_s[X]$, due to its left-continuity and non-decreasingness. Moreover, $Q_{\tau'}[X_n] \leqslant Q_\tau[X_n] \leqslant Q_{\tau+}[X_n] \leqslant Q_{\tau''}[X_n]$, for all $n \in \mathbb{N}$. Nevertheless, by our assumption, $\liminf_{n\in\mathbb{N}} Q_{\tau'}[X_n] = Q_{\tau'}[X]$ and $\limsup_{n\in\mathbb{N}} Q_{\tau''}[X_n] = Q_{\tau''}[X]$. Consequently,

$$Q_{\tau'}[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau''}[X].$$

Taking $\epsilon \downarrow 0$ and using the fact that $\lim_{\tau''\downarrow\tau} Q_{\tau''}[X] = Q_{\tau+}[X]$ and $\lim_{\tau'\uparrow\tau} Q_{\tau'}[X] = Q_\tau[X]$, then:

$$Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau+}[X].$$

($\Leftarrow$) Suppose that for all $\tau \in (0,1)$

$$Q_\tau[X] \leqslant \liminf_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_\tau[X_n] \leqslant \limsup_{n\in\mathbb{N}} Q_{\tau+}[X_n] \leqslant Q_{\tau+}[X].$$

For all $\tau \in (0,1)$ continuity point of $s \in (0,1) \mapsto Q_s[X]$, we know that $Q_\tau[X] = Q_{\tau+}[X]$. Therefore, the inequalities above reduce to equalities and we obtain the following:

$$Q_\tau[X] = \liminf_{n\in\mathbb{N}} Q_\tau[X_n] = \limsup_{n\in\mathbb{N}} Q_\tau[X_n] = Q_{\tau+}[X]$$

Thus, $Q_\tau[X] = \lim_{n\in\mathbb{N}} Q_\tau[X_n]$.

□

Delving further into the relation of the one-parameter family of quantiles to weak convergence, we generalize necessary and sufficient conditions for weak convergence. In this sense, we need first to define relative weak compactness.

Definition A.5.4. *Let* $\Pi = \{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of random variables in* $L^0(\Omega, \mathcal{F}, P)$*.* Π *is weakly relatively compact if every sequence of* $(X_n)_{n\in\mathbb{N}} \subset \Pi$ *admits a further weakly convergent subsequence.*

An equally valuable property when working with convergence in distribution is tightness.

Definition A.5.5. *Let* $\Pi = \{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of random variables in* $L^0(\Omega, \mathcal{F}, P)$*.* Π *is tight if there exists a, for all* $\epsilon > 0$*, there exists a compact* $K \subset \mathbb{R}$ *such that:*

$$\inf_{\lambda\in\Lambda} P[X_\lambda \in K] \geqslant 1 - \epsilon.$$

Equivalently,

$$\lim_{m\to+\infty} \inf_{\lambda\in\Lambda} \big(P[X_\lambda \leqslant m] - P[X_\lambda \leqslant -m]\big) = 1.$$

Under our hypothesis over $L^0(\Omega, \mathcal{F}, P)$, a necessary and sufficient condition for weak relative compactness of a family of random variables is tightness, which was demonstrated by Prohorov, Billingsley (1968). Our next result presents conditions for weak relative compactness, tightness of a family of random variables and Prohorov's Theorem in terms of quantiles.

Proposition A.5.6. *Let* $\Pi \subset L^0(\Omega, \mathcal{F}, P)$ *be a family of random variables.*

1. Π *is tight if, and only if, for all* $\epsilon \in (0, \frac{1}{2})$*, there is a* $m_\epsilon > 0$ *satisfying:*

$$\sup_{\substack{X\in\Pi \\ \tau\in(\epsilon,1-\epsilon]}} |Q_\tau[X]| \leqslant m_\epsilon \quad \text{and} \quad \sup_{\substack{X\in\Pi \\ \tau\in[\epsilon,1-\epsilon)}} |Q_{\tau+}[X]| \leqslant m_\epsilon. \tag{A.2}$$

2. Π *is weakly relatively compact if, and only if, for all sequence* $(X_n)_{n\in\mathbb{N}} \subset \Pi$ *there are a subsequence* $(X_{n_k})_{k\in\mathbb{N}}$ *and a random variable* $X \in L^0(\Omega, \mathcal{F}, P)$ *such that:*

$$\begin{aligned} Q_\tau[X] \leqslant \liminf_{k\in\mathbb{N}} Q_\tau[X_{n_k}] &\leqslant \limsup_{k\in\mathbb{N}} Q_\tau[X_{n_k}] \\ &\leqslant \limsup_{k\in\mathbb{N}} Q_{\tau+}[X_{n_k}] \leqslant Q_{\tau+}[X], \text{ for all } \tau \in (0,1). \end{aligned} \tag{A.3}$$

3. **Prohorov's Theorem**. *Equation (A.2) holds if, and only if, for all sequence* $(X_n)_{n\in\mathbb{N}} \subset \Pi$ *there is a subsequence* $(X_{n_k})_{k\in\mathbb{N}}$ *and a random variable* $X \in L^0(\Omega, \mathcal{F}, P)$ *such that equation (A.3) holds.*

Proof of Proposition A.5.6. Let $\Pi = \{X_\lambda\}_{\lambda\in\Lambda}$ be indexed by Λ, possibly uncountable set.

1. ($\Rightarrow$) From our definition for tightness, given $\epsilon \in (0, \frac{1}{2})$, there is a $m_\epsilon > 0$ such that:

$$\inf_{\lambda\in\Lambda} \big(P[X_\lambda \leqslant m_\epsilon] - P[X_\lambda \leqslant -m_\epsilon]\big) \geqslant 1 - \epsilon.$$

Since $P[X_\lambda \leqslant m_\epsilon] \leqslant 1$ and $P[X_\lambda \leqslant -m_\epsilon] \geqslant 0$, for all $\lambda \in \Lambda$, we have that:

$$\begin{aligned} P[X_\lambda \leqslant m_\epsilon] &\geqslant 1 - \epsilon \\ P[X_\lambda \leqslant -m_\epsilon] &\leqslant \epsilon, \end{aligned}$$

for all $\lambda \in \Lambda$. This implies that:

$$\begin{aligned} Q_{1-\epsilon}[X_\lambda] &\leqslant m_\epsilon \\ Q_{\epsilon+}[X_\lambda] &\geqslant -m_\epsilon \end{aligned}$$

for all $\lambda \in \Lambda$. Moreover,

$$\begin{aligned} Q_\tau[X_\lambda] \leqslant Q_{\tau+}[X_\lambda] \leqslant Q_{1-\epsilon}[X_\lambda] \leqslant m_\epsilon \\ Q_{\tau+}[X_\lambda] \geqslant Q_\tau[X_\lambda] \geqslant Q_{\epsilon+}[X_\lambda] \geqslant -m_\epsilon \end{aligned}$$

for all $\lambda \in \Lambda$ and $\tau \in (\epsilon, 1-\epsilon)$, and we obtain equation (A.2).

($\Leftarrow$) Take any $\epsilon \in (0, \frac{1}{2})$, then there is $m_{\frac{\epsilon}{2}} > 0$, so that

$$\begin{aligned} Q_{1-\frac{\epsilon}{2}}[X_\lambda] \leqslant m_{\frac{\epsilon}{2}} \\ Q_{\frac{\epsilon}{2}+}[X_\lambda] \geqslant -m_{\frac{\epsilon}{2}}, \text{ for all } \lambda \in \Lambda. \end{aligned}$$

Hence,

$$\begin{aligned} P\left[X_\lambda \leqslant m_{\frac{\epsilon}{2}}\right] \geqslant 1 - \frac{\epsilon}{2} \\ P\left[X_\lambda \leqslant -m_{\frac{\epsilon}{2}}\right] \leqslant \frac{\epsilon}{2}, \end{aligned}$$

for all $\lambda \in \Lambda$. Thus,

$$\inf_{\lambda \in \Lambda} \left(P\left[X_\lambda \leqslant m_{\frac{\epsilon}{2}}\right] - P\left[X_\lambda \leqslant -m_{\frac{\epsilon}{2}}\right] \right) \geqslant 1 - \epsilon.$$

Taking $\epsilon \downarrow 0$, we conclude the proof.

2. Π is weakly relatively compact if, and only if, for all sequence $(X_n)_{n \in \mathbb{N}} \subset \Pi$ there is a subsequence, $(X_{n_k})_{k \in \mathbb{N}}$, satisfying $X_{n_k} \Rightarrow X$, for some $X \in L^0(\Omega, \mathcal{F}, P)$. However, Theorem 2.4.8 states that $X_{n_k} \Rightarrow X$. if, and only if,

$$\begin{aligned} Q_\tau[X] \leqslant \liminf_{k \in \mathbb{N}} Q_\tau[X_{n_k}] &\leqslant \limsup_{k \in \mathbb{N}} Q_\tau[X_{n_k}] \\ &\leqslant \limsup_{k \in \mathbb{N}} Q_{\tau+}[X_{n_k}] \leqslant Q_{\tau+}[X], \end{aligned}$$

for all $\tau \in (0, 1)$, concluding the claim.

3. Prohorov's Theorem, Theorems 6.1 and 6.2 in Billingsley (1968), guarantees that Π is weakly relatively compact if, and only if, it is tight, since we are dealing with real valued random variables. Therefore, items 1 and 2 above, and Prohorov's Theorem assure that equation (A.2) holds if, and only if, for all sequence $(X_n)_{n \in \mathbb{N}} \subset \Pi$ there is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ and a random variable $X \in L^0(\Omega, \mathcal{F}, P)$ such that equation (A.3) holds.

□

A.6 Proofs of Section 2.5

A.6.1 Proofs of Subsection 2.5.1

Proof of Theorem 2.5.1. Firstly notice that item 1 and Proposition 2.3.7 assure the existence of a modification for $(Q_\tau[h(\bar{x}, Y)|\mathcal{G}])_{\bar{x} \in V \cap \mathcal{X}}$, which satisfies $Q_\tau[h(\bar{x}, Y)|\mathcal{G}] = h(\bar{x}, Q_\tau[Y|\mathcal{G}])$. Therefore, for all $\epsilon \in \mathbb{R}$ such that $x + \epsilon \in V \cap \mathcal{X}$:

$$\frac{Q_\tau[h(x+\epsilon, Y)|\mathcal{G}] - Q_\tau[h(x, Y)|\mathcal{G}]}{\epsilon} = \frac{h(x+\epsilon, Q_\tau[Y|\mathcal{G}]) - h(x, Q_\tau[Y|\mathcal{G}])}{\epsilon}.$$

Taking ϵ going to 0, using item 2, we conclude that:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, Q_\tau[Y|\mathcal{G}]).$$

Now, if in item 1 h is non-increasing, then Proposition 2.3.7 implies that there exists a modification of $(Q_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$, which satisfies $Q_\tau[h(\bar{x},Y)|\mathcal{G}] = h(\bar{x}, Q_{(1-\tau)+}[Y|\mathcal{G}])$. Hence, for all $\epsilon\in\mathbb{R}$ such that $x+\epsilon\in V\cap\mathcal{X}$:

$$\frac{Q_\tau[h(x+\epsilon,Y)|\mathcal{G}] - Q_\tau[h(x,Y)|\mathcal{G}]}{\epsilon} = \frac{h(x+\epsilon, Q_{(1-\tau)+}[Y|\mathcal{G}]) - h(x, Q_{(1-\tau)+}[Y|\mathcal{G}]}{\epsilon}.$$

Thus,

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, Q_{(1-\tau)+}[Y|\mathcal{G}]).$$

□

Proof of Corollary 2.5.2. If 1 and 2 hold Theorem 2.5.1 guarantees the existence of a modification of $(Q_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ such that:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, Q_\tau[Y|\mathcal{G}]).$$

Now, due to assumption 3, we apply Proposition 2.3.7 to $\frac{\partial h}{\partial x}(x, Q_\tau[Y|\mathcal{G}])$, to obtain the following:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

If in 3 $\frac{\partial h}{\partial x}(x,\cdot)$ is non-increasing, then Proposition 2.3.7 also implies that:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

Now suppose that h is non-increasing and left-continuous. Therefore, Theorem 2.5.1 assures there exists a modification of $(Q_\tau[h(\bar{x},Y)|\mathcal{G}])_{\bar{x}\in V\cap\mathcal{X}}$ satisfying:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = \frac{\partial h}{\partial x}(x, Q_{(1-\tau)+}[Y|\mathcal{G}]).$$

Consequently, item 3 and Proposition 2.3.7 imply that:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

However, if in 3 $\frac{\partial h}{\partial x}(x,\cdot)$ is non-increasing and left-continuous, then:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s.}$$

□

A.6.2 Proofs of Subsection 2.5.2

Proof of Theorem 2.5.3. First notice that by the assumptions on h and Theorem 2.3.6, then, for all V neighborhood of x, $(Q_\tau[h(\bar{x}, Y)]|\mathcal{G})_{\bar{x}\in V\cap\mathcal{X}}$ admits a modification such that:

$$Q_\tau[h(\bar{x}, Y)|\mathcal{G}] = \phi(\bar{x}) + \psi(\bar{x})Q_\tau[\eta(Y)|\mathcal{G}]\mathbb{1}_{[\psi(\bar{x})\geqslant 0]} + \psi(\bar{x})Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}]\mathbb{1}_{[\psi(\bar{x})<0]},$$

for all $\bar{x} \in V \cap \mathcal{X}$. From now on, we will work with this modification.

1. Taking the neighborhood V of x where $\psi(\bar{x}) \geqslant 0$, for all $\bar{x} \in V\cap\mathcal{X}$, we have that $Q_\tau[h(\bar{x}, Y)|\mathcal{G}] = \phi(\bar{x}) + \psi(\bar{x})Q_\tau[\eta(Y)|\mathcal{G}]$. Therefore, differentiability of ϕ and ψ at x implies differentiability of $\bar{x} \in V \cap \mathcal{X} \mapsto Q_\tau[h(\bar{x}, Y)|\mathcal{G}]$ at x, so that:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = \phi'(x) + \psi'(x)Q_\tau[\eta(Y)|\mathcal{G}].$$

   If $\psi'(x) \geqslant 0$, then Theorem 2.3.6 implies that:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

   On the other hand, if $\psi'(x) < 0$, then Theorem 2.3.6 implies that:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

2. Just notice that in $V\cap\mathcal{X}$, we have $Q_\tau[h(\bar{x}, Y)|\mathcal{G}] = \phi(\bar{x}) + \psi(\bar{x})Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}]$ and applying the same argument used in 1 we obtain:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = \phi'(x) + \psi'(x)Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}].$$

   If $\psi'(x) \geqslant 0$, then Theorem 2.3.6 implies that:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = Q_{(1-\tau)+}\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

   Nevertheless, if $\psi'(x) < 0$, then Theorem 2.3.6 implies that:

   $$\frac{d}{dx}Q_\tau[h(x, Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x, Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

3. Assuming that $\bar{x} \in V \cap \mathcal{X} \mapsto \psi(\bar{x})$ is non-decreasing, [2] then it is easy to show that the right and left-derivatives of $\bar{x} \in V \cap \mathcal{X} \mapsto Q_\tau[h(\bar{x}, Y)|\mathcal{G}]$ at x are:

   $$\lim_{\epsilon\downarrow 0}\frac{Q_\tau[h(x+\epsilon, Y)|\mathcal{G}] - Q_\tau[h(x, Y)|\mathcal{G}]}{\epsilon} = \phi'(x) + \psi'(x)Q_\tau[\eta(Y)|\mathcal{G}]$$
   $$\lim_{\epsilon\uparrow 0}\frac{Q_\tau[h(x+\epsilon, Y)|\mathcal{G}] - Q_\tau[h(x, Y)|\mathcal{G}]}{\epsilon} = \phi'(x) + \psi'(x)Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}].$$

[2] When ψ is non-increasing we invert the right and left-derivatives.

Thus, because $\psi'(x)Q_{(1-\tau)+}[\eta(Y)|\mathcal{G}] = \psi'(x)Q_\tau[\eta(Y)|\mathcal{G}]$, we obtain:

$$\frac{d}{dx}Q_\tau[h(x,Y)] = \varphi'(x) + \psi'(x)Q_\tau[\eta(Y)|\mathcal{G}], \text{ a.s..}$$

Provided $\bar{x} \in V \cap \mathcal{X} \mapsto \psi(\bar{x})$ is non-decreasing or non-increasing, either $\psi'(x) \geqslant 0$ or $\psi'(x) \leqslant 0$, respectively. Consequently, Theorem 2.3.6 allows us to interchange the conditional operator and the derivative:

$$\frac{d}{dx}Q_\tau[h(x,Y)|\mathcal{G}] = Q_\tau\left[\frac{\partial h}{\partial x}(x,Y)\middle|\mathcal{G}\right], \text{ a.s..}$$

□

A.7 Proofs of Section 2.6

A.7.1 Proofs of Subsection 2.6.1

Proof of Proposition 2.6.1. Let $\mathcal{H} = \{\emptyset, \Omega\}$, $\mathcal{F} = \{\emptyset, \Omega, A_1, A_2 \cup A_3\}$ and define $X = \sum_{i=1}^{3} i\mathbb{1}_{A_i}$. Therefore, an easy computation shows that $P[X \leqslant x|\mathcal{G}] = \mathbb{1}_{x\geqslant 1}\mathbb{1}_{A_1} + (\frac{p_2}{p_2+p3}\mathbb{1}_{2\leqslant x<3} + \mathbb{1}_{x\geqslant 3})\mathbb{1}_{A_2\cup A_3}$. Therefore, since $\frac{p_2}{p_2+p3} < \tau$, we obtain $Q_\tau[X|\mathcal{G}] = \mathbb{1}_{A_1} + 3\mathbb{1}_{A_2\cup A_3}$. Thus, $P[Q_\tau[X|\mathcal{G}] \leqslant x|\mathcal{H}] = P[Q_\tau[X|\mathcal{G}] \leqslant x] = p_1\mathbb{1}_{1\leqslant x<3} + \mathbb{1}_{x\geqslant 3}$. Consequently, $Q_\tau[Q_\tau[X|\mathcal{G}]\mathcal{H}] = 3$, since $p_1 < \tau$. However, $Q_\tau[X|\mathcal{H}] = Q_\tau[X] = 2$, since $p_1 + p_2 \geqslant \tau$ and $p_1 < \tau$. Hence, $Q_\tau[Q_\tau[X|\mathcal{G}]|\mathcal{H}] \neq Q_\tau[X|\mathcal{H}] = Q_\tau[Q_\tau[X|\mathcal{G}]|\mathcal{H}]$. □

Proof of Proposition 2.6.2.

1. For simplicity we will use $\pi_\mathcal{H}$ and $\pi_\mathcal{G}$ for the quantile projection operators. The first and second inclusions are trivial. To show that $L^1(\Omega,\mathcal{G},P) \subsetneq \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$ is strict, let $X \in L^1(\Omega,\mathcal{F},P)$ such that $X - E[X|\mathcal{G}]$ is independent of $\mathcal{G}$. Then,

$$\begin{aligned}
\pi_\mathcal{G}(\pi_\mathcal{H}(X)) &= \pi_\mathcal{G}\Big(\pi_\mathcal{H}\big(E[X|\mathcal{G}] + X - E[X|\mathcal{G}]\big)\Big),\\
&= \pi_\mathcal{G}\Big(E[X|\mathcal{G}] + \pi_\mathcal{H}\big(X - E[X|\mathcal{G}]\big)\Big),\\
&= E[X|\mathcal{G}] + Q_\tau\big[X - E[X|\mathcal{G}]\big],\\
&= \pi_\mathcal{H}\Big(E[X|\mathcal{G}] + \pi_\mathcal{G}\big(X - E[X|\mathcal{G}]\big)\Big),\\
&= \pi_\mathcal{H}\Big(\pi_\mathcal{G}\big(E[X|\mathcal{G}] + X - E[X|\mathcal{G}]\big)\Big),\\
&= \pi_\mathcal{H}(\pi_\mathcal{G}(X)),
\end{aligned}$$

where we use Proposition 2.9 items 4 and 5 in the previous computation. Observe that $X \notin L^1(\Omega,\mathcal{G},P)$, and this concludes the proof.

2. Given such $a \in L^1(\Omega,\mathcal{H},P)$, $b \in L^\infty(\Omega,\mathcal{H},P)$ and $X \in \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$, then it is immediate that $a + bX \in L^1(\Omega,\mathcal{F},P)$. Thus, applying Theorem 2.3.6, we have $[\pi_\mathcal{H},\pi_\mathcal{G}](a+bX) = b[\pi_\mathcal{H},\pi_\mathcal{G}](X) = 0$ a.s. Hence, $a + bX \in \mathcal{C}^\tau_{\mathcal{H},\mathcal{G}}$.

□

A.7.2 Proofs of Subsection 2.6.2

Proof of Example 2.6.3. For all $0 \leqslant n < m \in \mathbb{N}$:

$$\begin{aligned}
Q_\tau[\ldots Q_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_n] &= Q_\tau\left[\ldots Q_\tau\left[Q_\tau[X|\mathcal{F}_m]\middle|\mathcal{F}_{m-1}\right]\ldots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\ldots Q_\tau\left[Q_\tau\left[E[X|\mathcal{F}_m]+(X-E[X|\mathcal{F}_m])\,|\mathcal{F}_m\right]\middle|\mathcal{F}_{m-1}\right]\ldots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\ldots Q_\tau\left[E[X|\mathcal{F}_m]+Q_\tau\left[X-E[X|\mathcal{F}_m]\,|\mathcal{F}_m\right]\middle|\mathcal{F}_{m-1}\right]\ldots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\ldots Q_\tau\left[E[X|\mathcal{F}_m]+Q_\tau\left[X-E[X|\mathcal{F}_m]\right]\middle|\mathcal{F}_{m-1}\right]\ldots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[X-E[X|\mathcal{F}_m]\right]+Q_\tau\left[\ldots Q_\tau[E[X|\mathcal{F}_m]|\mathcal{F}_{m-1}]\ldots|\mathcal{F}_n\right].
\end{aligned}$$

The above computation holds since $E[X|\mathcal{F}_m] \in L^0(\Omega,\mathcal{F}_m,P)$, $Q_\tau[\cdot|\mathcal{F}_k]$ in $L^0(\Omega,\mathcal{F}_k,P)$, for all $k \in \mathbb{N}$, and $X - E[X|\mathcal{F}_m]$ is independent of $\mathcal{F}_m$. Moreover, if we repeat the argument above using that $E[X|\mathcal{F}_j] - E[X|\mathcal{F}_{j-1}]$ is independent of $\mathcal{F}_{j-1}$, we obtain:

$$Q_\tau[\ldots Q_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_n] = E[X|\mathcal{F}_n] + \sum_{j=1}^{m-n} Q_\tau\left[E[X|\mathcal{F}_{n+j}] - E[X|\mathcal{F}_{n+j-1}]\right] + Q_\tau\left[X - E[X|\mathcal{F}_m]\right].$$

In our example,

$$E[X|\mathcal{F}_{n+j}] - E[X|\mathcal{F}_{n+j-1}] = B_{t_{n+j}} - B_{t_{n+j-1}} \sim \sqrt{t_{n+j}-t_{n+j-1}}\mathrm{N}(0,1).$$

Therefore $Q_\tau\left[E[X|\mathcal{F}_{n+j}] - E[X|\mathcal{F}_{n+j-1}]\right] = \sqrt{t_{n+j}-t_{n+j-1}}Q_\tau[\mathrm{N}(0,1)]$, for all $1 \leqslant j \leqslant m-n$. By the same reason, $Q_\tau\left[X - E[X|\mathcal{F}_m]\right] = Q_\tau[B_T - B_{t_m}] = \sqrt{T-t_m}Q_\tau[\mathrm{N}(0,1)]$. Now observe that $t_m = \sum_{j=1}^m (t_j - t_{j-1}) = \sum_{j=1}^m \frac{6T}{\pi^2 n^2} \underset{m\to\infty}{\longrightarrow} T$. Thus, $Q_\tau\left[X - E[X|\mathcal{F}_m]\right] \underset{m\to\infty}{\longrightarrow} 0$, for all $\tau \in (0,1)$. However, notice that:

$$\begin{aligned}
Q_\tau[\ldots Q_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_n] &= B_{t_n} + \left(\sum_{j=1}^{m-n}\sqrt{t_{n+j}-t_{n+j-1}} + \sqrt{T-t_m}\right)Q_\tau[\mathrm{N}(0,1)],\\
&= B_{t_n} + \left(\frac{\sqrt{6T}}{\pi}\sum_{j=1}^{m-n}\frac{1}{j} + \sqrt{T-t_m}\right)Q_\tau[\mathrm{N}(0,1)].
\end{aligned}$$

Clearly, $\sum_{j=1}^{m-n}\frac{1}{j} \xrightarrow{m\to+\infty} +\infty$. Consequently, we obtain:

$$\lim_{m\to+\infty} Q_\tau[\ldots Q_\tau[X|\mathcal{F}_m]\ldots|\mathcal{F}_n] = \begin{cases} -\infty, & \text{if } \tau < \frac{1}{2},\\ B_{t_n}, & \text{if } \tau = \frac{1}{2},\\ +\infty, & \text{if } \tau > \frac{1}{2}.\end{cases}$$

□

Proof of Proposition 2.6.4. Fix $X \in H$ and $n \in \mathbb{N}$. Then, for all $m > n$, by independence of increments:

$$\begin{aligned}
Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] &= Q_\tau\left[\dots Q_\tau\left[Q_\tau[X|\mathcal{F}_m]\middle|\mathcal{F}_{m-1}\right]\dots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\dots Q_\tau\left[Q_\tau\left[E[X|\mathcal{F}_m]+(X-E\left[X|\mathcal{F}_m\right])\,|\mathcal{F}_m\right]\middle|\mathcal{F}_{m-1}\right]\dots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\dots Q_\tau\left[E[X|\mathcal{F}_m]+Q_\tau\left[X-E\left[X|\mathcal{F}_m\right]|\mathcal{F}_m\right]\middle|\mathcal{F}_{m-1}\right]\dots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\left[\dots Q_\tau\left[E[X|\mathcal{F}_m]+Q_\tau\left[X-E[X|\mathcal{F}_m]\right]\middle|\mathcal{F}_{m-1}\right]\dots\middle|\mathcal{F}_n\right],\\
&= Q_\tau\big[X-E[X|\mathcal{F}_m]\big]+Q_\tau\big[\dots Q_\tau[E[X|\mathcal{F}_m]|\mathcal{F}_{m-1}]\dots|\mathcal{F}_n\big],\\
&= E[X|\mathcal{F}_n]+\sum_{j=1}^{m-n} Q_\tau\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]+Q_\tau\big[X-E[X|\mathcal{F}_m]\big].
\end{aligned}$$

Firstly, notice that, since $\mathcal{F}=\mathcal{F}_\infty$ and $X\in L^1(\Omega,\mathcal{F},P)$, $X-E[X|\mathcal{F}_m]\xrightarrow{m\to+\infty}0$ in L^1, by discrete time martingale theory, Le Gall (2006). Therefore, $Q_\tau\big[X-E[X|\mathcal{F}_m]\big]\to Q_\tau[0]$ for every continuity point of $s\mapsto Q_s[0]$, by Corollary 4.8. However, at any $\tau\in(0,1)$ the map $s\mapsto Q_s[0]$ is continuous. Thus, $Q_\tau\big[X-E[X|\mathcal{F}_m]\big]\underset{m\to\infty}{\longrightarrow}0$ for every $\tau\in(0,1)$.

Secondly, observe that Proposition 2.2.9 item 5 implies that:

$$\begin{aligned}
\int_0^1\left|\sum_{j=1}^{m-n} Q_\tau\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right|d\tau &\leqslant \sum_{j=1}^{m-n}\int_0^1\left|Q_\tau\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right|d\tau,\\
&= \sum_{j=1}^{m-n} E\Big[\big|E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big|\Big],\\
&= \sum_{j=1}^{m-n}\|E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\|_{L^1},\\
&\leqslant \sum_{j\geqslant1}\|E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\|_{L^1}<+\infty.
\end{aligned}$$

Consequently, $\int_0^1\left|\sum_{j\geqslant1}Q_\tau\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right|d\tau<+\infty$, from which we conclude that

$$\sum_{j\geqslant1}Q_\tau\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]$$

is finite a.s. on $\tau\in(0,1)$. We claim that, in fact, this last sum is finite for every $\tau\in(0,1)$. Indeed, given $\tau\in(0,1)$ there are $0<\tau'<\tau<\tau''<1$, such that the series converges for both τ' and τ''. By the monotonicity of quantiles:

$$\begin{aligned}
\sum_{j\geqslant1}Q_{\tau'}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big] &\leqslant \sum_{j\geqslant1}Q_{\tau}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big],\\
&\leqslant \sum_{j\geqslant1}Q_{\tau''}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big].
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\sum_{j\geqslant1}Q_{\tau}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right| &\leqslant \left|\sum_{j\geqslant1}Q_{\tau''}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right|,\\
&+\left|\sum_{j\geqslant1}Q_{\tau'}\big[E[X|\mathcal{F}_{n+j}]-E[X|\mathcal{F}_{n+j-1}]\big]\right|<+\infty.
\end{aligned}$$

And the claim is proved. Thus, we demonstrate that:

$$\lim_{m\to+\infty} Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] = E[X|\mathcal{F}_n] + \sum_{j\geqslant 1} Q_\tau\big[E[X|\mathcal{F}_{n+j}] - E[X|\mathcal{F}_{n+j-1}]\big], \text{ a.s.}$$

□

Proof of Proposition 2.6.5. Fix $n \in \mathbb{N}\cup\{0\}$ and take any $m > n$. Then, for all $X \in L^\infty(\Omega, \mathcal{F}, P)$, $Q_\tau[X|\mathcal{F}_m] \in L^\infty(\Omega, \mathcal{F}_m, P)$, by Proposition 2.2.5 item 2, and $\|Q_\tau[X|\mathcal{F}_m]\|_\infty \leqslant \|X\|_\infty$ a.s., by Proposition 2.4.5. Therefore, applying recursively this argument, $Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] \in L^\infty(\Omega, \mathcal{F}_n, P)$ and there is a set $\Omega' \in \mathcal{F}_n$, with full probability, such that:

$$\|Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n]\|_\infty \leqslant \|X\|_\infty, \text{ on } \Omega'.$$

Hence,

$$\begin{aligned}
&\exists \liminf_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n](\omega) \in \mathbb{R},\\
&\exists \limsup_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n](\omega) \in \mathbb{R},
\end{aligned}$$

for $\omega \in \Omega'$. Furthermore, both limits are bounded by $\|X\|_\infty$ a.s. and we can extend them by 0 outside Ω'. Consequently:

$$\begin{aligned}
&\liminf_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[\cdot|\mathcal{F}_m]\dots|\mathcal{F}_n] : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_n,P),\\
&\limsup_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[\cdot|\mathcal{F}_m]\dots|\mathcal{F}_n] : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_n,P).
\end{aligned}$$

If $X \in \overline{\cup_{n\in\mathbb{N}\cup\{0\}} L^\infty(\Omega, \mathcal{F}_n, P)}$, let $(X_m)_{m\in\mathbb{N}} \subset L^\infty(\Omega,\mathcal{F},P)$ so that $X_m \in L^\infty(\Omega,\mathcal{F}_m,P)$, for all $m \in \mathbb{N}\cup\{0\}$, and $X_m \xrightarrow[m\to\infty]{\|\cdot\|_\infty} X$. First notice that, by Proposition 2.4.5:

$$\|Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] - Q_\tau[\dots Q_\tau[X_m|\mathcal{F}_m]\dots|\mathcal{F}_n]\|_\infty \leqslant \|X - X_m\|_\infty,$$

and

$$\begin{aligned}
&\|Q_\tau[\dots Q_\tau[X_m|\mathcal{F}_m]\dots|\mathcal{F}_n] - Q_\tau[\dots Q_\tau[X_k|\mathcal{F}_k]\dots|\mathcal{F}_n]\|_\infty =\\
&= \|Q_\tau[\dots Q_\tau[X_m|\mathcal{F}_k]\dots|\mathcal{F}_n] - Q_\tau[\dots Q_\tau[X_k|\mathcal{F}_k]\dots|\mathcal{F}_n]\|_\infty,\\
&\leqslant \|X_m - X_k\|_\infty, \text{ for all } m \leqslant k.
\end{aligned}$$

Thus, $(Q_\tau[\dots Q_\tau[X_m|\mathcal{F}_m]\dots|\mathcal{F}_n])_{m\geqslant n}$ is Cauchy in $L^\infty(\Omega,\mathcal{F},P)$ and, consequently, converges uniformly to some element $Z \in L^\infty(\Omega,\mathcal{F},P)$.

Since $(Q_\tau[\dots Q_\tau[X_m|\mathcal{F}_m]\dots|\mathcal{F}_n])_{m\geqslant n}$ and $(Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n])_{m\geqslant n}$ are uniformly close to each other, then $Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] \xrightarrow[m\to\infty]{\|\cdot\|_\infty} Z$. Thus, we may conclude that, under this assumption,

$$\liminf_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n] = \limsup_{m\in\mathbb{N}} Q_\tau[\dots Q_\tau[X|\mathcal{F}_m]\dots|\mathcal{F}_n].$$

□

Appendix B

Appendix of Chapter 3

In this appendix we shall provide the proofs of the main results. For that we shall start with some basic properties and definitions in order to make this work self contained.

B.1 Average Value at Risk Conditional to $\mathcal{G}$.

We shall now summarize some of the properties for the average value at risk conditional operator.

Proposition B.1.1. *The one-parameter family of operators $(\mathsf{AV@R}_\tau[\cdot|\mathcal{G}])_{\tau\in(0,1)}$ satisfies the following properties.*

1. *For any $X \in L^1(\Omega, \mathcal{F}, P)$ and $\omega \in \Omega$, the map $\tau \in (0,1) \mapsto \mathsf{AV@R}_\tau[X|\mathcal{G}](\omega) \in \mathbb{R}$ is continuous and non-increasing.*

2. *$\mathsf{AV@R}_\tau[\cdot|\mathcal{G}] : L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, \mathcal{G}, P)$, for any $\tau \in (0,1)$ and $p \in [1, +\infty]$.*

3. *$\mathsf{AV@R}_\tau[\cdot|\mathcal{G}]$ satisfies conditional translational invariance, monotonicity, conditional convexity, positive homogeneity, regularity and conditional law-invariance, for any $\tau \in (0,1)$.*

4. *For any $\Lambda \in L^\infty(\Omega, \mathcal{G}, P)$ satisfying $0 < \Lambda < 1$ a.s., the Λ-Average Value at Risk operator conditional to $\mathcal{G}$, $\mathsf{AV@R}_\Lambda : L^1(\Omega, \mathcal{F}, P) \to L^1(\Omega, \mathcal{G}, P)$, defined by*

$$\mathsf{AV@R}_\Lambda[X|\mathcal{G}] = \frac{1}{\Lambda}\int_0^\Lambda \mathsf{V@R}_\tau[X|\mathcal{G}]d\tau, \text{ for any } X \in L^1(\Omega, \mathcal{F}, P),$$

   is well-defined. Moreover $\mathsf{AV@R}_\Lambda[X|\mathcal{G}] \in L^0(\Omega, \mathcal{G}, P)$, and $\mathsf{AV@R}_\tau[\cdot|\mathcal{G}]$ is not certain on independent variables.

5. *For any $p \in [1, +\infty)$, if $(X_n)_{n\in\mathbb{N}} \subset L^p(\Omega, \mathcal{F}, P)$ satisfies $\sup_{n\in\mathbb{N}} |X_n| \in L^p(\Omega, \mathcal{F}, P)$, then, for any $\tau \in (0,1)$ fixed, the following holds a.s.:*

$$\mathsf{AV@R}_\tau[\limsup_{n\in\mathbb{N}} X_n|\mathcal{G}] \leqslant \liminf_{n\in\mathbb{N}} \mathsf{AV@R}_\tau[X_n|\mathcal{G}] \leqslant \limsup_{n\in\mathbb{N}} \mathsf{AV@R}_\tau[X_n|\mathcal{G}] \leqslant \mathsf{AV@R}_{\tau-}[\liminf_{n\in\mathbb{N}} X_n|\mathcal{G}].$$

   In particular, if $\liminf_{n\in\mathbb{N}} X_n = X = \limsup_{n\in\mathbb{N}} X_n$ and $\mathsf{AV@R}_\tau[X|\mathcal{G}] = \mathsf{AV@R}_{\tau-}[X|\mathcal{G}]$ a.s., we obtain an identity.

6. *For any $\tau \in (0,1)$, $\mathsf{AV@R}_\tau[\cdot|\mathcal{G}] : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ is 1-Lipschitz:*

$$\|\mathsf{AV@R}_\tau[X|\mathcal{G}] - \mathsf{AV@R}_\tau[Y|\mathcal{G}]\|_\infty \leqslant \|X - Y\|_\infty, \text{ for any } X, Y \in L^\infty(\Omega, \mathcal{F}, P).$$

7. *If* $X = (X_1, \dots, X_n) \in L^1(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ *is a* $\mathcal{G}$*-comonotonic random vector, then for any* $\tau \in (0,1)$ *fixed:*

$$AV@R_\tau \left[\sum_{i=1}^{n} X_i \middle| \mathcal{G} \right] = \sum_{i=1}^{n} AV@R_\tau [X_i | \mathcal{G}], \text{ a.s.}$$

B.2 Proofs of the Main Results

B.2.1 Proofs of Section 3.2.1

Proof of Proposition 3.2.3. It is straightforward to prove that $\sim_{\mathcal{G}}$ is an equivalence relation. Moreover, for any $X \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$, there exists a set $\Omega' \in \mathcal{G}$, such that $P[\Omega'] = 1$, and $P[X \in \mathcal{A}|\mathcal{G}](\omega) = \mathbb{1}_{[X \in \mathcal{A}]}(\omega)$ for every $\omega \in \Omega'$ and $\mathcal{A} \in \mathcal{B}(\mathbb{R})$. Thus, if $X, Y \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$ and $X \sim_{\mathcal{G}} Y$, there exists a set $\Omega' \in \mathcal{G}$, with $P[\Omega'] = 1$, such that:

$$\mathbb{1}_{[X \in \mathcal{A}]}(\omega) = \mathbb{1}_{[Y \in \mathcal{A}]}(\omega), \text{ for any } \omega \in \Omega' \text{ and } \mathcal{A} \in \mathcal{B}(\mathbb{R}).$$

Consequently, on Ω', we get that, for every $x \in \mathbb{R}$, $X(\omega) = x$ if, and only if, $Y(\omega) = x$, implying that $X = Y$ a.s.

If $X \in L^p(\Omega, \mathcal{F}, P)$ and $\bar{X} \in L^0(\Omega, \mathcal{F}, P)$, so that $\bar{X} \sim_{\mathcal{G}} X$, then there exists a set $\Omega' \in \mathcal{G}$, with full probability, such that $P[X \in \mathcal{A}|\mathcal{G}](\omega) = P[\bar{X} \in \mathcal{A}|\mathcal{G}](\omega)$, for any $\omega \in \Omega$ and $\mathcal{A} \in \mathcal{B}(\mathbb{R})$. In particular, since $X \in L^p(\Omega, \mathcal{F}, P)$:

$$E[|\bar{X}|^p|\mathcal{G}](\omega) = \int |x|^p P[\bar{X} \in \cdot|\mathcal{G}](\omega) = \int |x|^p P[X \in \cdot|\mathcal{G}](\omega) = E[|X|^p|\mathcal{G}](\omega), \text{ for any } \omega \in \Omega'.$$

Hence, $E[|\bar{x}|^p] = E[|X|^p] < +\infty$, and $\bar{X} \in L^p(\Omega, \mathcal{F}, P)$.

Finally, given $X, \bar{X} \in L^0(\Omega, \mathcal{F}, P)$, it is immediate to check that, if $\bar{X} + Y \sim_{\mathcal{G}} X + Y$, for any $Y \in L^0(\Omega, \mathcal{G}, P)$, then $\bar{X} \sim_{\mathcal{G}} X$. On the other hand, if $\bar{X} \sim_{\mathcal{G}} X$ and $Y \in L^0(\Omega, \mathcal{G}, P)$, we can extract a set $\Omega' \in \mathcal{G}$, with probability one, such that:

$$P[\bar{X} + Y \in \mathcal{A}|\mathcal{G}](\omega) = P[\bar{X} \in \mathcal{A} - Y(\omega)|\mathcal{G}](\omega) = P[X \in \mathcal{A} - Y(\omega)|\mathcal{G}](\omega) = P[X + Y \in \mathcal{A}|\mathcal{G}](\omega),$$

for any $\omega \in \Omega'$ and $\mathcal{A} \in \mathcal{B}(\mathbb{R})$. $\square$

Proof of Proposition 3.2.6. Assume that ρ is conditionally law-invariant. Then, for any $X \in \mathcal{A}_\rho$ fixed, let $\bar{X} \sim_{\mathcal{G}} X$. From the definition of conditional law-invariance:

$$\rho(\bar{X}) = \rho(X) \leqslant 0, \text{ a.s.}$$

Therefore, $\bar{X} \in \mathcal{A}_\rho$ and $\mathcal{A}_\rho$ is invariant under $\sim_{\mathcal{G}}$.

On the other hand, if $\mathcal{A}_\rho$ is $\sim_{\mathcal{G}}$-invariant, taking any $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $\bar{X} \sim_{\mathcal{G}} X$, then Proposition 3.2.3 implies $\bar{X} + \rho(X) \sim_{\mathcal{G}} X + \rho(X)$ and $\bar{X} + \rho(\bar{X}) \sim_{\mathcal{G}} X + \rho(\bar{X})$. Moreover, $\rho\left(X + \rho(X)\right) = \rho(X) - \rho(X) = 0$ and $\rho\left(\bar{X} + \rho(\bar{X})\right) = \rho(\bar{X}) - \rho(\bar{X}) = 0$, by conditional translational invariance. Thus, both $\bar{X} + \rho(\bar{X}), X + \rho(X) \in\ A_\rho$ and, by the $\sim_{\mathcal{G}}$-invariance of A_ρ, $\bar{X} + \rho(X), X + \rho(\bar{X}) \in A_p$. Consequently,

$$\rho(X) - \rho(\bar{X}) = \rho\left(X + \rho(\bar{X})\right) \leqslant 0,$$
$$\rho(\bar{X}) - \rho(X) = \rho\left(\bar{X} + \rho(X)\right) \leqslant 0,$$

from where we conclude $\rho(X) = \rho(\bar{X})$. $\square$

B.2.2 Proofs of Section 3.3.1

Proof of Lemma 3.3.3. Our proof is similar to the one presented in Föllmer and Schied (2002) for the unconditional framework. We start by showing that $\mathsf{E}[\bar{X}Y|\mathcal{G}] \leqslant \int_0^1 Q_\tau[\bar{X}|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau$ a.s., for any $\bar{X} \sim_{\mathcal{G}} X$. Indeed, by Lemma A.5.2 in Chapter 2, given any $\bar{X} \sim_{\mathcal{G}} X$, then:

$$\mathsf{E}\left[(\bar{X}-Y)^2|\mathcal{G}\right] \geqslant \int_0^1 \left(Q_\tau[\bar{X}|\mathcal{G}] - Q_\tau[Y|\mathcal{G}]\right)^2 d\tau, \text{ a.s.}$$

Consequently,

$$\mathsf{E}[\bar{X}Y|\mathcal{G}] \leqslant \int_0^1 Q_\tau[\bar{X}|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau, \text{ a.s.}$$

For the converse inequality, we need to first demonstrate that, assuming the existence of $Y \in L^1(\Omega, \mathcal{F}, P)$ so that $P[Y \in \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$ a.s., then there exists an $U \in L^0(\Omega, \mathcal{F}, P)$ with $P[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s. Indeed, let $F_{Y|\mathcal{G}} : \mathbb{R} \times \Omega \to [0,1]$ be the conditional c.d.f. of Y given $\mathcal{G}$, i.e. $F_{Y|\mathcal{G}}(y,\omega) = P[Y \leqslant y|\mathcal{G}](\omega)$. Thus, it is trivial to show that $F_{Y|\mathcal{G}}$ is $\mathcal{G} \otimes \mathcal{B}(\mathbb{R})$-measurable. Define $U : \Omega \to \mathbb{R}$ by $U(\omega) = F_{Y|\mathcal{G}}(Y(\omega), \omega)$ will imply that $U \in L^0(\Omega, \mathcal{F}, P)$, since it is the composition of measurable and finite maps.

Assume, without lost of generality, that there exists a set $\Omega' \in \mathcal{G}$, with full probability measure, such that, for any $\tau \in (0,1)$, $P[Y \leqslant Q_\tau[Y|\mathcal{G}]|\mathcal{G}] = F_{Y|\mathcal{G}}(Q_\tau[Y|\mathcal{G}], \cdot) = \tau$. This is possible since $Q_\tau[Y|\mathcal{G}] \in L^0(\Omega, \mathcal{G}, P)$, $Q_\tau[Y|\mathcal{G}] = \inf\{y \in \mathbb{R} : P[Y \leqslant y|\mathcal{G}] \geqslant \tau\}$, and $y \in \mathbb{R} \mapsto P[Y \leqslant y|\mathcal{G}]$ is in $\mathcal{C}(\mathbb{R})$ a.s. Hence, for any $\omega \in \Omega'$ and $\tau \in (0,1)$:

$$\begin{aligned} P[U \geqslant \tau|\mathcal{G}](\omega) &= P\left[\omega' \in \Omega : F_{Y|\mathcal{G}}(Y(\omega'), \omega') \geqslant \tau \middle| \mathcal{G}\right] = P\left[Y \geqslant Q_\tau[Y|\mathcal{G}] \middle| \mathcal{G}\right](\omega), \\ &= 1 - P\left[Y < Q_\tau[Y|\mathcal{G}] \middle| \mathcal{G}\right](\omega) = 1 - P\left[Y \leqslant Q_\tau[Y|\mathcal{G}] \middle| \mathcal{G}\right](\omega), \\ &= 1 - \tau, \end{aligned}$$

and our first claim is proved.

Recall now that, because $\tau \in (0,1) \mapsto Q_\tau[X|\mathcal{G}] \in L^0(\Omega, \mathcal{G}, P)$ is cad-lag, then $(\tau, \omega) \in (0,1) \times \Omega \mapsto Q_\tau[X|\mathcal{G}] \in \mathbb{R}$ is $\mathcal{B}((0,1)) \otimes \mathcal{G}$ measurable. Thus, we can define $\bar{X} = Q_U[X|\mathcal{G}]$. Moreover, $\bar{X} \in L^0(\Omega, \mathcal{F}, P)$ and, for any $q \in \mathbb{Q}$, there exists $\Omega_q \in \mathcal{G}$, with full probability measure, such that:

$$P\left[U \leqslant F_{X|\mathcal{G}}(q, \cdot) \middle| \mathcal{G}\right](\omega) = F_{X|\mathcal{G}}(q, \omega), \text{ for } \omega \in \Omega_q.$$

Taking $\Omega'' = \cap_{q \in \mathbb{Q}} \Omega_q$, then for any $q \in \mathbb{Q}$ and $\omega \in \Omega''$:

$$\begin{aligned} P[\bar{X} \leqslant q|\mathcal{G}](\omega) &= P\left[Q_U[X|\mathcal{G}] \leqslant q \middle| \mathcal{G}\right](\omega) = P\left[U \leqslant F_{X|\mathcal{G}}(q, \cdot) \middle| \mathcal{G}\right](\omega) = F_{X|\mathcal{G}}(q, \omega), \\ &= P\left[X \leqslant q|\mathcal{G}\right](\omega). \end{aligned}$$

Consequently, $X \sim_{\mathcal{G}} \bar{X}$. If we set $\bar{Y} := Q_U[Y|\mathcal{G}]$, then $Y = \bar{Y}$ a.s. Indeed, notice that $Y \geqslant Q_U[Y|\mathcal{G}]$ due to the definition of $Q_\tau[Y|\mathcal{G}]$:

$$\{\omega \in \Omega : Y(\omega) \geqslant Q_U[Y|\mathcal{G}](\omega)\} = \{\omega \in \Omega : F_{Y|\mathcal{G}}(Y(\omega), \omega) \geqslant U(\omega)\} = \Omega.$$

Furthermore, $\mathsf{E}[Y] = \mathsf{E}\left[Q_U[Y|\mathcal{G}]\right]$, since:

$$\mathsf{E}[Y] = \mathsf{E}[\mathsf{E}[Y|\mathcal{G}]] = \mathsf{E}\left[\int_0^1 Q_\tau[Y|\mathcal{G}]d\tau\right] = \mathsf{E}\left[\int_0^1 Q_\tau[Y|\mathcal{G}]P[U \in d\tau|\mathcal{G}]\right] = \mathsf{E}\left[\mathsf{E}[Q_U[Y|\mathcal{G}]\right] = \mathsf{E}\left[Q_U[Y|\mathcal{G}]\right],$$

where the second equality follows from Proposition 2.2.9 item 6 in Chapter 2. Because $\mathsf{E}[Y] = \mathsf{E}[\bar{Y}]$ and $Y \geqslant \bar{Y}$, then $\bar{Y} = Y$ a.s. Besides this, $Y = \bar{Y}$ a.s. also implies that $\mathsf{E}[Y\bar{X}|\mathcal{G}] = \mathsf{E}[\bar{Y}\bar{X}|\mathcal{G}]$ a.s. Hence,

$$\mathsf{E}[Y\bar{X}|\mathcal{G}] = \mathsf{E}[\bar{Y}\bar{X}|\mathcal{G}] = \mathsf{E}\left[Q_U[Y|\mathcal{G}]Q_U[Y|\mathcal{G}]|\,\mathcal{G}\right] = \int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau, \text{ a.s.}$$

Therefore, we demonstrated the desired equality provided that $\mathsf{P}[Y \leqslant \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$ a.s.

Now, suppose that $Y \in L^1(\Omega, \mathcal{F}, \mathsf{P})$ takes only a countable number of values. Moreover, fix $Z \in L^1(\Omega, \mathcal{F}, \mathsf{P})$, positive, and satisfying $\mathsf{P}[Z \leqslant \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$. The existence of such Z is guaranteed by the following argument. Let $f \in \mathcal{C}(\mathbb{R})$ non-decreasing function whose support lies in $\mathbb{R}_+$, taking values on $[0,1]$, with $\lim_{x\downarrow 0} f(x) = 0$, $\lim_{x\uparrow +\infty} f(x) = 1$ and such that $\int_0^{+\infty}(1 - f(x))dx < +\infty$. Then, we can define $q_\tau[f] = \inf\{x \in \mathbb{R} : f(x) \geqslant \tau\}$ for any $\tau \in (0,1)$. Moreover, we can also define $Z = q_U[f]$, where $U \in L^0(\Omega, \mathcal{F}, \mathsf{P})$ and $\mathsf{P}[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s. It is straightforward to verify that $Z \in L^0(\Omega, \mathcal{F}, \mathsf{P})$ as well as it is positive. Besides this, $Z \in L^1(\Omega, \mathcal{F}, \mathsf{P})$ because:

$$\mathsf{E}[Z] = \mathsf{E}\left[\mathsf{E}[Z|\mathcal{G}]\right] = \mathsf{E}\left[\int_0^1 q_U[f]\mathsf{P}[U \in d\tau|\mathcal{G}]\right] = \mathsf{E}\left[\int_0^1 q_\tau[f]d\tau\right] = \int_0^{+\infty}(1 - f(x))dx < +\infty.$$

Fix $n \in \mathbb{N}$ and let $Y_n = Y + \frac{1}{n}Z$. We claim that $\mathsf{P}[Y_n \leqslant \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$ a.s. Indeed, because $Im(Y)$ is countable, from the monotone convergence theorem for conditional expectations and basic properties of transition kernels, we get that, for any $q \in \mathbb{Q}$ and $r \in \mathbb{Q}_+$, there exists $\Omega_{q,r} \in \mathcal{G}$, with probability measure one, satisfying:

$$\begin{aligned}
\mathsf{P}[Y_n \leqslant q + r|\mathcal{G}](\omega) - \mathsf{P}[Y_n \leqslant q|\mathcal{G}](\omega) &= \mathsf{P}[q < Y_n \leqslant q + r|\mathcal{G}](\omega), \\
&= \mathsf{E}[\mathbb{1}_{[q<Y_n\leqslant q+r]}|\mathcal{G}](\omega), \\
&= \mathsf{E}\left[\sum_{y\in Im(Y)} \mathbb{1}_{\left[Y=y, Z\in\left(n(q-y), n(q+r-y)\right]\right]}\middle|\,\mathcal{G}\right](\omega), \\
&= \sum_{y\in Im(Y)} \mathsf{E}\left[\mathbb{1}_{\left[Y=y, Z\in\left(n(q-y), n(q+r-y)\right]\right]}|\mathcal{G}\right](\omega), \\
&= \sum_{y\in Im(Y)} \mathsf{P}\left[Y = y, Z \in \left(n(q-y), n(q+r-y)\right]\middle|\mathcal{G}\right](\omega),
\end{aligned}$$

for any $\omega \in \Omega_{q,r}$. Moreover, we can also require for any $\omega \in \Omega_{q,r}$:

$$\mathsf{P}\left[Y = y, Z \in (n(q-y), n(q+r-y))\middle|\mathcal{G}\right](\omega) \leqslant \mathsf{P}[Y = y|\mathcal{G}](\omega)$$

Let $\Omega' \in \mathcal{G}$, with $\mathsf{P}[\Omega'] = 1$, be such that $\mathsf{P}[Z \leqslant \cdot|\mathcal{G}](\omega) \in \mathcal{C}(\mathbb{R})$ for any $\omega \in \Omega'$. Then, it is immediate to see that $\Omega'' = \left(\cap_{(q,r)\in\mathbb{Q}\times\mathbb{Q}_+}\Omega_{q,r}\right) \cap \Omega' \in \mathcal{G}$ has full probability measure. Furthermore, due to the previous equations, Weierstrass M-test and the fact that $\mathsf{P}[Y_n \leqslant \cdot|\mathcal{G}]$ is cad-lag:

$$\mathsf{P}[Y_n \leqslant y' + \epsilon|\mathcal{G}](\omega) - \mathsf{P}[Y_n \leqslant y'|\mathcal{G}](\omega) \leqslant \sum_{y\in Im(Y)} \mathsf{P}\left[Y = y, Z \in (n(y'-y), n(y'+\epsilon-y))\middle|\mathcal{G}\right](\omega)$$

for any $\omega \in \Omega''$, $y \in \mathbb{R}$ and $\epsilon > 0$. Finally, since $\mathsf{P}[Z \leqslant \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$ on Ω'', then Weierstrass M-test imply:

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\mathsf{P}[Y_n\leqslant y'+\epsilon|\mathcal{G}](\omega)-\mathsf{P}[Y_n\leqslant y'|\mathcal{G}](\omega) &= \lim_{\epsilon\downarrow 0}\sum_{y\in \mathrm{Im}(Y)}\mathsf{P}\left[Y=y, Z\in(n(y'-y),n(y'+\epsilon-y)\middle|\mathcal{G}\right](\omega),\\ &= \sum_{y\in \mathrm{Im}(Y)}\lim_{\epsilon\downarrow 0}\mathsf{P}\left[Y=y, Z\in(n(y'-y),n(y'+\epsilon-y))\middle|\mathcal{G}\right](\omega),\\ &= 0,\ \text{for any } \omega\in\Omega'' \text{ and } y\in\mathbb{R},\end{aligned}$$

proving our claim.

Assume, without loss of generality, $X \geqslant 0$ a.s. Moreover, because Z is positive, $Y_n \geqslant Y$. From item 4 in Proposition 2.2.9 Chapter 2, we know that $Q_\tau[Y_n|\mathcal{G}] \geqslant Q_\tau[Y|\mathcal{G}]$ a.s. for any $n \in \mathbb{N}$. Thus,

$$\begin{aligned}\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau &\leqslant \liminf_{n\in\mathbb{N}}\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y_n|\mathcal{G}]d\tau,\\ &\leqslant \liminf_{n\in\mathbb{N}}\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}Y_n|\mathcal{G}],\ \text{a.s.}\end{aligned}$$

For any $n \in \mathbb{N}$, our previous computations showed that there exists $\bar{X}_n \sim_{\mathcal{G}} X$, such that $\mathsf{E}[\bar{X}_n Y_n|\mathcal{G}] = \operatorname{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}Y_n|\mathcal{G}]$ a.s. Besides, Holder inequality ensures that $|\mathsf{E}[\bar{X}_n Y_n|\mathcal{G}]-\mathsf{E}[\bar{X}_n Y|\mathcal{G}]| \leqslant \frac{\|X\|_{+\infty}\mathsf{E}[|Z||\mathcal{G}]}{n}$ a.s., for any $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}\liminf_{n\in\mathbb{N}}\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}Y_n|\mathcal{G}] = \liminf_{n\in\mathbb{N}}\mathsf{E}[\bar{X}_n Y_n|\mathcal{G}] &\leqslant \liminf_{n\in\mathbb{N}}\left(\mathsf{E}[\bar{X}_n Y|\mathcal{G}]+\left|\mathsf{E}[\bar{X}_n Y_n|\mathcal{G}]-\mathsf{E}[\bar{X}_n Y|\mathcal{G}]\right|\right),\\ &\leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}Y|\mathcal{G}],\ \text{a.s.,}\end{aligned}$$

proving the identity for $X \in L^\infty(\Omega, \mathcal{F}, \mathsf{P})$ positive and Y simple.

If $Y \in L^1(\Omega, \mathcal{F}, \mathsf{P})$, let $(Y_n)_{n\in\mathbb{N}} \subset L^1(\Omega, \mathcal{F}, \mathsf{P})$ be a sequence of simple random variables such that $Y_n \geqslant Y$, $Y_n \to Y$ a.s. and in L^1. As we proved before, we can construct a sequence $(W_n)_{n\in\mathbb{N}} \subset L^1(\Omega, \mathcal{F}, \mathsf{P})$, so that $\mathsf{P}[W_n \in \cdot|\mathcal{G}] \in \mathcal{C}(\mathbb{R})$ a.s., $W_n \geqslant Y_n$, for every $n \in \mathbb{N}$, and $W_n \to Y_n$ in L^1. Furthermore, for any $n \in \mathbb{N}$, there is a $\bar{X}_n \sim_{\mathcal{G}} \bar{X}$ so that:

$$\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}W_n|\mathcal{G}] = \mathsf{E}[\bar{X}_n W_n|\mathcal{G}].$$

Thus,

$$\begin{aligned}\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau \leqslant \liminf_{n\in\mathbb{N}}\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[W_n|\mathcal{G}]d\tau &= \liminf_{n\in\mathbb{N}}\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}W_n|\mathcal{G}],\\ &= \liminf_{n\in\mathbb{N}}\mathsf{E}[\bar{X}_n W_n|\mathcal{G}],\ \text{a.s.}\end{aligned}$$

On the other hand, $|\mathsf{E}[\bar{X}_n Y|\mathcal{G}]-\mathsf{E}[\bar{X}_n W_n|\mathcal{G}]| \leqslant \|X\|_{+\infty}\mathsf{E}[|W_n - Y|\big|\mathcal{G}]$ a.s., for any $n \in \mathbb{N}$. At least along a subsequence, we know that $\mathsf{E}[|W_n - Y|\big|\mathcal{G}] \to 0$ a.s. Therefore:

$$\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau \leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X}\mathsf{E}[\bar{X}Y|\mathcal{G}],\ \text{a.s..}$$

Finally, for a general $X \in L^\infty(\Omega, \mathcal{F}, P)$, apply the previous equation to $X + \|X\|_\infty$. Then, since $\bar{X} \sim_{\mathcal{G}} X$ if, and only if, $\bar{X} + \|X\|_\infty \sim_{\mathcal{G}} X + \|X\|_\infty$ – Proposition 3.2.3, we have that:

$$\begin{aligned}\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau + \|X\|_\infty \int_0^1 Q_\tau[Y|\mathcal{G}]d\tau &= \int_0^1 Q_\tau[X + \|X\|_\infty|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau, \\ &\leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} E\left[\left(\bar{X} + \|X\|_\infty\right)Y|\mathcal{G}\right], \\ &\leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E[\bar{X}Y|\mathcal{G}] + \|X\|_\infty E[Y|\mathcal{G}]\right), \\ &\leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E[\bar{X}Y|\mathcal{G}] + \|X\|_\infty \int_0^1 Q_\tau[Y|\mathcal{G}]d\tau\right), \text{ a.s.,}\end{aligned}$$

However, since $\|X\|_\infty \int_0^1 Q_\tau[Y|\mathcal{G}]d\tau$ is a $\mathcal{G}$-measurable random variable that it is not affected by changes on $\bar{X}$, we finally get that:

$$\int_0^1 Q_\tau[X|\mathcal{G}]Q_\tau[Y|\mathcal{G}]d\tau \leqslant \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} E[\bar{X}Y|\mathcal{G}], \text{ a.s.,}$$

which concludes the proof. ☐

Proof. Proof of Theorem 3.3.4

($\Rightarrow$) Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ be a conditionally convex and continuous from above risk measure. Then, Theorem 3.3.1 shows that there exists $\alpha_* : \mathcal{P}_{\mathcal{G}} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}}_+)$, so that:

$$\alpha_*(Q) = \sup_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(E^Q[-X|\mathcal{G}] - \rho(X)\right), \text{ for any } Q \in \mathcal{P}_{\mathcal{G}},$$

and,

$$\rho(X) = \sup_{Q\in\mathcal{P}_{\mathcal{G}}} \left(E^Q[-X|\mathcal{G}] - \alpha_*(Q)\right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Then, because ρ is conditionally law-invariant and Lemma 3.3.3, we have that:

$$\begin{aligned}\alpha_*(Q) &= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(E^Q[-X|\mathcal{G}] - \rho(X)\right) = \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E^Q[-\bar{X}|\mathcal{G}] - \rho(\bar{X})\right), \\ &= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E^Q[-\bar{X}|\mathcal{G}]\right) - \rho(X)\right) = \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E^Q[-\bar{X}|\mathcal{G}]\right) - \rho(X)\right), \\ &= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(\operatorname*{esssup}_{\bar{X}\sim_{\mathcal{G}}X} \left(E\left[-\bar{X}\frac{dQ}{dP}\Big|\mathcal{G}\right]\right) - \rho(X)\right), \\ &= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},P)} \left(\int_0^1 Q_\tau[-X|\mathcal{G}]Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right]d\tau - \rho(X)\right).\end{aligned}$$

Fixing a $X \in L^\infty(\Omega, \mathcal{F}, P)$, and letting $Q \in \mathcal{P}_{\mathcal{G}}$, the previous identity ensures that:

$$\rho(X) \geqslant \int_0^1 Q_\tau[-X|\mathcal{G}]Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right]d\tau - \alpha_*(Q), \text{ a.s.}$$

Moreover, Lemma 3.3.3 also guarantees that that:

$$\mathsf{E}^{\mathsf{Q}}[-\mathsf{X}|\mathcal{G}] = \mathsf{E}\left[-\mathsf{X}\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] \leqslant \int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau.$$

Both inequalities imply that:

$$\operatorname*{esssup}_{\mathsf{Q}\in\mathcal{P}_\mathcal{G}}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \alpha_*(\mathsf{Q})\right) \leqslant \rho(\mathsf{X}) \leqslant \operatorname*{esssup}_{\mathsf{Q}\in\mathcal{P}_\mathcal{G}}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \alpha_*(\mathsf{Q})\right),$$

establishing the desired identity.

Now, let Q and $\bar{\mathsf{Q}} \in \mathcal{P}_\mathcal{G}$, such that $\frac{d\mathsf{Q}}{d\mathsf{P}} \sim_\mathcal{G} \frac{d\bar{\mathsf{Q}}}{d\mathsf{P}}$. Then, $\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] = \mathsf{Q}_\tau\left[\frac{d\bar{\mathsf{Q}}}{d\mathsf{P}}\middle|\mathcal{G}\right]$ for any $\tau \in (0,1)$ in a fixed set of full probability measure. As a consequence of the characterization of α_* in terms of conditional quantiles, we obtain that $\alpha_*(\bar{\mathsf{Q}}) = \alpha_*(\mathsf{Q})$.

($\Leftarrow$) Let ρ be representable as in Theorem 3.3.4. Fixed $\mathsf{X} \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$, take any $\bar{\mathsf{X}} \sim_\mathcal{G} \mathsf{X}$. Then, $\mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}] = \mathsf{Q}_\tau[-\bar{\mathsf{X}}|\mathcal{G}]$ for any $\tau \in (0,1)$ in a set of full probability measure. Consequently, for any $\mathsf{Q} \in \mathcal{P}_\mathcal{G}$, we conclude that:

$$\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau = \int_0^1 \mathsf{Q}_\tau[-\bar{\mathsf{X}}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau, \text{ a.s.}$$

Thus, by its representation representation, we get that $\rho(\mathsf{X}) = \rho(\bar{\mathsf{X}})$.

□

Proof. Proof of Corollary 3.3.5.

($\Rightarrow$) Suppose that ρ is coherent, continuous from above and conditionally law-invariant. Then, Theorem 3.3.4 states that, for any $\mathsf{Q} \in \mathcal{P}_\mathcal{G}$:

$$\alpha_*(\mathsf{Q}) = \operatorname*{esssup}_{\mathsf{X}\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \rho(\mathsf{X})\right).$$

Furthermore, fixed $\mathsf{X} \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})$, for any $\lambda > 0$, since $\mathsf{Q}_\tau[-\lambda\mathsf{X}|\mathcal{G}] = \lambda\mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]$ and $\rho(\lambda\mathsf{X}) = \lambda\rho(\mathsf{X})$, for any $\tau \in (0,1)$ in a set of full probability measure, we observe that:

$$\begin{aligned}
\alpha_*(\mathsf{Q}) &= \frac{\lambda}{\lambda}\operatorname*{esssup}_{\mathsf{X}\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \rho(\mathsf{X})\right), \\
&= \lambda\operatorname*{esssup}_{\mathsf{X}\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})}\left(\int_0^1 \frac{1}{\lambda}\mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \frac{1}{\lambda}\rho(\mathsf{X})\right), \\
&= \lambda\operatorname*{esssup}_{\mathsf{X}\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})}\left(\int_0^1 \mathsf{Q}_\tau\left[-\frac{1}{\lambda}\mathsf{X}\middle|\mathcal{G}\right]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \rho(\frac{1}{\lambda}\mathsf{X})\right), \\
&= \lambda\operatorname*{esssup}_{\mathsf{Y}\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P})}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{Y}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau - \rho(\mathsf{Y})\right), \\
&= \lambda\alpha_*(\mathsf{Q}).
\end{aligned}$$

Because the above identity holds for any $\lambda > 0$, then either $\alpha_*(\mathsf{Q}) = +\infty$ or 0. Therefore, we may define $\mathcal{Q} = \{\mathsf{Q} \in \mathcal{P}_\mathcal{G} : \alpha_*(\mathsf{Q}) = 0 \text{ a.s.}\}$, leading to the representation:

$$\rho(\mathsf{X}) = \operatorname*{esssup}_{\mathsf{Q}\in\mathcal{Q}}\left(\int_0^1 \mathsf{Q}_\tau[-\mathsf{X}|\mathcal{G}]\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau\right), \text{ for any } \mathsf{X} \in \mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P}).$$

($\Leftarrow$) If ρ satisfies:

$$\rho(X) = \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{G}}} \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] d\tau \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

By Theorem 3.3.4, ρ is conditionally law-invariant, convex and continuous from above. Moreover, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $\Lambda \in L^\infty(\Omega, \mathcal{G}, P)$, so that $\Lambda \geqslant 0$ a.s., Theorem 2.3.6 in Chapter 2 shows that $Q_\tau[-\Lambda X|\mathcal{G}] = \Lambda Q_\tau[-X|\mathcal{G}]$, for any $\tau \in (0,1)$ in a fixed set with full probability. Therefore,

$$\begin{aligned}
\rho(\Lambda X) &= \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{G}}} \left(\int_0^1 Q_\tau[-\Lambda X|\mathcal{G}] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] d\tau \right) = \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{G}}} \left(\int_0^1 \Lambda Q_\tau[-X|\mathcal{G}] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] d\tau \right), \\
&= \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{G}}} \Lambda \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] d\tau \right) = \Lambda \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{G}}} \left(\int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] d\tau \right), \\
&= \Lambda \rho(X).
\end{aligned}$$

Thus, ρ is coherent.

$\square$

B.2.3 Proofs of Section 3.3.2

Proof of Theorem 3.3.9. We first show that there is bijection between $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{M}^{\mathcal{G}}_{(0,1]}$. Take any $Q \in \mathcal{P}_{\mathcal{G}}$, then Proposition 2.2.9 in Chapter 2 shows that $\tau \in (0,1) \mapsto Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right] \in \mathbb{R}$ is left-continuous with right-limits. Moreover, fixing $\omega \in \Omega$ define ν by:

$$\nu\left(\omega, (1-\tau, 1]\right) = Q_\tau \left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega), \text{ for any } \tau \in (0,1).$$

Thus, $\nu(\omega, \cdot)$ can uniquely be extended to a σ-finite measure on $\mathcal{B}\left((0,1]\right)$. Moreover, this construction holds for any $\omega \in \Omega$.

Notice also that, due to Proposition 2.2.9 item 6 in Chapter 2, there is a set $\Omega' \in \mathcal{G}$, with probability one, so that:

$$\int_0^1 Q_s \left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega) ds = E\left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega), \text{ for any } \omega \in \Omega'.$$

Since $Q|_{\mathcal{G}} = P|_{\mathcal{G}}$, we can assume that on Ω' we have that $E\left[\frac{dQ}{dP} \middle| \mathcal{G} \right] = \frac{d\,Q|_{\mathcal{G}}}{d\,P|_{\mathcal{G}}} = 1$.

Now, define $\mu : \Omega \times \mathcal{B}\left((0,1]\right) \to [0,1]$ by:

$$\begin{aligned}
\mu(\omega, A) &= \int_A s\nu(\omega, ds), \text{ for any } A \in \mathcal{B}\left((0,1]\right) \text{ and } \omega \in \Omega'. \\
\mu(\omega, A) &= \delta_{\frac{1}{2}}(A), \text{ for any } A \in \mathcal{B}\left((0,1]\right) \text{ and } \omega \in \Omega \cap (\Omega')^c.
\end{aligned}$$

If we prove that $\mu\left(\omega, (0,1]\right) = 1$ for any $\omega \in \Omega'$, then the fact that ν is transition kernel is enough to derive that μ is also a transition probability kernel. However, in Ω':

$$\begin{aligned}
\mu\left(\omega, (0,1]\right) &= \int_{(0,1]} s\nu(\omega, ds) = \int_0^1 \nu\left(\omega, (s,1]\right) ds = \int_0^1 Q_{s+} \left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega) ds = \int_0^1 Q_s \left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega) ds, \\
&= E^P \left[\frac{dQ}{dP} \middle| \mathcal{G} \right](\omega) = 1,
\end{aligned}$$

which ensures that $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$.

Given $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, define $\psi : \Omega \times (0,1) \to \mathbb{R}_+$ by:

$$\psi(\omega,\tau) = \int_{(1-\tau,1]} \frac{1}{s}\mu(\omega, ds), \text{ for any } \omega \in \Omega \text{ and } \tau \in (0,1).$$

Then, $\psi(\cdot,\tau) \in L^0(\Omega,\mathcal{G},P)$, for any $\tau \in (0,1)$. Moreover, by its definition, $\tau \in (0,1) \mapsto \psi(\tau,\omega)$ is left-continuous with right-limits, for every $\omega \in \Omega$. Since there exists a random variable $U \in L^0(\Omega,\mathcal{F},P)$ so that $P[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s., then the composition $\psi(\cdot, U(\cdot)) \in L^0(\Omega,\mathcal{F},P)$. Consequently, we may define $Q : \mathcal{F} \to [0,1]$ through $\frac{dQ}{dP} = \psi(\cdot, U(\cdot))$.

It is immediate to verify that Q will be a measure. To ensure that it is a probability measure in $\mathcal{P}_{\mathcal{G}}$, we shall demonstrate that $Q[\Omega] = 1$ and $Q[A] = P[A]$, for any $A \in \mathcal{G}$. This accomplished by noticing that:

$$E^P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right](\omega) = \int \psi(\omega,\tau)P[U \in d\tau|\mathcal{G}](\omega) = \int_0^1 \psi(\omega,\tau)d\tau, \text{ for } \omega \in \Omega',$$

where $\Omega' \in \mathcal{G}$ is a set of P-probability one. Due to the definition of ψ:

$$\begin{aligned}\int_0^1 \psi(\omega,\tau)d\tau &= \int_0^1 \left(\int_{(1-\tau,1]} \frac{1}{s}\mu(\omega,ds)\right)d\tau = \int_{(0,1]} \frac{1}{s}\left(\int_{(1-s,1]} d\tau\right)\mu(\omega,ds) = \mu(\omega,(0,1]),\\ &= 1, \text{ for any } \omega \in \Omega.\end{aligned}$$

Thus, $E^P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] = 1$ a.s., which implies $Q[A] = P[A]$, for any $A \in \mathcal{G}$, including Ω, guaranteeing that $Q \in \mathcal{P}_{\mathcal{G}}$.

As it can be seen from the construction carried above, the map that identifies $Q \in \mathcal{P}_{\mathcal{G}}$ to $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ satisfies the following.

1. $\forall Q \in \mathcal{P}_{\mathcal{G}}$, there exists a $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ constructed as before. This μ is uniquely determined except in a $\mathcal{G}$-measurable set of zero P-probability measure.

2. $\forall \mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, there exists a unique $Q \in \mathcal{P}_{\mathcal{G}}$ constructed as before.

3. If $Q \in \mathcal{P}_{\mathcal{G}}$ is fixed, let $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ be defined as in item 1, and $\bar{Q} \in \mathcal{P}_{\mathcal{G}}$ be the associated probability measure to μ obtained through item 2. Then, because $P[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s., we can extract a set $\Omega' \in \mathcal{G}$, with full probability measure, such that in Ω':

$$\begin{aligned}Q_\tau\left[\frac{d\bar{Q}}{dP}\middle|\mathcal{G}\right] &= \psi(\cdot,\tau) = \int_{(1-\tau,1]} \frac{1}{s}\mu(\cdot,ds) = \int_{(1-\tau,1]} \nu(\cdot,ds) = \nu(\cdot,(1-\tau,1]),\\ &= Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right], \text{ for any } \tau \in (0,1).\end{aligned}$$

   Thus, $P\left[\frac{d\bar{Q}}{dP} \in \cdot\middle|\mathcal{G}\right] = P\left[\frac{dQ}{dP} \in \cdot\middle|\mathcal{G}\right]$ a.s. This is enough to conclude that $\bar{Q}[A] = Q[A]$, for any $A \in \mathcal{F}$. Therefore, the map is injective.

4. Conversely, if $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ is fixed, and $Q \in \mathcal{P}_{\mathcal{G}}$ is constructed as in item 2, then, let $\bar{\mu} \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ be the one obtained from Q as in item 1. From its construction, it is possible to select a set

$\Omega' \in \mathcal{G}$, with probability one, where $\bar{\nu} : \Omega \times \mathcal{B}\left((0,1]\right) \to \mathbb{R}_+$, obtained from $Q \mapsto \bar{\mu}$ in item 1, satisfies:

$$\bar{\nu}\left(\cdot, (1-\tau, 1]\right) = \int_{(1-\tau,1]} \frac{1}{s}\mu(\cdot, ds), \text{ for any } \tau \in (0,1).$$

Hence, in Ω', we obtain that:

$$\bar{\mu}(\cdot, A) = \int_A s\bar{\nu}(\cdot, ds) = \mu(\cdot, A), \text{ for any } A \in \mathcal{B}\left((0,1]\right).$$

Consequently, we proved that the map constructed is a surjective, proving that it is a bijection between $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{M}^{\mathcal{G}}_{(0,1]}$.

Now, notice that, for each fixed $Q \in \mathcal{P}_{\mathcal{G}}$, letting $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ be the mapped measure obtained above, there is a set $\Omega' \in \mathcal{G}$, with probability one, so that:

$$\begin{aligned}
\int_0^1 Q_\tau[-X|\mathcal{G}](\omega) Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right](\omega)d\tau &= \int_0^1 Q_{(1-\tau)}[-X|\mathcal{G}](\omega) Q_{(1-\tau)}\left[\frac{dQ}{dP}\middle|\mathcal{G}\right](\omega)d\tau, \\
&= \int_0^1 V@R_\tau[X|\mathcal{G}](\omega)\nu\left(\omega, (1-\tau, 1]\right)d\tau, \\
&= \int_0^1 V@R_\tau[X|\mathcal{G}](\omega)\left(\int_{(1-\tau,1]} \frac{1}{s}\mu(\omega, ds)\right)d\tau, \\
&= \int_{(0,1]} \frac{1}{s}\left(\int_{(1-s,1]} V@R_\tau[X|\mathcal{G}](\omega)d\tau\right)\mu(\omega, ds), \\
&= \int_{(0,1]} AV@R_s[X|\mathcal{G}](\omega)\mu(\omega, ds),
\end{aligned}$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $\omega \in \Omega'$.

Hence, we can define $\beta_* : \mathcal{M}^{\mathcal{G}}_{(0,1]} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}}_+)$, by:

$$\beta_*(\mu) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int_{(0,1]} AV@R_s[X|\mathcal{G}](\omega)\mu(\omega, ds) - \rho(X)\right),$$

such that, if $Q \in \mathcal{P}_{\mathcal{G}}$ is the measure derived from item 2, then $\beta_*(\mu) = \alpha_*(Q)$. Furthermore, we also obtained that:

$$\rho(X) = \operatorname*{esssup}_{\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}} \left(\int_{(0,1]} AV@R_s[X|\mathcal{G}](\omega)\mu(\omega, ds) - \beta_*(\mu)\right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

□

Proof of Corollary 3.3.10. Under the conditions of Corollary 3.3.10, ρ is conditionally law invariant if, and only if, the representation Theorem 3.3.9 holds, since ρ is convex.

Moreover, due to conditional positive homogeneity we get that:

$$\beta_*(\mu) = \lambda\beta_*(\mu), \text{ for any } \lambda > 0 \text{ and } \mu \in \mathcal{M}^{\mathcal{G}}_{[0,1]}.$$

Therefore, either $\beta_*(\mu) = +\infty$ or 0, concluding the if part.

The other direction follows from the positive homogeneity of $AV@R_\tau[\cdot|\mathcal{G}]$, item 4 in Proposition B.1.1, and the linearity of the integral. □

B.2.4 Proofs of Section 3.3.3

Proof of Lemma 3.3.12. Let Φ be the map defined above. We first show that it is well-defined for any $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ as an element of $\mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$.

Fix $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, and let $\omega \in \Omega$ be arbitrarily chosen. Then, $\mu(\omega, \cdot) : \mathcal{B}((0,1]) \to [0,1]$ is a probability measure. Moreover, for any $\tau \in (0,1)$, $\mu(\omega, (\tau, 1])$ and $\int_{(\tau,1]} \frac{\tau}{s}\mu(\omega, ds)$ are positive and finite. Thus, $\psi(\omega, \tau) \in \mathbb{R}$. Notice also that $\tau \in [0,1] \mapsto \psi(\omega,\tau)$ is continuous. Indeed, if $\tau \in [0,1]$, then the following holds:

1. If $\tau_n \downarrow \tau \in [0,1)$, $\psi(\omega, \tau_n) \to \psi(\omega,\tau)$, by dominated convergence and the continuity of measures.

2. If $\tau_n \uparrow \tau \in (0,1]$, then

$$\psi(\omega, \tau_n) \to 1 - \mu(\omega, [\tau, 1]) + \int_{[\tau,1]} \frac{\tau}{s}\mu(\omega, ds) = \psi(\omega,\tau) - \mu(\omega, \{\tau\}) + \mu(\omega, \{\tau\}) = \psi(\omega, \tau).$$

Moreover, it is trivial to show that:

$$\psi(\omega,\tau) = 1 - \int_\tau^1 \int_{(t,1]} \frac{1}{s}\mu(\omega, ds)dt, \text{ for any } \tau \in (0,1).$$

Consequently, $\psi(\omega, \cdot)$ admits right-derivatives in $(0,1)$, so that:

$$\psi'_+(\omega,\tau) = \int_{(\tau,1]} \frac{1}{s}\mu(\omega, ds), \text{ for any } \tau \in (0,1).$$

Since $\tau \in (0,1) \mapsto \psi'_+(\omega,\tau)$ is greater or equal to zero and non-increasing, $\psi(\omega, \cdot)$ is concave and non-decreasing.

For $\tau \in \{0,1\}$ is obvious that $\psi(\cdot, \tau)$ is $\mathcal{G}$-measurable. If $\tau \in (0,1)$, $\psi(\cdot,\tau) \in L^0(\Omega, \mathcal{G}, P)$ because μ is a transition kernel with respect to $\mathcal{G}$, and $\frac{\tau}{s}\mathbb{1}_{(\tau,1]}(s)$ is a bounded function – see Le Gall (2006). Hence, the map Φ is well-defined.

The inverse map of Φ is equally straightforwardly defined. For any $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, let $\psi'_+ : \Omega \times (0,1) \to \mathbb{R}_+$ be its right-derivative process, which exists since any path is concave and non-decreasing. For every $\omega \in \Omega$, $\tau \in (0,1) \mapsto \psi'_+(\omega,\tau)$ is, therefore, non-increasing, left-continuous with right-limits. Moreover, since it may be obtained as the limit of $\mathcal{G}$ measurable maps, $\psi'_+(\cdot,\tau) \in L^0(\Omega, \mathcal{G}, P)$ for any $\tau \in (0,1)$.

We can define the following transition kernel, $\nu : \Omega \times \mathcal{B}((0,1]) \to \mathbb{R}_+$, by $\nu(\omega, (\tau, 1])) := \psi'_+(\omega,\tau)$, for any $\tau \in (0,1)$ and $\omega \in \Omega$. Obviously, for any fixed $\omega \in \Omega$, $\nu(\omega, \cdot)$ has a unique extension as σ-finite measure on $\mathcal{B}((0,1])$. Additionally, ν is indeed a transition kernel, since for any $\mathcal{A} \in \mathcal{B}((0,1])$, then $\nu(\cdot, \mathcal{A})$ is the limit of differences of $\psi'_+(\cdot, \tau)$, for deterministic τ, assuring the $\mathcal{G}$-measurability of $\nu(\cdot, \mathcal{A})$.

With this transition kernel we can define $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ by:

$$\mu(\omega, \mathcal{A}) = \int_{\mathcal{A}} s\nu(\omega, ds), \text{ for any } \omega \in \Omega \text{ and } \mathcal{A} \in \mathcal{B}((0,1]).$$

Because it is the integral of a bounded, deterministic and positive function, μ is a transition kernel. Besides that, it is also a transition probability since:

$$\mu(\omega, (0,1]) = \int_{(0,1]} s\nu(\omega, ds) = \int_0^1 \nu(\omega, (\tau, 1])\, d\tau = \int_0^1 \psi'_+(\omega,\tau)\, d\tau = \psi(\omega, 1) - \psi(\omega, 0) = 1.$$

Moreover, this map is, indeed, the inverse of Φ, because the iteration of these maps, $\mu \mapsto \psi \mapsto \bar{\mu}$, implies:

$$\int_{(\tau,1]} \frac{1}{s}\mu(\omega, ds) = \psi'_+(\omega,\tau) = \int_{(\tau,1]} \frac{1}{s}\bar{\mu}(\omega, ds),$$

for any $\omega \in \Omega$ and $\tau \in (0,1)$. Hence, $\mu(\omega,\cdot) = \bar{\mu}(\omega,\cdot)$ are equal measures for any $\omega \in \Omega$, the map Φ is injective and $\Phi \circ \Phi^{-1} = \mathrm{Id}_{\mathcal{M}^{\mathcal{G}}_{(0,1]}}$.

On the other hand, the iteration $\Phi^{-1} \circ \Phi$, $\psi \mapsto \mu \mapsto \bar{\psi}$ will provide:

$$\psi'_+(\omega,\tau) = \int_{(\tau,1]} \frac{1}{s}\mu(\omega, ds) = \bar{\psi}'_+(\omega,\tau),$$

for any $\omega \in \Omega$ and $\tau \in (0,1)$. Since $\psi(\omega,0) = \bar{\psi}(\omega,0) = 0$, $\psi(\omega,1) = \bar{\psi}(\omega,1) = 1$, and both functions are continuous, then $\psi(\omega,\tau) = \bar{\psi}(\omega,\tau)$, for any $\omega \in \Omega$ and $\tau \in (0,1)$. From this, we conclude that Φ is surjective, with $\Phi^{-1} \circ \Phi = \mathrm{Id}_{\mathrm{Conc}(\Omega,\mathcal{G},[0,1])}$.

□

Proof of Theorem 3.3.13. Fixed $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, let $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ be the function obtained in Lemma 3.3.12. In the proof of Lemma 3.3.12, it was shown that $\psi'_+(\omega,\tau) = \int_{(\tau,1]} \frac{1}{s}\mu(\omega, ds)$, for any $\tau \in (0,1)$ and $\omega \in \Omega$. Therefore, given any fixed $X \in L^\infty(\Omega, \mathcal{F}, P)$, $X \geqslant 0$, we get that:

$$\begin{aligned}
\rho_\mu(-X)(\omega) &= \int_0^1 AV@R_\tau[-X|\mathcal{G}](\omega) d\mu(\omega,\tau) = \int_{(0,1]} \left(\frac{1}{\tau}\int_0^\tau Q_{1-s}[X|\mathcal{G}](\omega) ds\right) d\mu(\omega,\tau),\\
&= \int_0^1 Q_{1-s}[X|\mathcal{G}](\omega) \left(\int_{(s,1]} \frac{1}{\tau} d\mu(\omega,\tau)\right) ds = \int_0^1 Q_{1-s}[X|\mathcal{G}](\omega)\psi'_+(\omega,s) ds,\\
&= \int_0^1 Q_{(1-s)+}[X|\mathcal{G}](\omega)\psi'_+(\omega,s) ds,
\end{aligned}$$

since, for any $\omega \in \Omega$, the number of $s \in (0,1)$ such that $Q_{1-s}[X|\mathcal{G}](\omega) \neq Q_{(1-s)+}[X|\mathcal{G}](\omega)$ is at most countable. Moreover, by the definition of conditional quantile, and Proposition 2.2.9 item 1 in Chapter 2:

$$\begin{aligned}
\rho_\mu(-X) &= \int_0^1 Q_{(1-s)+}[X|\mathcal{G}](\omega)\psi'_+(\omega,s) ds = \int_0^1 \left(\int_0^{+\infty} \mathbb{1}_{[P[X\leqslant x|\mathcal{G}](\omega)\leqslant 1-s]}(x) dx\right) \psi'_+(\omega,s) ds,\\
&= \int_0^{+\infty} \left(\int_0^{1-P[X\leqslant x|\mathcal{G}](\omega)} \psi'_+(\omega,s) ds\right) dx = \int_0^{+\infty} \psi\left(\omega, P[X > x|\mathcal{G}](\omega)\right) dx, \text{ a.s.}
\end{aligned}$$

Now, if $X \in L^\infty(\Omega, \mathcal{F}, P)$ is fixed, then $X + \|X\|_\infty \geqslant 0$. Proposition 2.2.9 item 2 in Chapter 2 ensures that, in a $\mathcal{G}$-set of probability one, $P[X > \|X\|_{+\infty}|\mathcal{G}] = 0$ and $P[X > -\|X\|_{+\infty}|\mathcal{G}] = 1$. Conditional translational invariance, $\psi(\cdot,0) = 0$ and $\psi(\cdot,1) = 1$ imply, then, that the following holds a.s.:

$$\begin{aligned}
\|X\|_\infty + \rho_\mu(-X) &= \int_0^{+\infty} \psi\left(\omega, P[X + \|X\|_\infty > x|\mathcal{G}](\omega)\right) dx = \int_0^{+\infty} \psi\left(\omega, P[X > x - \|X\|_\infty|\mathcal{G}](\omega)\right) dx,\\
&= \int_{-\|X\|_\infty}^{+\infty} \psi\left(\omega, P[X > x|\mathcal{G}](\omega)\right) dx,\\
&= \|X\|_\infty + \int_{-\infty}^0 \left(\psi\left(\omega, P[X > x|\mathcal{G}](\omega)\right) - 1\right) dx + \int_0^{+\infty} \psi\left(\omega, P[X > x|\mathcal{G}](\omega)\right) dx,
\end{aligned}$$

proving the desired result. □

Proof of Corollary 3.3.16. Given $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$ and letting and $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ be its corresponding concave distortion as in Lemma 3.3.12. Then, Corollary 3.3.10 guarantees that the conditional risk measure, $\rho_\mu : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$,

$$\rho_\mu(X) = \int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

is conditionally coherent, conditionally law-invariant and continuous. Thus, Theorem 3.3.1 implies that ρ_μ admits robust representation given by

$$\rho_\mu(X) = \operatorname*{esssup}_{Q\in\mathcal{Q}} E^Q[-X|\mathcal{G}], \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{Q} \subset \mathcal{P}_{\mathcal{G}}$.

Fixed any $Q \in \mathcal{Q}$ and $X \in L^\infty(\Omega, \mathcal{F}, P)$, Corollary 3.3.5 and the proof of Theorem 3.3.13 ensure that:

$$\int_0^1 Q_\tau\left[X|\mathcal{G}\right] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau \leqslant \rho_\mu(-X) = \int_0^1 Q_{1-\tau}[X|\mathcal{G}](\omega)\psi'_+(\omega,\tau)d\tau, \text{ a.s.}$$

Since there exists $U \in L^\infty(\Omega, \mathcal{F}, P)$, so that $P[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s., for any $t \in (0,1)$, the random variable $X = \mathbb{1}_{[\tau\leqslant U\leqslant 1]} \in L^\infty(\Omega, \mathcal{F}, P)$ is such that $Q_\tau[X|\mathcal{G}] = \mathbb{1}_{[t\leqslant\tau\leqslant 1]}$, for any $\tau \in (0,1)$, in a $\mathcal{G}$-measurable set with full probability. Therefore, there exists a set $\Omega' \in \mathcal{G}$, with $P[\Omega'] = 1$, so that in Ω':

$$\int_t^1 Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right](\omega)d\tau \leqslant \rho_\mu(-X) = \int_t^1 \psi'_+(\omega, 1-\tau)d\tau = \psi(\omega, 1-t), \text{ for any } t \in (0,1).$$

Now, suppose that $Q \in \mathcal{P}_{\mathcal{G}}$, and the above inequality holds for any $t \in (0,1)$ in a fixed $\mathcal{G}$-measurable set with probability one. Then, because $Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right]$ is left-continuous, we obtain that:

$$Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] = \lim_{h\downarrow 0}\frac{1}{h}\int_{\tau-h}^{\tau} Q_s\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] ds \leqslant \lim_{h\downarrow 0}\frac{\psi(\cdot, 1-\tau+h) - \psi(\cdot, 1-\tau)}{h} = \psi'_+(\cdot, 1-\tau),$$

for any $\tau \in (0,1)$, a.s.

Recall that, from Proposition 2.2.9 in Chapter 2, for any negative $X \in L^\infty(\Omega, \mathcal{F}, P)$, $Q_\tau[-X|\mathcal{G}] \geqslant 0$, for any $\tau \in (0,1)$, in a fixed $\mathcal{G}$-measurable set of full probability. Therefore,

$$\int_0^1 Q_\tau\left[-X|\mathcal{G}\right] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau \leqslant \int_0^1 Q_{1-s}[-X|\mathcal{G}](\omega)\psi'_+(\omega,s)ds = \rho_\mu(X), \text{ a.s.}$$

Translational invariance of both sides of the above equation establishes the inequality for any $X \in L^\infty(\Omega, \mathcal{F}, P)$. By Theorem 3.3.4, the penalty function α_* associated to ρ_μ in Q is:

$$\alpha_*(Q) = \sup_{X\in L^\infty(\Omega,\mathcal{F},P)}\left(\int_0^1 Q_\tau\left[X|\mathcal{G}\right] Q_\tau\left[\frac{dQ}{dP}\Big|\mathcal{G}\right] d\tau - \rho_\mu(X)\right) \leqslant 0,$$

which forces $\alpha_*(Q) = 0$, and we conclude that $Q \in \mathcal{Q}$. $\square$

Proof of Example 3.3.17. Notice first that $\mathsf{AV@R}_\Lambda[\cdot|\mathcal{G}]$ is of the form ρ_μ, with $\mu = \delta_\Lambda$. Thus,

$$\psi(\omega, \tau) = \frac{\tau}{\Lambda(w)} \mathbb{1}_{[\tau \in [0, \Lambda(\omega)]]} + \mathbb{1}_{[\tau \in (\Lambda(\omega), 1]]}$$

Observe then that $\psi(\omega, \cdot)$ is differentiable at 0 for any $\omega \in \Omega$, and its derivative is $\psi'(\omega, 0) = \frac{1}{\Lambda(\omega)}$.

Let $\mathsf{Q} \in \mathcal{Q}$, and suppose that

$$\mathsf{P}\left[\mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda}\middle|\mathcal{G}\right] > 0\right] > 0.$$

This assumption implies that there exists a set $\Omega' \in \mathcal{G}$, with full probability, such that:

$$\mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda}\middle|\mathcal{G}\right](\omega) = \mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda(\omega)}\middle|\mathcal{G}\right](\omega), \text{ for any } \omega \in \Omega'.$$

Then, for every $\omega \in \left[\mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda}\middle|\mathcal{G}\right] > 0\right] \cap \Omega'$ fixed, there exists a $\tau \in (0,1)$ such that:

$$\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right](\omega) > \frac{1}{\Lambda(\omega)},$$

by the definition of $\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right]$. We can, without loss of generality, assume that $0 < 1-\tau < \Lambda(\omega)$ and $\tau > \frac{1}{2}$. Therefore,

$$\frac{\tau}{\Lambda(\omega)} < \int_{1-\tau}^{1} \mathsf{Q}_s\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right](\omega)ds \leqslant \psi(\omega, 1-\tau) = \frac{1-\tau}{\Lambda(\omega)},$$

which is an absurd. Thus,

$$\mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda}\middle|\mathcal{G}\right] = 0, \text{ a.s.},$$

from where we conclude that

$$\mathsf{P}\left[\frac{d\mathsf{Q}}{d\mathsf{P}} > \frac{1}{\Lambda}\right] = 0.$$

On the other hand, if $\mathsf{Q} \in \left\{\mathsf{Q} \in \mathcal{P}_\mathcal{G} : \frac{d\mathsf{Q}}{d\mathsf{P}} \leqslant \frac{1}{\Lambda} \text{ a.s.}\right\}$, then:

$$\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] \leqslant \frac{1}{\Lambda} \text{ for any } \tau \in (0,1), \text{ a.s.},$$

by Propositions 2.2.9 items 3 and 5 in Chapter 2. Consequently,

$$\mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right]\mathbb{1}_{[\tau \in [1-\Lambda, 1]]} \leqslant \frac{1}{\Lambda}\mathbb{1}_{[\tau \in [1-\Lambda, 1]]}, \text{ a.s.}$$

Recall, from Proposition 2.2.9 items 1 and 6 in Chapter 2, and $\mathsf{E}\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] = 1$ a.s., that

$$\int_0^1 \mathsf{Q}_\tau\left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau = 1, \text{ a.s.},$$

and $\tau \in (0,1) \mapsto \mathsf{Q}_\tau \left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right]$ is non-decreasing and positive.

Hence, we may conclude that

$$\int_t^1 \mathsf{Q}_\tau \left[\frac{d\mathsf{Q}}{d\mathsf{P}}\middle|\mathcal{G}\right] d\tau \leqslant \int_t^1 \frac{1}{\Lambda}\mathbb{1}_{[\tau\in[1-\Lambda,1]]} d\tau, \text{ for any } t \in (0,1), \text{ a.s.}$$

Finally, since

$$\psi(\cdot, 1-t) = \int_t^1 \frac{1}{\Lambda}\mathbb{1}_{[\tau\in[1-\Lambda,1]]} d\tau,$$

we conclude that $\mathsf{Q} \in \mathcal{Q}$ by Corollary 3.3.16. $\square$

Proof of Theorem 3.3.14. Let $\rho : L^\infty(\Omega, \mathcal{F}, \mathsf{P}) \to L^\infty(\Omega, \mathcal{G}, \mathsf{P})$ be convex, continuous from above conditional risk measure. Theorem 3.3.9 ensures that ρ is conditionally law-invariant if, and only, there exists $\beta_* : \mathcal{M}^{\mathcal{G}}_{(0,1]} \to L^0(\Omega, \mathcal{G}, \mathsf{P}; \bar{\mathbb{R}}_+)$ such that:

$$\rho(X) = \operatorname*{esssup}_{\mu\in\mathcal{M}^{\mathcal{G}}_{(0,1]}} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) - \beta_*(\mu)\right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, \mathsf{P}).$$

Fix $X \in L^\infty(\Omega, \mathcal{F}, \mathsf{P})$. Recall from Theorem 3.3.13 that, for any $\mu \in \mathcal{M}^{\mathcal{G}}_{(0,1]}$, there exists a unique $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, such that:

$$\int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) = \int_{-\infty}^0 \psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) dx + \int_0^{+\infty} \left(\psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) - 1\right) dx, \text{ a.s.}$$

Consequently, we obtain that:

$$\begin{aligned}
\beta_*(\mu) &= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},\mathsf{P})} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) - \rho(X)\right) \\
&= \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},\mathsf{P})} \left(\int_{-\infty}^0 \psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) dx + \int_0^{+\infty} \left(\psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) - 1\right) dx - \rho(X)\right).
\end{aligned}$$

Because $\mathcal{M}^{\mathcal{G}}_{(0,1]}$ and $\mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ are bijectively related, due to Lemma 3.3.12, the above identity enables us to define a map $\gamma_* : \mathrm{Conc}(\Omega, \mathcal{G}, [0,1]) \to L^0(\Omega, \mathcal{G}, \mathsf{P}; \bar{\mathbb{R}}_+)$ satisfying $\gamma_*(\psi) = \beta_*\left(\Phi^{-1}(\psi)\right)$ and:

$$\gamma_*(\psi) = \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F},\mathsf{P})} \left(\int_{-\infty}^0 \psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) dx + \int_0^{+\infty} \left(\psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) - 1\right) dx - \rho(X)\right),$$

for any $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$. Moreover, we also derive that, for any $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, if $\mu = \Phi^{-1}(\psi)$, then:

$$\int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) - \beta_*(\mu) = \int_{-\infty}^0 \psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) dx + \int_0^{+\infty} \left(\psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) - 1\right) dx - \gamma_*(\psi), \text{ a.s.}$$

Taking the essential supremum in the above identity and, using the fact that $\mathcal{M}^{\mathcal{G}}_{(0,1]}$ an $\mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$ are bijectively related through Φ, we conclude that:

$$\begin{aligned}
\rho(X) &= \operatorname*{esssup}_{\mu\in\mathcal{M}^{\mathcal{G}}_{(0,1]}} \left(\int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) - \beta_*(\mu)\right), \\
&= \operatorname*{esssup}_{\psi\in\mathrm{Conc}(\Omega,\mathcal{G},[0,1])} \left(\int_{-\infty}^0 \psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) dx + \int_0^{+\infty} \left(\psi\left(\cdot, \mathsf{P}[X < x|\mathcal{G}]\right) - 1\right) dx - \gamma_*(\psi)\right),
\end{aligned}$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$, and the result is proved.

□

Proof of Corollary 3.3.15. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ be coherent and continuous from above conditional risk measure. Then, Theorem 3.3.14 ensures that ρ is conditionally law-invariant if, and only if, there exists $\gamma_* : \mathrm{Conc}(\Omega, \mathcal{G}, [0,1]) \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}}_+)$ satisfying $\gamma_*(\psi) = \beta_*\left(\Phi^{-1}(\psi)\right)$, where Φ is the bijection in Lemma 3.3.12 and β_* is the penalty function in Theorem 3.3.9, and:

$$\gamma_*(\psi) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int_{-\infty}^{0} \psi\left(\cdot, P[X < x|\mathcal{G}]\right) dx + \int_{0}^{+\infty} \left(\psi\left(\cdot, P[X < x|\mathcal{G}]\right) - 1\right) dx - \rho(X) \right),$$

for any $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$. Moreover, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$,

$$\rho(X) = \operatorname*{esssup}_{\psi \in \mathrm{Conc}(\Omega,\mathcal{G},[0,1])} \left(\int_{-\infty}^{0} \psi\left(\cdot, P[X < x|\mathcal{G}]\right) dx + \int_{0}^{+\infty} \left(\psi\left(\cdot, P[X < x|\mathcal{G}]\right) - 1\right) dx - \gamma_*(\psi) \right).$$

Now, Theorem 3.3.10 also guarantees that, for any $\mu \in \mathcal{M}^{\mathcal{G}}_{[0,1]}$, either $\beta_*(\mu) = 0$ or $+\infty$. Moreover $\beta(\mu) = 0$ if, and only if, $\mu \in \mathcal{M}$. Since $\gamma_*(\psi) = \beta_*\left(\Phi^{-1}(\psi)\right)$ and Φ is a bijection, if we define $\mathcal{C} = \Phi(\mathcal{M})$, then $\gamma_*(\psi) = 0$ or $+\infty$, for any $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, so that $\gamma_*(\psi)$ equals 0 if, and only if, $\psi \in \mathcal{C}$.

Therefore, we conclude that:

$$\rho(X) = \operatorname*{esssup}_{\psi \in \mathcal{C}} \left(\int_{-\infty}^{0} \psi\left(\cdot, P[X < x|\mathcal{G}]\right) dx + \int_{0}^{+\infty} \left(\psi\left(\cdot, P[X < x|\mathcal{G}]\right) - 1\right) dx \right),$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$, proving the desired result.

□

B.2.5 Proofs of Section 3.3.4

In order to demonstrate Theorem 3.3.23, we need first the following technical lemma. Similarly to when $\mathcal{G} = \{\emptyset, \Omega\}$, Lemma B.2.1 will be useful when computing Choquet integrals of simple variables.

Lemma B.2.1. *Let $X, Y \in L^\infty(\Omega, \mathcal{G}, P)$, with $0 \leqslant X \leqslant Y$ a.s., and $A \in \mathcal{F}$. Then X and $(Y - X)\mathbb{1}_A$ are $\mathcal{G}$-comonotonic random variables. Moreover, for any pair of sets $A, B \in \mathcal{F}$, then $\mathbb{1}_{A\cap B}$ and $\mathbb{1}_{A\cup B}$ are $\mathcal{G}$-comonotonic.*

Proof of Lemma B.2.1. First observe that we can construct a set $\Omega' \in \mathcal{G}$, with probability one, such that the following holds on Ω':

1. $P\left[(Y - X)\mathbb{1}_A \leqslant q_1, X \leqslant q_2 |\mathcal{G}\right] =:$

q_2 \ q_1	$(-\infty, 0)$	$\left[0, \lVert Y - X\rVert_\infty\right)$	$\left[\lVert Y - X\rVert_\infty, +\infty\right)$
$(-\infty, 0)$	0	0	0
$\left[0, \lVert X\rVert_\infty\right)$	0	$E[\mathbb{1}_A\lvert\mathcal{G}]\mathbb{1}_{[Y-X\leqslant q_1, X\leqslant q_2]}$	$1_{[X\leqslant q_2]}$
$\left[\lVert X\rVert_\infty, +\infty\right)$	0	$E[\mathbb{1}_A\lvert\mathcal{G}]\mathbb{1}_{[Y-X\leqslant q_1]}$	1

2. $\mathsf{P}\left[(\mathsf{Y}-\mathsf{X})\mathbb{1}_{\mathsf{A}} \leqslant \mathsf{q}_1|\mathcal{G}\right] =:$

$q_2 \backslash q_1$	$(-\infty, 0)$	$\left[0, \|\mathsf{Y}-\mathsf{X}\|_\infty\right)$	$\left[\|\mathsf{Y}-\mathsf{X}\|_\infty, +\infty\right)$
$(-\infty, 0)$	0	$\mathsf{E}[\mathbb{1}_{\mathsf{A}}\|\mathcal{G}]\mathbb{1}_{[\mathsf{Y}-\mathsf{X}\leqslant \mathsf{q}_1]}$	1
$\left[0, \|\mathsf{X}\|_\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{\mathsf{A}}\|\mathcal{G}]\mathbb{1}_{[\mathsf{Y}-\mathsf{X}\leqslant \mathsf{q}_1]}$	1
$\left[\|\mathsf{X}\|_\infty, +\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{\mathsf{A}}\|\mathcal{G}]\mathbb{1}_{[\mathsf{Y}-\mathsf{X}\leqslant \mathsf{q}_1]}$	1

3. $\mathsf{P}\left[\mathsf{X} \leqslant \mathsf{q}_2|\mathcal{G}\right] =:$

$q_2 \backslash q_1$	$(-\infty, 0)$	$\left[0, \|\mathsf{Y}-\mathsf{X}\|_\infty\right)$	$\left[\|\mathsf{Y}-\mathsf{X}\|_\infty, +\infty\right)$
$(-\infty, 0)$	0	0	0
$\left[0, \|\mathsf{X}\|_\infty\right)$	$1_{[\mathsf{X}\leqslant \mathsf{q}_2]}$	$1_{[\mathsf{X}\leqslant \mathsf{q}_2]}$	$1_{[\mathsf{X}\leqslant \mathsf{q}_2]}$
$\left[\|\mathsf{X}\|_\infty, +\infty\right)$	1	1	1

4. $\mathsf{P}\left[(\mathsf{Y}-\mathsf{X})\mathbb{1}_{\mathsf{A}} \leqslant \mathsf{q}_1|\mathcal{G}\right] \wedge \mathsf{P}\left[\mathsf{X} \leqslant \mathsf{q}_2|\mathcal{G}\right] =:$

$q_2 \backslash q_1$	$(-\infty, 0)$	$\left[0, \|\mathsf{Y}-\mathsf{X}\|_\infty\right)$	$\left[\|\mathsf{Y}-\mathsf{X}\|_\infty, +\infty\right)$
$(-\infty, 0)$	0	0	0
$\left[0, \|\mathsf{X}\|_\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{\mathsf{A}}\|\mathcal{G}]\mathbb{1}_{[\mathsf{Y}-\mathsf{X}\leqslant \mathsf{q}_1, \mathsf{X}\leqslant \mathsf{q}_2]}$	$1_{[\mathsf{X}\leqslant \mathsf{q}_2]}$
$\left[\|\mathsf{X}\|_\infty, +\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{\mathsf{A}}\|\mathcal{G}]\mathbb{1}_{[\mathsf{Y}-\mathsf{X}\leqslant \mathsf{q}_1]}$	1

Consequently, because $\mathsf{P}[(\mathsf{Y}-\mathsf{X})\mathbb{1}_{\mathsf{A}} \leqslant \mathsf{q}_1, \mathsf{X} \leqslant \mathsf{q}_2|\mathcal{G}] = \mathsf{P}[(\mathsf{Y}-\mathsf{X})\mathbb{1}_{\mathsf{A}} \leqslant \mathsf{q}_1|\mathcal{G}] \wedge \mathsf{P}[\mathsf{X} \leqslant \mathsf{q}_2|\mathcal{G}]$ in a $\mathcal{G}$-measurable set with full probability, Lemma 2.3.3 in Chapter 2 ensures that these random variables are $\mathcal{G}$-comonotonic.

Now, if $\mathsf{A}, \mathsf{B} \in \mathcal{F}$, we can repeat the construction above to obtain a $\mathcal{G}$-measurable set, with full probability, such that in it the following holds:

1. $\mathsf{P}\left[\mathbb{1}_{\mathsf{A}\cup\mathsf{B}} \leqslant \mathsf{q}_1, 1_{\mathsf{A}\cap\mathsf{B}} \leqslant \mathsf{q}_2|\mathcal{G}\right] =:$

$q_2 \backslash q_1$	$(-\infty, 0)$	$[0, 1)$	$[1, +\infty)$
$(-\infty, 0)$	0	0	0
$\left[0, 1\right)$	0	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cup\mathsf{B})^c}\|\mathcal{G}]$	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cap\mathsf{B})^c}\|\mathcal{G}]$
$\left[1, +\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cup\mathsf{B})^c}\|\mathcal{G}]$	1

2. $\mathsf{P}\left[\mathbb{1}_{\mathsf{A}\cup\mathsf{B}} \leqslant \mathsf{q}_1|\mathcal{G}\right] =:$

$q_2 \backslash q_1$	$(-\infty, 0)$	$[0, 1)$	$[1, +\infty)$
$(-\infty, 0)$	0	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cup\mathsf{B})^c}\|\mathcal{G}]$	1
$\left[0, 1\right)$	0	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cup\mathsf{B})^c}\|\mathcal{G}]$	1
$\left[1, +\infty\right)$	0	$\mathsf{E}[\mathbb{1}_{(\mathsf{A}\cup\mathsf{B})^c}\|\mathcal{G}]$	1

3. $\mathsf{P}\left[\mathbb{1}_{\mathsf{A}\cap\mathsf{B}} \leqslant \mathsf{q}_2|\mathcal{G}\right] =:$

q_2 \ q_1	$(-\infty,0)$	$[0,1)$	$[1,+\infty)$
$(-\infty,0)$	0	0	0
$[0,1)$	$E[\mathbb{1}_{(A\cap B)^c}\|\mathcal{G}]$	$E[\mathbb{1}_{(A\cap B)^c}\|\mathcal{G}]$	$E[\mathbb{1}_{(A\cap B)^c}\|\mathcal{G}]$
$[1,+\infty)$	1	1	1

4. $P\left[\mathbb{1}_{A\cup B} \leqslant q_1|\mathcal{G}\right] \wedge P\left[\mathbb{1}_{A\cap B} \leqslant q_2|\mathcal{G}\right] =:$

q_2 \ q_1	$(-\infty,0)$	$[0,1)$	$[1,+\infty)$
$(-\infty,0)$	0	0	0
$[0,1)$	0	$E[\mathbb{1}_{(A\cup B)^c}\|\mathcal{G}]$	$E[\mathbb{1}_{(A\cap B)^c}\|\mathcal{G}]$
$[1,+\infty)$	0	$E[\mathbb{1}_{(A\cup B)^c}\|\mathcal{G}]$	1

Then, $P[(Y-X)\mathbb{1}_A \leqslant q_1, X \leqslant q_2|\mathcal{G}] = P[(Y-X)\mathbb{1}_A \leqslant q_1|\mathcal{G}] \wedge P[X \leqslant q_2|\mathcal{G}]$ in a $\mathcal{G}$-measurable set with full probability. Thus, Lemma 2.3.3 in Chapter 2 implies the result.

□

Proof of Theorem 3.3.23. ($\Rightarrow$) Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ be conditionally coherent, conditionally law-invariant, continuous from above and conditionally comonotonic risk measure. Besides, let $U \in L^\infty(\Omega, \mathcal{F}, P)$ be such that $P[U \in \cdot|\mathcal{G}] = U(0,1)$ a.s. Then, there exists a set $\Omega' \in \mathcal{G}$, so that $P[\Omega'] = 1$ and, for any pair $(q, q') \in \mathbb{Q}^2 \cap [0,1]^2$, with $q < q'$, we have:

1. Conditional law-invariance:

$$\rho\left(-\mathbb{1}_{[U\leqslant q]}\right)(\omega) + \rho\left(-\mathbb{1}_{[U\leqslant q']}\right)(\omega) = \rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant \frac{q'+q}{2}\right]}\right)(\omega) + \rho\left(-\mathbb{1}_{[U\leqslant q']}\right)(\omega),$$

2. Conditional comononoticity of $\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant \frac{q'+q}{2}\right]}$ and $\mathbb{1}_{[U\leqslant q']}$, taking $A = \left[U \leqslant \frac{q'+q}{2}\right]$ and $B = \left[\frac{q'-q}{2} \leqslant U \leqslant q'\right]$ in Lemma B.2.1:

$$\rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant \frac{q'+q}{2}\right]}\right)(\omega) + \rho\left(-\mathbb{1}_{[U\leqslant q']}\right)(\omega) = \rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant \frac{q'+q}{2}\right]} - \mathbb{1}_{[U\leqslant q']}\right)(\omega),$$

3. $\mathbb{1}_{A\cup B} + \mathbb{1}_{A\cap B} = \mathbb{1}_A + \mathbb{1}_B$:

$$\rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant \frac{q'+q}{2}\right]} - \mathbb{1}_{[U\leqslant q']}\right)(\omega) = \rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]} - \mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant q'\right]}\right)(\omega),$$

4. Conditional coherence:

$$\rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]} - \mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant q'\right]}\right)(\omega) \leqslant \rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]}\right)(\omega) + \rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant q'\right]}\right)(\omega),$$

5. Conditional law-invariance:

$$\rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]}\right)(\omega) + \rho\left(-\mathbb{1}_{\left[\frac{q'-q}{2}\leqslant U\leqslant q'\right]}\right)(\omega) = \rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]}\right)(\omega) + \rho\left(-\mathbb{1}_{\left[U\leqslant \frac{q'+q}{2}\right]}\right)(\omega),$$

Thus,

$$\rho\left(-\mathbb{1}_{[U\leqslant q]}\right)(\omega)+\rho\left(-\mathbb{1}_{[U\leqslant q']}\right)(\omega)\leqslant 2\rho\left(-\mathbb{1}_{\left[\frac{q'+q}{2}\leqslant U\leqslant q'\right]}\right)(\omega),\ \text{for all } \omega\in\Omega'.$$

Moreover, we can also require that, in Ω', the following holds due to monotonicity, coherence and normalization:

$$\begin{aligned}\rho\left(-\mathbb{1}_{[U\leqslant q]}\right)(\omega)&\leqslant\rho\left(-\mathbb{1}_{[U\leqslant q']}\right)(\omega),\\ \rho(1)(\omega)&=1,\\ \rho(0)(\omega)&=0,\end{aligned}$$

This allows us to define the random function, $\psi:\Omega\times[0,1]\to[0,1]$, by:

$$\psi(\omega,\tau)=\begin{cases}\rho(-1_{[U\leqslant\tau]})(\omega), & \text{if } \omega\in\Omega' \text{ and } \tau\in\mathbb{Q}\cap[0,1],\\ \lim_{q\downarrow\tau}\psi(\omega,q), & \text{if } \omega\in\Omega' \text{ and } \tau\in[0,1]\cap\mathbb{Q}^c,\\ \tau, & \text{otherwise.}\end{cases}$$

Notice that ψ has the following properties:

1. For any $\omega\in\Omega$, $\psi(\omega,0)=0$ and $\psi(\omega,1)=1$. Moreover, $\tau\in[0,1]\mapsto\psi(\omega,\tau)$ is non-decreasing.
2. For any $\tau,\tau'\in[0,1]$, such that $\tau<\tau'$, and $\omega\in\Omega$, then:

$$\frac{\psi(\omega,\tau)+\psi(\omega,\tau')}{2}\leqslant\psi\left(\omega,\frac{\tau+\tau'}{2}\right).$$

   Thus, $\tau\in[0,1]\mapsto\psi(\omega,\tau)$ is concave and, consequently, continuous, for every $\omega\in\Omega$.
3. For any $\tau\in[0,1]$, $\psi(\cdot,\tau)\in L^\infty(\Omega,\mathcal{G},P)$. Hence, ψ is $\mathcal{G}\otimes\mathcal{B}([0,1])$-measurable. Moreover, there exists $\psi'_+:\Omega\times(0,1)\to\mathbb{R}_+$, which is also $\mathcal{G}\otimes\mathcal{B}((0,1))$-measurable, right-continuous with left-limits and integrable, i.e

$$\int_0^1\psi'_+(\omega,\tau)d\tau=1,\text{for any } \omega\in\Omega.$$

4. For any $A\in\mathcal{B}([0,1])$, let $\lambda(A)$ be its Lebesgue measure. Then it is trivial to show that $\mathbb{1}_{[U\in\mathcal{A}]}\sim_{\mathcal{G}}\mathbb{1}_{[U\leqslant\lambda(A)]}$. Conditional law-invariance of ρ implies that:

$$\begin{aligned}\rho\left(-\mathbb{1}_{[U\in A]}\right)&=\rho\left(-\mathbb{1}_{[U\leqslant\lambda(A)]}\right),\\ &=\psi(\cdot,\lambda(A)),\\ &=\int_{1-\lambda(A)}^1\psi'_+(\cdot,1-\tau)d\tau,\\ &=\int_0^1 Q_\tau[\mathbb{1}_{[U\in A]}|\mathcal{G}]\psi'_+(\cdot,1-\tau)d\tau,\ \text{a.s.}\end{aligned}$$

Furthermore, Lemma B.2.1, the conditional comonotonicity, conditional coherence, conditional law-invariance and item 4 above imply that, for every $X\leqslant Y\in L^\infty(\Omega,\mathcal{G},P)$ and

$A, B \in \mathcal{B}([0,1])$, such that $A \cup B = [0,1]$:

$$\begin{aligned}\rho\left(-X\mathbb{1}_{[U\in A]} - Y\mathbb{1}_{[U\in B]}\right) &= \rho\left(-(Y-X)\mathbb{1}_{[U\in B]} - X\mathbb{1}_{[U\in A\cup B]}\right),\\ &= \rho\left(-(Y-X)\mathbb{1}_{[U\in B]}\right) + \rho\left(-X\mathbb{1}_{[U\in A\cup B]}\right),\\ &= \int_0^1 Q_\tau[(Y-X)\mathbb{1}_{[U\in B]}|\mathcal{G}]\psi'_+(\cdot, 1-\tau)d\tau +\\ &+ \int_0^1 Q_\tau[X\mathbb{1}_{[U\in A\cup B]}|\mathcal{G}]\psi'_+(\cdot, 1-\tau)d\tau,\\ &= \int_0^1 \left(Q_\tau[(Y-X)\mathbb{1}_{[U\in B]}|\mathcal{G}] + Q_\tau[X\mathbb{1}_{[U\in A\cup B]}|\mathcal{G}]\right)\psi'_+(\cdot, 1-\tau)d\tau, \text{ a.s.}\end{aligned}$$

Now, due to Corollary 2.3.5 in Chapter 2,

$$\begin{aligned}Q_\tau[(Y-X)\mathbb{1}_{[U\in B]}|\mathcal{G}] + Q_\tau[X\mathbb{1}_{[U\in A\cup B]}|\mathcal{G}] &= Q_\tau[(Y-X)\mathbb{1}_{[U\in B]} + X\mathbb{1}_{[U\in A\cup B]}|\mathcal{G}],\\ &= Q_\tau[Y\mathbb{1}_{[U\in B]} + X\mathbb{1}_{[U\in A]}|\mathcal{G}], \text{ for every } \tau \in (0,1), \text{ a.s.}\end{aligned}$$

Therefore,

$$\rho\left(-X\mathbb{1}_{[U\in A]} - Y\mathbb{1}_{[U\in B]}\right) = \int_0^1 Q_\tau[Y\mathbb{1}_{[U\in B]} + X\mathbb{1}_{[U\in A]}|\mathcal{G}]\psi'_+(\cdot, 1-\tau)d\tau, \text{ a.s.}$$

Proceeding inductively, for any random variable $\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]}$, where $(X_n)_{n\leqslant m} \subset L^\infty(\Omega, \mathcal{G}, P)$, $X_n \leqslant X_{n+1}$, and $(A_n)_{n\leqslant m} \subset \mathcal{B}([0,1])$ disjoint decomposition of Ω, then:

$$\rho\left(-\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]}\right) = \int_0^1 Q_\tau\left[\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]}\middle|\mathcal{G}\right]\psi'_+(\cdot, 1-\tau)d\tau, \text{ a.s.}$$

Now, if $X \in L^\infty(\Omega, \mathcal{F}, P)$ is positive, then there exists a sequence of random variables as above, so that:

$$\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]} \downarrow Q_U[X|\mathcal{G}], \text{ a.s.}$$

As a consequence, Theorem 2.4.1 in Chapter 2 guarantees that there exists a $\mathcal{G}$-measurable set with full probability, such that in this set:

$$\begin{aligned}Q_\tau[Q_U[X|\mathcal{G}]|\mathcal{G}] &\leqslant \liminf_{m\in\mathbb{N}} Q_\tau\left[\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]}\middle|\mathcal{G}\right] \leqslant \limsup_{m\in\mathbb{N}} Q_\tau\left[\sum_{i=1}^m X_n\mathbb{1}_{[U\in A_n]}\middle|\mathcal{G}\right],\\ &\leqslant Q_{\tau+}\left[Q_U[X|\mathcal{G}]\middle|\mathcal{G}\right], \text{ for any } \tau \in (0,1).\end{aligned}$$

Since $Q_U[X|\mathcal{G}] \sim_{\mathcal{G}} X$, then $Q_\tau[Q_U[X|\mathcal{G}]|\mathcal{G}] = Q_\tau[X|\mathcal{G}]$ for any $\tau \in (0,1)$, in a $\mathcal{G}$-measurable set of probability one. Thus, because of Fatou's Lemma for integrals and $\psi'_+(\cdot, \tau) \geqslant 0$, we get that:

$$\begin{aligned}
\int_0^1 Q_\tau[X|\mathcal{G}]\psi'_+(\cdot,1-\tau)d\tau &\leqslant \int_0^1 \left(\liminf_{m\in\mathbb{N}} Q_\tau\left[\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\Big|\mathcal{G}\right]\right)\psi'_+(\cdot,1-\tau)d\tau,\\
&\leqslant \liminf_{m\in\mathbb{N}} \int_0^1 Q_\tau\left[\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\Big|\mathcal{G}\right]\psi'_+(\cdot,1-\tau)d\tau,\\
&= \liminf_{m\in\mathbb{N}} \rho\left(-\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\right),\\
&\leqslant \limsup_{m\in\mathbb{N}} \rho\left(-\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\right),\\
&= \limsup_{m\in\mathbb{N}} \int_0^1 Q_\tau\left[\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\Big|\mathcal{G}\right]\psi'_+(\cdot,1-\tau)d\tau,\\
&\leqslant \int_0^1 \left(\limsup_{m\in\mathbb{N}} Q_\tau\left[\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\Big|\mathcal{G}\right]\right)\psi'_+(\cdot,1-\tau)d\tau,\\
&\leqslant \int_0^1 \left(Q_{\tau+}\left[\limsup_{m\in\mathbb{N}} \sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\Big|\mathcal{G}\right]\right)\psi'_+(\cdot,1-\tau)d\tau,\\
&= \int_0^1 Q_{\tau+}\left[Q_U[X|\mathcal{G}]\Big|\mathcal{G}\right]\psi'_+(\cdot,1-\tau)d\tau,\\
&= \int_0^1 Q_{\tau+}\,[X|\mathcal{G}]\,\psi'_+(\cdot,1-\tau)d\tau,\\
&= \int_0^1 Q_\tau\,[X|\mathcal{G}]\,\psi'_+(\cdot,1-\tau)d\tau,\ \text{a.s.}
\end{aligned}$$

Continuity from above of ρ ensures that:

$$\rho\left(-Q_U[X|\mathcal{G}]\right) = \lim_{m\in\mathbb{N}} \rho\left(-\sum_{i=1}^m X_n \mathbb{1}_{[U\in A_n]}\right) = \int_0^1 Q_\tau\,[X|\mathcal{G}]\,\psi'_+(\cdot,1-\tau)d\tau,\ \text{a.s.}$$

On the other hand, conditional law-invariance implies that $\rho(-X) = \rho\left(-Q_U[X|\mathcal{G}]\right)$. Therefore,

$$\rho(-X) = \int_0^1 Q_\tau\,[X|\mathcal{G}]\,\psi'_+(\cdot,1-\tau)d\tau,\ \text{a.s.}$$

If $X \in L^\infty(\Omega,\mathcal{F},P)$, then conditional translational invariance of the above equality when applied to $X + \|X\|_\infty$ is enough to conclude that:

$$\rho(X) = \int_0^1 Q_\tau\,[-X|\mathcal{G}]\,\psi'_+(\cdot,1-\tau)d\tau,\ \text{for any } X\in L^\infty(\Omega,\mathcal{F},P).$$

In the proof of Theorem 3.3.13 we demonstrated that the above representation is equivalent to, for every $X \in L^\infty(\Omega,\mathcal{F},P)$,

$$\rho(X) = \int_0^{+\infty} \left(\psi\left(\cdot, P[X<x|\mathcal{G}]\right) - 1\right)dx + \int_{-\infty}^0 \psi\left(\cdot, P[X<x|\mathcal{G}]\right)dx,\ \text{a.s.}$$

Then, we can define the following family of transition capacities, $C(\mathcal{G}) = \{c_X\}_{\{X \in L^\infty(\Omega,\mathcal{F},P;\mathbb{R}^d), d\in\mathbb{N}\}}$. For any $d \in \mathbb{N}$ and $X \in\in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^d)$:

$$c_X(\omega, A) = \psi(\omega, P[X \in A|\mathcal{G}](\omega)), \text{ for any } A \in \mathcal{B}(\mathbb{R}^d) \text{ and } \omega \in \Omega.$$

By construction, we get that:

$$\rho(X) = \int (-x) c_X(\omega, dx), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Besides, fixed $X \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^d)$, this map has the following properties.

1. For every $\omega \in \Omega$, $A \in \mathcal{B}(\mathbb{R}^d) \mapsto c_X(\omega, A)$ is monotone. Indeed, if $A \subset B$, then $P[X \in A|\mathcal{G}] \leqslant P[X \in B|\mathcal{G}]$. Moreover, $\tau \in [0,1] \mapsto \psi(\cdot, \tau)$ is non-decreasing. Thus, $c_X(\omega, A) \leqslant c_X(\omega, B)$.
2. For any $A \in \mathcal{B}(\mathbb{R}^d)$, then $\omega \in \Omega \mapsto c_X(\omega, A)$ is $\mathcal{G}$-measurable. To prove this, let $A \in \mathcal{B}(\mathbb{R}^d)$, then $\omega \in \Omega \mapsto P[X \in A|\mathcal{G}](\omega)$ is $\mathcal{G}$-measurable, since it is a transition kernel. Moreover, ψ is $\mathcal{G} \otimes \mathcal{B}([0,1])$-measurable. Hence, the composition of both maps is $\mathcal{G}$-measurable.
3. Let $K = \prod_{i=1}^d [-\|X_i\|_\infty, \|X_i\|_\infty]$, where $X = (X_1, \ldots, X_d)$. Then, it is trivial to show that $P[X \in K^c|\mathcal{G}] = 0$. Since $\psi(\cdot, 0) = 0$, we get that $c_X(\cdot, K^c) = 0$ a.s., i.e. it is $\|\cdot\|_\infty$-compactly supported.
4. Let $A, B \in \mathcal{B}(\mathbb{R}^d)$, and define $r : \Omega \to [0,1]$ by $r(\omega) = P[X \in A|\mathcal{G}](\omega) - P[X \in A \cap B|\mathcal{G}](\omega) = P[X \in A \cup B|\mathcal{G}](\omega) - P[X \in B|\mathcal{G}](\omega)$. If $\omega \in \{r = 0\}$, then notice that $c_X(\omega, A) = c_X(\omega, A \cap B)$ and $c_X(\omega, A \cup B) = c_X(\omega, B)$. Then,

$$c_X(\omega, A \cup B) + c_X(\omega, A \cap B) \leqslant c_X(\omega, A) + c_X(\omega, B).$$

If $\omega \in \{r > 0\}$ and $P[X \in A|\mathcal{G}](\omega) \leqslant P[X \in B|\mathcal{G}](\omega)$, then the following holds:

$$P[X \in A \cap B|\mathcal{G}](\omega) < P[X \in A|\mathcal{G}](\omega) \leqslant P[X \in B|\mathcal{G}](\omega) < P[X \in A \cup B|\mathcal{G}](\omega).$$

Then, concavity of $\psi(\omega, \cdot)$ implies that:

$$\frac{\psi(\omega, P[X \in A \cup B|\mathcal{G}](\omega)) - \psi(\omega, P[X \in B|\mathcal{G}](\omega))}{r(\omega)} \leqslant \frac{\psi(\omega, P[X \in A|\mathcal{G}](\omega)) - \psi(\omega, P[X \in A \cap B|\mathcal{G}](\omega))}{r(\omega)},$$

which ensures that:

$$c_X(\omega, A \cup B) - c_X(\omega, B) \leqslant c_X(\omega, A) - c_X(\omega, A \cap B).$$

Finally, if $\omega \in \{r > 0\}$ so that $P[X \in A|\mathcal{G}](w) > P[X \in B|\mathcal{G}](w)$, repeat the same computation made above, exchanging the roles of A and B, and substituting $r(\omega)$ by $r(\omega) + P[X \in A|\mathcal{G}](\omega) - P[X \in B|\mathcal{G}](\omega) > 0$. Then, c_X is submodular.
5. Fix $A \in \mathcal{B}(\mathbb{R}^d)$, and let $(A_n)_{n\in\mathbb{N}}$ such that $A_n \downarrow A$. For any $\omega \in \Omega$ $B \in \mathcal{B}(\mathbb{R}) \mapsto P[X \in B|\mathcal{G}](\omega)$ is a probability measure. Thus, $P[X \in A_n|\mathcal{G}](\omega) \downarrow P[X \in A|\mathcal{G}](\omega)$, for any $\omega \in \Omega$. The same argument holds if $A_n \uparrow A$. Consequently, by continuity of the sample paths of ψ, we get that c_X is continuous.

6. $C(\mathcal{G})$ is consistent, since for any $\phi : \mathbb{R}^d \to \mathbb{R}^k$ Borel measurable function and $X \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^d)$, there exists a set $\Omega_{X,\phi} \in \mathcal{G}$, such that:

$$P[\phi(X) \in A|\mathcal{G}](\omega) = P[X \in \phi^{-1}(A)|\mathcal{G}](\omega), \text{ for any } \omega \in \Omega_{X,\phi} \text{ and } A \in \mathcal{B}(\mathbb{R}^k).$$

   Consequently, by taking the composition with $\psi(\omega, \cdot)$:

$$c_{\phi(X)}(\omega, A) = c_X\left(\omega, \phi^{-1}(A)\right), \text{ for any } \omega \in \Omega_{X,\phi} \text{ and } A \in \mathcal{B}(\mathbb{R}^k).$$

7. If $X \sim_{\mathcal{G}} Y$, then there exists a set $\Omega' \in \mathcal{G}$, with probability one, so that $P[X \in A|\mathcal{G}](\omega) = P[Y \in A|\mathcal{G}](\omega)$, for any $\omega \in \Omega'$ and $A \in \mathcal{B}(\mathbb{R})$. Thus, $c_X(\omega, A) = c_Y(\omega, B)$, for any $\omega \in \Omega'$ and $A \in \mathcal{B}(\mathbb{R})$, and $C(\mathcal{G})$ is conditionally law-invariant.
8. $C(\mathcal{G})$ is conditionally translational invariant. Indeed, if $X \in L^\infty(\Omega, \mathcal{F}, P)$ and $Y \in L^\infty(\Omega, \mathcal{G}, P)$, then there is a set $\Omega_{X,Y} \in \mathcal{G}$, with probability one, such that:

$$P[X + Y \in A|\mathcal{G}](\omega) = P[X \in A - Y(\omega)|\mathcal{G}](\omega), \text{ for any } \omega \in \Omega_{X,Y} \text{ and } A \in \mathcal{B}(\mathbb{R}).$$

   Thus, by taking the composition with $\psi(\omega, \cdot)$, we get that $C(\mathcal{G})$ is, indeed, conditionally translational invariant.

Therefore, $C(\mathcal{G})$ is a spectral family and the result is proved.

($\Leftarrow$) Suppose now that $C(\mathcal{G})$ is a spectral family of transition capacities. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{G}, P)$ be the associated conditional risk measure defined by:

$$\rho(X) = \int (-x) c_X(\cdot, dx), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

This map is well-defined, since c_X is a compactly supported and continuous transition capacity for any $X \in L^\infty(\Omega, \mathcal{F}, P)$.

Let $U \in L^\infty(\Omega, \mathcal{F}, P)$ be such that $P[U \in \cdot|\mathcal{G}] = U(0, 1)$. We will repeat the construction made previously to generate a concave distortion, substituting ρ by c_U. Indeed, fix $(q, q') \in \mathbb{Q}^2 \cap [0, 1]^2$, such that $q < q'$. Then, because $c_U(\omega, \cdot)$ is submodular, the law-invariance property and consistency of $C(\mathcal{G})$, we may extract a $\mathcal{G}$-measurable set of full probability satisfying:

$$\begin{aligned}
c_U(\cdot,(-\infty,q]) + c_U\left(\cdot,(-\infty,q']\right) &= c_U(\cdot,[0,q]) + c_U\left(\cdot,[0,q']\right), \\
&= c_{-\mathbb{1}_{[U\leqslant q]}}(\cdot,(-\infty,0)) + c_U\left(\cdot,[0,q']\right), \\
&= c_{-\mathbb{1}_{\left[U\in\left[\frac{q'-q}{2},\frac{q'+q}{2}\right]\right]}}(\cdot,(-\infty,0)) + c_U\left(\cdot,[0,q']\right), \\
&= c_U\left(\cdot,\left[\frac{q'-q}{2},\frac{q'+q}{2}\right]\right) + c_U\left(\cdot,[0,q']\right), \\
&= c_U\left(\cdot,\left[0,\frac{q'+q}{2}\right]\cap\left[\frac{q'-q}{2},q'\right]\right) + \\
&+ c_U\left(\cdot,\left[0,\frac{q'+q}{2}\right]\cup\left[\frac{q'-q}{2},q'\right]\right), \\
&\leqslant c_U\left(\cdot,\left[0,\frac{q'+q}{2}\right]\right) + c_U\left(\cdot,\left[\frac{q'-q}{2},q'\right]\right) \\
&= 2c_U\left(\cdot,\left[0,\frac{q'+q}{2}\right]\right), \\
&= 2c_U\left(\cdot,\left(-\infty,\frac{q'+q}{2}\right]\right),
\end{aligned}$$

Let $\Omega' \in \mathcal{G}$, with probability measure one, such that the above inequality holds for any $(q,q') \in \mathbb{Q}^2 \cap [0,1]^2$. Furthermore, since c_U is compactly supported in supp$U$, we can assume that $c_U(\cdot,(-\infty,0]) = 0$ and $c_U(\cdot,(-\infty,1]) = 1$ in Ω'. Therefore, we can define $\psi : \Omega \times [0,1] \to [0,1]$ by:

$$\psi(\omega,\tau) = \begin{cases} c_U(\omega,(-\infty,\tau]), & \text{if } \omega\in\Omega' \text{ and } \tau\in\mathbb{Q}\cap[0,1], \\ \lim_{q\downarrow\tau}\psi(\omega,q), & \text{if } \omega\in\Omega' \text{ and } \tau\in[0,1]\cap\mathbb{Q}^c, \\ \tau, & \text{otherwise.} \end{cases}$$

Notice that the sample paths of ψ are concave and, consequently, continuous. Furthermore, $\psi(\cdot,\tau) \in L^\infty(\Omega,\mathcal{G},P)$, for any $\tau\in[0,1]$. Thus, ψ is $\mathcal{G}\otimes\mathcal{B}([0,1])$-measurable. Finally, notice that by continuity of c_U, then $c_U(\omega,(-\infty,\tau]) = \psi(\omega,\tau)$, for any $\tau\in[0,1]$ and $\omega\in\Omega'$.

We now claim that, for every $X\in L^\infty(\Omega,\mathcal{F},P)$, there exists a set $\Omega''\in\mathcal{G}$, with full probability measure, such that:

$$c_X(\omega,(-\infty,x]) = \psi(\omega,P[X\leqslant x|\mathcal{G}]), \text{for any } x\in\mathbb{R} \text{ and } \omega\in\Omega''.$$

Indeed, let $D\subset\mathbb{R}$ be a dense and countable subset of $\mathbb{R}$. Then, consistency, conditional law-invariance and translational invariance imply that, for any $x\in D$, the following holds a.s.:

$$\begin{aligned}
c_X(\cdot,(-\infty,x]) &= c_{Q_U[X|\mathcal{G}]}(\cdot,(-\infty,x]) = c_{-1_{[Q_U[X|\mathcal{G}]\leqslant x]}}(\cdot,(-\infty,0)), \\
&= c_{-1_{[U\leqslant P[X\leqslant x|\mathcal{G}]]}}(\cdot,(-\infty,0)) = c_{-1_{[U-P[X\leqslant x|\mathcal{G}]\leqslant 0]}}(\cdot,(-\infty,0)), \\
&= c_{U-P[X\leqslant x|\mathcal{G}]}(\cdot,(-\infty,0]) = c_U(\cdot,(-\infty,P[X\leqslant x|\mathcal{G}](\omega)]), \\
&= \psi(\cdot,P[X\leqslant x|\mathcal{G}](\omega)).
\end{aligned}$$

Since this equality holds for a fixed $x \in D$, we may intersect all the sets where it holds for different $x \in D$ to obtain a $\mathcal{G}$-measurable set, with full probability set, so that:

$$c_X(\omega, (-\infty, x]) = \psi(\omega, P[X \leqslant x|\mathcal{G}](\omega)), \text{for any } x \in D \text{ and } \omega \in \Omega'.$$

However, notice that $x \in \mathbb{R} \mapsto c_X(\omega, (-\infty, x])$ and $x \in \mathbb{R} \mapsto \psi(\omega, P[X \leqslant x|\mathcal{G}](\omega))$ are bounded, right-continuous with left-limits, for any $\omega \in \Omega'$, then they are equal in a dense of $\mathbb{R}$. Nevertheless, this is possible if, and only if, they equal everywhere. In particular, due to continuity properties of c_X, $P[X \in \cdot|\mathcal{G}]$ and ψ, we derive that:

$$c_X(\omega, (-\infty, x)) = \psi(\omega, P[X < x|\mathcal{G}](\omega)), \text{ for any } x \in \mathbb{R} \text{ and } \omega \in \Omega'.$$

As a consequence, we obtain that:

$$\begin{aligned}
\rho(X) &= \int (-x) c_X(\cdot, dx), \\
&= \int_0^{+\infty} (c_X(\cdot, (-\infty, x)) - 1)\, dx + \int_{-\infty}^0 c_X(\cdot, (-\infty, x))\, dx, \\
&= \int_0^{+\infty} (\psi(\cdot, P[X < x|\mathcal{G}]) - 1)\, dx + \int_{-\infty}^0 \psi(\cdot, P[X < x|\mathcal{G}])\, dx, \text{ a.s.}
\end{aligned}$$

By the proof given above and Theorem 3.3.13, ρ is a conditionally law-invariant, coherent, and continuous from above risk measure.

It remains to show, hence, that it is conditionally comonotonic. Recall from the proof of Theorem 3.3.9 and Theorem 3.3.13 that, since $\psi \in \mathrm{Conc}(\Omega, \mathcal{G}, [0,1])$, there exists a $Q \in \mathcal{P}_{\mathcal{G}}$, such that, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$:

$$\int_0^{+\infty} (\psi(\cdot, P[X < x|\mathcal{G}]) - 1)\, dx + \int_{-\infty}^0 \psi(\cdot, P[X < x|\mathcal{G}])\, dx = \int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] d\tau, \text{ a.s.}$$

Moreover, if $(X, Y) \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^2)$ are $\mathcal{G}$-comonotonic, then $(-X, -Y)$ is also $\mathcal{G}$-comonotonic and, by Corollary 2.3.5 in Chapter 2, $Q_\tau[-X-Y|\mathcal{G}] = Q_\tau[-X|\mathcal{G}] + Q_\tau[-Y|\mathcal{G}]$, for any $\tau \in (0,1)$, in a fixed $\mathcal{G}$-measurable set of probability one. Thus, the above identities forces that:

$$\begin{aligned}
\rho(X+Y) &= \int_0^1 Q_\tau[-X-Y|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] d\tau, \\
&= \int_0^1 (Q_\tau[-X|\mathcal{G}] + Q_\tau[-Y|\mathcal{G}])\, Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] d\tau, \\
&= \int_0^1 Q_\tau[-X|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] d\tau + \int_0^1 Q_\tau[-Y|\mathcal{G}] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{G}\right] d\tau, \\
&= \rho(X) + \rho(Y).
\end{aligned}$$

This concludes the first part of the theorem.

To characterize $\mathcal{Q}$, fix $X \in L^\infty(\Omega, \mathcal{F}, P)$ and let $c_X \in C(\mathcal{G})$ be its corresponding transition capacity. Since ρ is a conditionally coherent and continuous from above, as well as $C(\mathcal{G})$ is consistent, Corollary 3.3.5 implies that $Q \in \mathcal{Q}$ if, and only if,

$$E^Q[Y|\mathcal{G}] \leqslant \rho(-Y) = \int y c_Y(\cdot, dy), \text{ for every } Y \in L^\infty(\Omega, \mathcal{F}, P) \text{ a.s.}$$

In particular, for any $x \in \mathbb{R}$, conditional law-invariance of $C(\mathcal{G})$ forces:

$$\begin{aligned} Q[X < x|\mathcal{G}] &= E^Q\left[\mathbb{1}_{[X<x]}|\mathcal{G}\right], \\ &\leqslant \int_0^{+\infty} c_{\mathbb{1}_{[X<x]}}\left(\cdot, (\bar{x}, +\infty)\right) d\bar{x}, \\ &= c_X\left(\cdot, (-\infty, x)\right), \text{ a.s.} \end{aligned}$$

Then, we can extract a $\mathcal{G}$-measurable set, with full probability measure, such that the above identity holds for any $x \in \mathbb{Q}$. Moreover, left-continuity of both $c_X(\cdot, (\infty, x))$ and $Q[X < \cdot|\mathcal{G}]$ imply that:

$$Q[X < x|\mathcal{G}] \leqslant c_X\left(\cdot, (-\infty, x)\right), \text{ for any } x \in \mathbb{R} \text{ and } \omega \in \Omega_X$$

where $\Omega_X \in \mathcal{G}$, $P[\Omega_X] = 1$.

The converse is obviously true. Indeed, if Q is such that:

$$Q[X < x|\mathcal{G}] \leqslant c_X\left(\cdot, (-\infty, x)\right), \text{ for any } x \in \mathbb{R} \text{ and } \omega \in \Omega_X,$$

where $\Omega_X \in \mathcal{G}$, $P[\Omega_X] = 1$, then:

$$\begin{aligned} \rho(X) = \int (-x) c_X(\cdot, dx) &= \int_0^{+\infty} \left(c_X\left(\cdot, (-\infty, x)\right) - 1\right) dx + \int_{-\infty}^0 c_X\left(\cdot, (-\infty, x)\right) dx, \\ &\geqslant \int_0^{+\infty} \left(Q[X < x|\mathcal{G}] - 1\right) dx + \int_{-\infty}^0 Q[X < x|\mathcal{G}] dx = E^Q[-X|\mathcal{G}], \text{ a.s.}, \end{aligned}$$

implying that $Q \in \mathcal{Q}$. □

Proof of Proposition 3.3.25. ($\Longrightarrow$) Suppose c admits a disintegration with respect to P conditioned to $\mathcal{G}$. Let $C(\mathcal{G}, c)$ be its associated spectral family. Then, $c_{\mathbb{1}_\Omega}((-\infty, 1]) = 1$ a.s., and:

$$P[A] = E\left[\mathbb{1}_A c_{\mathbb{1}_\Omega}\Big((-\infty, 1]\Big)\right] = c\Big(A, [\mathbb{1}_\Omega \leqslant 1]\Big) = c(A), \text{ for any } A \in \mathcal{G}.$$

Moreover, if $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$, such that $X \sim_{\mathcal{G}} Y$, and $B \in \mathcal{B}(\mathbb{R})$, then conditional law-invariance of $C(\mathcal{G}, c)$ implies that $c_X(B) = c_Y(B)$ a.s. Therefore,

$$c(X \in B) = E[c_X(B)] = E[c_Y(B)] = c(Y \in B).$$

Finally, fixed $\tau \in (0, 1)$ and letting $U \in L^\infty(\Omega, \mathcal{F}, P)$ be conditionally uniform, theset function $A \in \mathcal{G} \mapsto c(A, U \leqslant \tau)$ satisfies:

$$c(A, U \leqslant \tau) = E\left[\mathbb{1}_A c_U\Big((-\infty, \tau]\Big)\right], \text{ for any } A \in \mathcal{G},$$

from where we conclude it is a measure in $(\Omega, \mathcal{G})$ which is absolutely continuous with respect to P.

($\Longleftarrow$) Assume now that c satisfies the three conditions in Proposition 3.3.25. For any $\tau \in \mathbb{Q} \cap (0, 1)$, let $\psi(\cdot, \tau) \in L^0(\Omega, \mathcal{G}, P)$ be the Radon-Nykodyn derivative of $c(\cdot, U \leqslant \tau)$ with respect to P. Then, $0 \leqslant \psi(\cdot, \tau) \leqslant 1$ a.s., since $c(A, U \leqslant \tau) \leqslant c(A) = P[A]$, by monotonicity and item 1, for any $A \in \mathcal{G}$. Moreover, for any pair $\tau < \tau' \in (0,1)$, we derive from property 2 and submodularity that:

$$\begin{aligned}
\mathsf{E}\left[\Big(\psi(\cdot,\tau)+\psi(\cdot,\tau')\Big)\mathbb{1}_A\right] &= \mathsf{E}\left[\psi(\cdot,\tau)\mathbb{1}_A\right]+\mathsf{E}\left[\psi(\cdot,\tau')\mathbb{1}_A\right], \\
&= c\Big(A, U\leqslant\tau\Big)+c\Big(A,U\leqslant\tau'\Big), \\
&= c\Big(A, U\leqslant\tau'\Big)+c\left(A,\frac{\tau'-\tau}{2}\leqslant U\leqslant\frac{\tau'+\tau}{2}\right), \\
&\leqslant c\left(A, U\leqslant\frac{\tau'+\tau}{2}\right)+c\left(A,\frac{\tau'-\tau}{2}\leqslant U\leqslant\tau'\right), \\
&= 2c\left(A, U\leqslant\frac{\tau'+\tau}{2}\right), \\
&= \mathsf{E}\left[\mathbb{1}_A\psi\left(\cdot,\frac{\tau'+\tau}{2}\right)\right], \text{ for any } A\in\mathcal{G}.
\end{aligned}$$

In particular, this implies that:

$$\psi(\cdot,\tau)+\psi(\cdot,\tau')\leqslant\psi\left(\cdot,\frac{\tau'+\tau}{2}\right), \text{ a.s.}$$

Moreover, monotonicity of c implies that $\Big(\psi(\cdot,\tau)\Big)_{\tau\in(0,1)\cap\mathbb{Q}}$ is almost surely non-decreasing. Continuity of c and the monotone convergence theorem forces that:

$$\begin{aligned}
\mathsf{E}\left[\lim_{\tau\downarrow 0}\psi(\cdot,\tau)\right] &= \lim_{\tau\downarrow 0}\mathsf{E}[\psi(\cdot,\tau)] = \lim_{\tau\downarrow 0} c(U\leqslant\tau) = c(U\leqslant 0) = 0, \\
\mathsf{E}\left[\lim_{\tau\uparrow 1}\psi(\cdot,1)\right] &= \lim_{\tau\uparrow 1}\mathsf{E}[\psi(\cdot,\tau)] = \lim_{\tau\uparrow 1} c(U\leqslant\tau) = c(U< 1) = 1.
\end{aligned}$$

Thus, $\Big(\psi(\cdot,\tau)\Big)_{\tau\in(0,1)\cap\mathbb{Q}}$ admits an extension to $[0,1]$, so that $\psi\in\mathrm{Conc}(\Omega,\mathcal{G},[0,1])$. Following the proof of Theorem 3.3.23, we can define the associated spectral family, $\mathrm{C}(\mathcal{G},c)$, as:

$$c_X(\omega,B)=\psi(\omega,\mathsf{P}[X\in B|\mathcal{G}]), \text{ for any } \omega\in\Omega,\ X\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P};\mathbb{R}^d) \text{ and } B\in\mathcal{B}(\mathbb{R}^d).$$

It remains to show that $\mathrm{C}(\mathcal{G},c)$ is a disintegration of c. Fixed any $A\in\mathcal{G}$, $X\in\mathsf{L}^\infty(\Omega,\mathcal{F},\mathsf{P};\mathbb{R}^d)$ and $B\in\mathcal{B}(\mathbb{R}^d)$, then $c_X(B)=c_{\mathbb{1}_{U\leqslant \mathsf{P}[X\in B|\mathcal{G}]}}\Big((0,1]\Big)$ a.s., by conditional law-invariance and consistency of a spectral family. Beyond that, $c(A,X\in B)=c\Big(A,U\leqslant\mathsf{P}[X\in B|\mathcal{G}]\Big)$, by property 2. Consequently,

$$\mathsf{E}\left[c_X(B)\mathbb{1}_A\right]=\mathsf{E}\left[c_{\mathbb{1}_{U\leqslant \mathsf{P}[X\in B|\mathcal{G}]}}\Big((0,1]\Big)\mathbb{1}_A\right]=c\Big(A,U\leqslant\mathsf{P}[X\in B|\mathcal{G}]\Big)=c(A,X\in B),$$

concluding the proof. □

Proof of Theorem 3.3.26. As showed in the proofs of Theorem 3.3.9, Lemma 3.3.12, Theorem 3.3.23 and Proposition 3.3.25, the following chain of bijections hold

$$\mathsf{Q}\in\mathcal{P}_{\mathcal{G}}\longleftrightarrow\mu\in\mathcal{M}^{\mathcal{G}}_{(0,1]}\longleftrightarrow\psi\in\mathrm{Conc}(\Omega,\mathcal{G},[0,1])\longleftrightarrow c\in\mathrm{C},$$

such that, for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ we have a.s.

$$\begin{aligned} E^Q[-X|\mathcal{G}] &= \int_0^1 AV@R_\tau[X|\mathcal{G}]d\mu(\tau) = \int_{-\infty}^0 \psi(\cdot, P[X<x|\mathcal{G}])\,dx + \int_0^{+\infty} (\psi(\cdot, P[X<x|\mathcal{G}]) - 1)\,dx \\ &= \int (-x)c_X(dx), \end{aligned}$$

where $c_X \in C(\mathcal{G}, c)$.

Let ρ be a convex, continuous from above and conditionally law-invariant risk measure. From its representation in Theorem 3.3.4, the previous bijection and formula, construct $\delta_* : C \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}})$ by setting $\delta_*(c) = \alpha_*(Q)$, where $Q \in \mathcal{P}_\mathcal{G}$ and $c \in C$ are bijectively connected as above. Then,

$$E^Q[-X|\mathcal{G}] - \alpha_*(Q) = \int (-x)c_X(dx) - \delta_*(c), \text{ a.s. and for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

The previous bijection, again, ensures us that, by taking the essential supremum in both sides of the above identity, we get:

$$\rho(X) = \operatorname*{esssup}_{c \in C} \left(\int (-x)c_X(dx) - \delta_*(c) \right), \text{ for any } X \in L^\infty(\Omega, \mathcal{F}, P).$$

Moreover, we also conclude that:

$$\delta_*(c) = \alpha_*(Q) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(E^Q[-X|\mathcal{G}] - \rho(X) \right) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F},P)} \left(\int (-x)c_X(dx) - \rho(X) \right).$$

Conversely, if ρ is representable as in Theorem 3.3.26, then we can repeat the argument given above, now in the opposite direction, to construct $\alpha_* : \mathcal{P}_\mathcal{G} \to L^0(\Omega, \mathcal{G}, P; \bar{\mathbb{R}})$ to show that ρ is representable as in Theorem 3.3.4. Therefore, it is convex, continuous from above and conditionally law-invariant. □

B.2.6 Proofs of Section 3.4.3

Proof of Theorem 3.4.5. Item 1 holds if, and only if, for any fixed $t \in \{0, t_1, \ldots, t_{n-1}\}$, ρ_t restricted to $L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P)$ is conditionally law-invariant, convex and continuous from above. As Theorems 3.3.4, 3.3.9 and 3.3.14 show, this is possible if, and only if, ρ_t admits the following representations:

1.

$$\alpha_*^t(Q) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)} \left(\int_0^1 Q_\tau[-X|\mathcal{F}_t] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] d\tau - \rho_t(X) \right), \text{ a.s., for any } Q \in \mathcal{P}_{\mathcal{F}_t, \mathcal{F}_{t_{k+1}}},$$

and,

$$\rho_t(X) = \operatorname*{esssup}_{Q \in \mathcal{P}_{\mathcal{F}_t, \mathcal{F}_{t_{k+1}}}} \left(\int_0^1 Q_\tau[-X|\mathcal{F}_t] Q_\tau\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] d\tau - \alpha_*^t(Q) \right), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P).$$

2.

$$\beta_*^t(\mu) = \operatorname*{esssup}_{X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)} \left(\int_0^1 AV@R_\tau[-X|\mathcal{F}_t] d\mu(\tau) - \rho_t(X) \right), \text{ a.s., for any } \mu \in \mathcal{M}_{(0,1]}^{\mathcal{F}_t},$$

and,

$$\rho_t(X) = \operatorname*{esssup}_{\mu\in\mathcal{M}_{(0,1]}^{\mathcal{F}_t}} \left(\int_0^1 AV@R_\tau[-X|\mathcal{F}_t]d\mu(\tau) - \beta_*^t(\mu)\right), \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P).$$

3.
$$\gamma_*^t(\psi) = \operatorname*{esssup}_{X\in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)} \left(\int_{\mathbb{R}} \left(\psi\left(\cdot, P[X<x|\mathcal{F}_t]\right) - \mathbb{1}_{[0,+\infty)}(x)\right)dx - \rho_t(X)\right),$$

for any $\psi \in \mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])$, and

$$\rho_t(X) = \operatorname*{esssup}_{\psi\in\mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])} \left(\int_{\mathbb{R}} \left(\psi\left(\cdot, P[X<x|\mathcal{F}_t]\right) - \mathbb{1}_{[0,+\infty)}(x)\right)dx - \gamma_*^t(\psi)\right),$$

for any $X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$.

Now observe that $\rho_{t_{k+1}} : L^\infty(\Omega,\mathcal{F},P) \to L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$ is surjective. Thus, time-consistency allows to take the essential supremums in $1-3$ for α_*^t, β_*^t and γ_*^t over $L^\infty(\Omega,\mathcal{F},P)$, substituting $-X$ by $\rho_{t_{k+1}}(X)$ in 1 and 2, and X by $-\rho_{t_{k+1}}(X)$ in 3. Moreover, when calculating ρ_t, time-consistency, $\rho_t(X) = \rho_t\left(-\rho_{t_{k+1}}(X)\right)$, enables us to change $-X$ by $\rho_{t_{k+1}}(X)$ in $1-2$, and X by $-\rho_{t_{k+1}}(X)$ in 3. This proves the result.

Finally, item 1 holds with $(\rho_t)_{t\in\{t_0,\dots,t_{n-1}\}}$ coherent, if and only if, the characterization above holds and the penalty functions, α_*^t, β_*^t and γ_*^t take either 0 or $+\infty$, due to Corollaries 3.3.5, 3.3.10 and 3.3.15. Furthermore, these Corollaries also guarantee that, for any $t \in \{0,t_1,\dots,t_{n-1}\}$, we can define sets $\mathcal{P}^t \subset \mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}}$, $\mathcal{M}^t \subset \mathcal{M}_{(0,1]}^{\mathcal{F}_t}$ and $\mathcal{C}^t \subset \mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])$ where the respective penalty function is 0 and the essential supremum is taken, concluding the proof.

□

Proof of Theorem 3.4.6. Item 1 holds if, and only if, for any fixed $t \in \{0,t_1,\dots,t_{n-1}\}$, ρ_t restricted to $L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$ is conditionally law-invariant, coherent, continuous from above and $\mathcal{F}_t$-comonotonic. Then, Theorem 3.3.23 states that this condition is equivalent to the existence of a spectral family of transitional capacities $C_t(\mathcal{F}_t)$, such that:

$$\rho_t(X) = \int (-x)\, c_X(\cdot,dx), \text{ a.s., for any } X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P).$$

Furthermore, ρ_t admits the following robust representation:

$$\rho_t(X) = \operatorname*{esssup}_{Q\in\mathcal{Q}^t} E^Q[-X|\mathcal{F}_t], \text{ a.s., for any } L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P).$$

such that $Q \in \mathcal{Q}^t$ if, and only if, $Q \in \mathcal{P}_{\mathcal{F}_t,\mathcal{F}_{t_{k+1}}}$ and for any $X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$, there exists $\Omega_X \in \mathcal{F}_t$, with $P[\Omega_X] = 1$, satisfying

$$Q_\tau[X<x|\mathcal{F}_t](\omega) \leqslant c_X\left(\omega,(-\infty,x)\right), \text{ for any } x \in \mathbb{R} \text{ and } \omega \in \Omega_X.$$

Therefore, time-consistency implies that 1 and 4 are equivalent.

Now, in the proof of Theorem 3.3.23 it was shown that, for each spectral family of transition capacities, $C_t(\mathcal{F}_t)$, there exists an $\psi_t \in \mathrm{Conc}(\Omega,\mathcal{F}_t,[0,1])$, such that, for any $X \in L^\infty(\Omega,\mathcal{F}_{t_{k+1}},P)$

$$\rho_t(X) = \int_{-\infty}^0 \left(\psi_t\left(\cdot, P[X<x|\mathcal{F}_t]\right) - 1\right)dx + \int_0^{+\infty} \psi_t\left(\cdot, P[X<x|\mathcal{F}_t]\right)dx, \text{ a.s.}$$

Once again, time-consistency implies that 4 and 3 are equivalent.

Finally, from Lemma 3.3.12 and Theorem 3.3.13, we can define $\mu_t = \Phi^{-1}(\psi_t) \in \mathcal{M}^{\mathcal{F}_t}_{(0,1]}$, for any $t \in \{0, t_1, \ldots, t_{n-1}\}$, so that:

$$\rho_t(X) = \int_0^1 AV@R_\tau[-X|\mathcal{F}_t]\mu_t(\tau), \text{ a.s., for any } X \in L^\infty(\Omega, \mathcal{F}_{t_{k+1}}, P).$$

As before, time-consistency guarantees that 3 and 2 are equivalent. □

Proof of Proposition 3.4.8. Fixed $t = T-1$, and let $A \in \mathcal{F}$, such that $P[A] > 0$, and $\epsilon > 0$. Recall that, from Proposition B.1.1 item 1 and Proposition 2.2.9 item 6 in Chapter 2, for any positive $X \in L^\infty(\Omega, \mathcal{F}, P)$, then:

$$AV@R_\tau[-X|\mathcal{F}_t] \geqslant AV@R_1[-X|\mathcal{F}_t] = E[X|\mathcal{F}_t] = AV@R_\tau\left[-E[X|\mathcal{F}_t]\middle|\mathcal{F}_t\right], \text{ a.s.}$$

Therefore, for any $\mu \in \mathcal{M}^{\mathcal{F}_t}_{(0,1]}$,

$$\int_{(0,1]} AV@R_\tau[-X|\mathcal{F}_t]d\mu(\tau) - \beta^t_*(\mu) \geqslant \int_{(0,1]} AV@R_\tau\left[-E[X|\mathcal{F}_t]\middle|\mathcal{F}_t\right]d\mu_t(\tau) - \beta^t_*(\mu), \text{ a.s.}$$

Taking the essential supremum in both side of the above inequality and using monotonicity of ρ_t:

$$\rho_t(-X) \geqslant \rho_t(E[-X|\mathcal{F}_t]) = E[X|\mathcal{F}_t], \text{ a.s.}$$

By monotonicity of ρ_t, we already know that $\rho_t(-\epsilon\mathbb{1}_A) \geqslant 0$. However, the equation above guarantees that $E[\rho_t(-\epsilon\mathbb{1}_A)] \geqslant \epsilon P[A] > 0$. Hence, ρ_t is relevant.

Now, suppose by induction that, given $t \in \{t_1, \ldots, t_{n-1}, T\}$, for any $s \geqslant t$, $s \in \Pi$, ρ_s is relevant. Hence, for any $A \in \mathcal{F}$, so that $P[A] > 0$, and $\epsilon > 0$, we get that $\rho_s(-\epsilon\mathbb{1}_A) \geqslant 0$ and $P[\rho_s(-\epsilon\mathbb{1}_A) > 0] > 0$. Employing the representation in Theorem 3.4.5 item 3, using the strongly time-consistency and repeating the argument aboe, we obtain that:

$$\rho_{t-1}(-\epsilon\mathbb{1}_A) = \rho_{t-1}(-\rho_t(-\epsilon\mathbb{1}_A)) \geqslant E[\rho_t(-\epsilon\mathbb{1}_A)|\mathcal{F}_{t-1}], \text{ a.s.}$$

Due to monotonicity of ρ_{t-1}, $\rho_{t-1}(-\epsilon\mathbb{1}_A) \geqslant 0$. Then, taking the expected value in the equation above we get that $E[\rho_{t-1}(-\epsilon\mathbb{1}_A)] \geqslant E[\rho_t(-\epsilon\mathbb{1}_A)] > 0$, by the induction hypothesis. Thus, the induction holds and ρ_{t-1} is relevant. □

Proof of Proposition 3.4.10. For any $p \in [1, +\infty]$, we denote the set of progressively measurable stochastic processes with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and taking values on L^p as $\mathcal{H}^p(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$.

Given $t \in [0,T]$, the conditional risk measure $\eta_t : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_t, P)$ defined by

$$\eta_t(Y) = \left(\int_{-\infty}^0 \psi_t\left(\cdot, P\left[Y < y|\mathcal{F}_t\right]\right)dy + \int_0^{+\infty}\left(\psi_t\left(\cdot, P\left[Y < y|\mathcal{F}_t\right] - 1\right)\right)dy\right)$$

admits an extension to $L^2(\Omega, \mathcal{F}, P)$ with values on $L^2(\Omega, \mathcal{F}_t, P)$. Indeed, let $Y \in L^2(\Omega, \mathcal{F}, P)$, recall that, from conditional Chebsyshev's inequality, there exists a $\mathcal{G}$-measurable set, with full probability measure, such that on it, for any $y \in (-\infty, 0)$:

$$P[Y < y|\mathcal{F}_t] \leqslant \frac{E[|Y|^2|\mathcal{F}_t]}{y^2} \wedge 1.$$

Moreover, since $\psi_t(\omega,\cdot):[0,1]\to[0,1]$ is concave, $\psi(\cdot,0)=0$ and $\psi_t(\cdot,1)=1$, then $\psi_t(\omega,\tau)\geqslant\tau$, for any $\tau\in[0,1]$ and $\omega\in\Omega$. In particular, for any $y\in[0,+\infty)$:

$$1-\psi_t\left(\omega,\mathsf{P}[Y<y|\mathcal{F}_t](\omega)\right)\leqslant 1-\mathsf{P}[Y<y|\mathcal{F}_t](\omega),\ \text{for any }\omega\in\Omega.$$

Consequently, we get that:

$$\begin{aligned}\int_{-\infty}^{0}\psi_t\left(\cdot,\mathsf{P}\left[Y<y|\mathcal{F}_t\right]\right)dy&+\int_0^{+\infty}\left|\psi_t\left(\cdot,\mathsf{P}\left[Y<y|\mathcal{F}_t\right]-1\right)\right|dy\\ &\leqslant\int_{-\infty}^{0}\psi_t\left(\cdot,\frac{\mathsf{E}[|Y|^2|\mathcal{F}_t]}{y^2}\wedge 1\right)dy+\int_0^{+\infty}\left(1-\mathsf{P}\left[Y<y|\mathcal{F}_t\right]\right)dy,\\ &=\int_{-\infty}^{-\sqrt{\mathsf{E}[|Y|^2|\mathcal{F}_t]}}\psi_t\left(\cdot,\frac{\mathsf{E}[|Y|^2|\mathcal{F}_t]}{y^2}\right)dy+\sqrt{\mathsf{E}[|Y|^2|\mathcal{F}_t]}+\mathsf{E}[Y_+|\mathcal{F}_t],\\ &\leqslant\left(2+\int_0^1\frac{\psi_t(\cdot,\tau)}{2\tau\sqrt{\tau}}d\tau\right)\sqrt{\mathsf{E}[|Y|^2|\mathcal{F}_t]},\ \text{a.s.}\end{aligned}$$

Therefore, we can define $\eta_t(Y)$ as:

$$\eta_t(Y)=\left(\int_{-\infty}^{0}\psi_t\left(\cdot,\mathsf{P}\left[Y<y|\mathcal{F}_t\right]\right)dy+\int_0^{+\infty}\left(\psi_t\left(\cdot,\mathsf{P}\left[Y<y|\mathcal{F}_t\right]-1\right)\right)dy\right),$$

which is finite a.s. and $\mathcal{F}_t$-measurable. Besides that, $\eta_t(Y)\in L^2(\Omega,\mathcal{F}_t,\mathsf{P})$, since, by Cauchy-Schwarz:

$$\mathsf{E}[|\eta_t(Y)|^2]\leqslant\mathsf{E}\left[\left(2+\int_0^1\frac{\psi_t(\cdot,\tau)}{2\tau\sqrt{\tau}}d\tau\right)^2\right]\mathsf{E}[|Y|^2].$$

Let $\mu_t\in\mathcal{M}_{(0,1}^{\mathcal{F}_t}$ be the the measure associate to ψ_t in Lemma 3.3.12. Then, it is trivial to show that:

$$\int_0^1\frac{\psi_t(\cdot,\tau)}{2\tau\sqrt{\tau}}d\tau+1=2\int_{(0,1]}\frac{1}{\sqrt{s}}d\mu_t(s),$$

implying that $\int_{(0,1]}\frac{1}{\sqrt{s}}d\mu_t(s)\in L^2(\Omega,\mathcal{F}_t,\mathsf{P})$. Since for any $Y\in L^2(\Omega,\mathcal{F},\mathsf{P})$ there exists a $\mathcal{F}_t$-measurable set with full probability such that for every $\tau\in(0,1]$ $|AV@R_\tau[Y|\mathcal{F}_t]|\leqslant\frac{\mathsf{E}[|Y|^2|\mathcal{F}_t]}{\sqrt{s}}$, we conclude that:

$$\int_{(0,1]}AV@R_\tau[Y|\mathcal{F}_t]d\mu_t(\tau)\leqslant\left(\int_{(0,1]}\frac{1}{\sqrt{s}}d\mu_t(s)\right)\mathsf{E}[|Y|^2|\mathcal{F}_t],\ \text{a.s.}$$

Thus, we can repeat the argument in Theorem 3.3.13 to demonstrate that:

$$\eta_t(Y)=\int_{(0,1]}AV@R_\tau[Y|\mathcal{F}_t]d\mu_t(\tau),\ \text{for any }Y\in L^2(\Omega,\mathcal{F},\mathsf{P}).$$

In particular, we obtain that η_t is coherent in $L^2(\Omega,\mathcal{F},\mathsf{P})$. Hence, for any $Y_1,Y_2\in L^2(\Omega,\mathcal{F},\mathsf{P})$:

$$|\eta_t(Y_1)-\eta_t(Y_2)|\leqslant|\eta_t(Y_1-Y_2)|\leqslant\left(2+\int_0^1\frac{\psi_t(\cdot,\tau)}{2\tau\sqrt{\tau}}d\tau\right)\sqrt{\mathsf{E}[|Y_1-Y_2|^2|\mathcal{F}_t]},\ \text{a.s.}$$

As a consequence, we obtain that, for any $s\geqslant t\in[0,T]$ and $Z\in\mathbb{R}$, the following holds:

$$g(\omega,t,z)=\eta_t\left(-z\frac{(B_s-B_t)}{\sqrt{s-t}}\right)(\omega),\ \text{for any } w\in\Omega.$$

Hence, g is a well-defined and finite function satisfying:

$$|g(\omega,t,z_1)-g(\omega,t,z_2)|\leqslant|z_1-z_2|\frac{1}{\sqrt{2\pi}}\left(\int_0^{\frac{1}{2}}\psi_t(\omega,\tau)e^{\frac{Q_\tau[N(0,1)]^2}{2}}d\tau+\int_{\frac{1}{2}}^1(\psi_t(\omega,\tau)-1)\,e^{\frac{Q_\tau[N(0,1)]^2}{2}}d\tau\right),$$

$$g(\omega,t,rz)=rg(\omega,t,z),$$

$$g(\omega,t,z_1+z_2)\leqslant g(\omega,t,z_1)+g(\omega,t,z_2),\ \text{for any } z,z_1,z_2\in\mathbb{R}, r>0 \text{ and } t\in[0,T].$$

Furthermore, $t\mapsto g(\omega,t,z)$ is continuous in $[0,T]$, for any $\omega\in\Omega$ and $z\in\mathbb{R}$. To see this, notice that:

$$\begin{aligned}
&|g(\omega,t,z)-g(\omega,s,z)|\\
&\leqslant|z|\frac{1}{\sqrt{2\pi}}\left(\int_0^{\frac{1}{2}}\frac{|\psi_t(\omega,\tau)-\psi_s(\omega,\tau)|}{\tau}\tau e^{\frac{Q_\tau[N(0,1)]^2}{2}}d\tau+\int_{\frac{1}{2}}^1\frac{|\psi_t(\omega,\tau)-\psi_s(\omega,\tau|)}{\tau}\tau e^{\frac{Q_\tau[N(0,1)]^2}{2}}d\tau\right),\\
&\leqslant\frac{2|z|}{\sqrt{2\pi}}\sup_{\tau\in[0,1]}\frac{|\psi_t(\omega,\tau)-\psi_s(\omega,\tau)|}{\tau}.
\end{aligned}$$

Under these hypothesis, if $X\in L^\infty(\Omega,\mathcal{F},P)$ is fixed, Pardoux and Peng (1990) ensure the existence of a solution $(Y_t,Z_t)_{t\in[0,T]}\in\mathcal{H}^2(\Omega,(\mathcal{F}_t)_{t\in[0,T]},P)\times\mathcal{H}^2(\Omega,(\mathcal{F}_t)_{t\in[0,T]},P)$ of Equation (3.4).

Fixed any $n\in\mathbb{N}$, let $(Y^n_{t^n_i},Z^n_{t^n_i})_{i\in\{0,\dots,k_n\}}$ be the following recursively defined sequence of random variables:

$$\begin{aligned}
Y^n_{t^n_{k_n}}&=-X,\\
Z^n_{t^n_{k_n}}&=0,
\end{aligned}$$

and, for any $i\in\{0,\dots,k_n-1\}$, $(Z^n_t)_{t\in(t^n_i,t^n_{i+1}]}\in\mathcal{H}^2(\Omega,(\mathcal{F}_t)_{t\in(t^n_i,t^n_{i+1}]},P)$ satisfying:

$$\begin{aligned}
\int_{t^n_i}^{t^n_{i+1}}Z^n_s dB_s&=Y^n_{t^n_{i+1}}-E\left[Y^n_{t^n_{i+1}}\middle|\mathcal{F}_{t^n_i}\right],\\
Z^n_{t_i}\left(t^n_{i+1}-t^n_i\right)&=E\left[Y^n_{t^n_{i+1}}\left(B_{t^n_{i+1}}-B^n_{t_i}\right)\middle|\mathcal{F}_{t^n_i}\right],\\
Y^n_{t^n_i}&=E\left[Y^n_{t^n_{i+1}}\middle|\mathcal{F}_{t^n_i}\right]+g\left(t^n_i,Z^n_{t^n_i}\right)\left(t^n_{i+1}-t^n_i\right)
\end{aligned}$$

Its interpolation is the stochastic process:

$$Y^n_t=Y^n_{t^n_i}-g\left(t^n_i,Z^n_{t^n_i}\right)(t-t^n_i)+\int_{t^n_i}^t Z^n_s dB_s,\ \text{for any } t\in[t_i,t_{i+1}).$$

Moreover, Bouchard and Touzi (2004) prove that there exists a $K>0$ such that:

$$\sup_{t\in[0,T]}\left(E\left[|Y_t-Y^n_t|^2\right]+E\left[\int_0^T|Z_t-Z^n_t|^2dt\right]\right)<K|\Pi^{(n)}|,$$

$$\max_{i\in\{0,\dots,k_n-1\}}\sup_{t\in[t^n_i,t^n_{i+1})}\left(E\left[|Y_t-Y_{t^n_i}|^2\right]\right)<K|\Pi^{(n)}|,$$

for any $n \in \mathbb{N}$. Thus, to prove the desired result, it suffices to show that there exists a fixed $L > 0$ so that:

$$\sup_{i \in \{0,\dots,k_n\}} E\left[|\rho^n_{t^n_i}(X) - Y^n_{t^n_i}|^2\right] \leqslant L|\Pi^{(n)}|, \text{ for every } n \in \mathbb{N},$$

since:

$$\sup_{t \in [t^n_i, t^n_{i+1})} E\left[|\rho^n_{t^n_i}(X) - Y_t|^2\right] \leqslant 9\left(E\left[|\rho^n_{t^n_i}(X) - Y^n_{t^n_i}|^2\right] + E\left[|Y_{t^n_i} - Y^n_{t^n_i}|^2\right] + E\left[|Y_{t^n_i} - Y_t|^2\right]\right).$$

In order to demonstrate this bound, first observe that:

$$\rho^n_{t^n_{k_n}}(X) - \bar{Y}^n_{t^n_{k_n}}(X) = 0.$$

Indeed, for $i \in \{0, \dots, k_n - 1\}$, notice that:

$$\begin{aligned}
\rho^n_{t^n_i}(X) - Y^n_{t^n_i} &= E\left[\rho^n_{t^n_{i+1}}(X)|\mathcal{F}_{t^n_i}\right] - Y^n_{t^n_i} + \left(t^n_{i+1} - t^n_i\right)\rho^n_{t^n_i, t^n_{i+1}}\left(-\frac{\left(\rho_{t^n_{i+1}}(X) - E\left[\rho_{t^n_{i+1}(X)}|\mathcal{F}_{t^n_i}\right]\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right), \\
&= E\left[\rho^n_{t^n_{i+1}}(X)|\mathcal{F}_{t^n_i}\right] - E\left[Y^n_{t^n_{i+1}}|\mathcal{F}_{t^n_i}\right] \\
&+ \left(t^n_{i+1} - t^n_i\right)\rho^n_{t^n_i, t^n_{i+1}}\left(-\frac{\left(\rho_{t^n_{i+1}}(X) - E\left[\rho_{t^n_{i+1}(X)}|\mathcal{F}_{t^n_i}\right]\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right) \\
&- \left(t^n_{i+1} - t^n_i\right)\rho^n_{t^n_{i+1}, t^n_i}\left(-\frac{\left(Y^n_{t^n_{i+1}} - E\left[Y^n_{t^n_{i+1}}|\mathcal{F}_{t^n_i}\right]\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right) \\
&+ \left(t^n_{i+1} - t^n_i\right)\left(\rho^n_{t^n_{i+1}, t^n_i}\left(-\frac{\left(Y^n_{t^n_{i+1}} - E\left[Y^n_{t^n_{i+1}}|\mathcal{F}_{t^n_i}\right]\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right) - \rho^n_{t^n_{i+1}, t^n_i}\left(-\frac{Z^n_{t^n_i}\left(B_{t^n_{i+1}} - B_{t^n_i}\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right)\right) \\
&+ \left(t^n_{i+1} - t^n_i\right)\left(\rho^n_{t^n_{i+1}, t^n_i}\left(-\frac{Z^n_{t^n_i}\left(B_{t^n_{i+1}} - B_{t^n_i}\right)}{\sqrt{\left(t^n_{i+1} - t^n_i\right)}}\right) - g\left(t^n_i, Z^n_{t^n_i}\right)\right)
\end{aligned}$$

Consequently, we obtain the following bound:

$$\|\rho^n_{t^n_i}(X) - Y^n_{t^n_i}\|^2_{L^2} \leqslant \left(t^n_{i+1} - t^n_i\right) ME\left[\int_{t^n_i}^{t^n_{i+1}} |Z^n_s - Z^n_{t^n_i}|^2 ds\right] + \left(1 + 4M\left(t^n_{i+1} - t^n_i\right)\right)\|\rho^n_{t^n_{i+1}}(X) - Y^n_{t^n_{i+1}}\|^2_{L^2}.$$

Discrete Grownall Lemma implies then that:

$$\|\rho^n_{t^n_i}(X) - Y^n_{t^n_i}\|^2_{L^2} \leqslant \left(1 + 4M\left(T - t^n_i\right)\right) e^{4M(T - t^n_i)} \sum_{j=i+1}^{k_n - 1} E\left[\int_{t^n_j}^{t^n_{j+1}} |Z^n_s - Z^n_{t^n_j}|^2 ds\right].$$

The previous bounds and equation 3.10 in Bouchard and Touzi (2004) ensures that:

$$\|\rho^n_{t^n_i}(X) - Y^n_{t^n_i}\|^2_{L^2} \leqslant 4\left(1 + 4M\left(T - t^n_i\right)\right) e^{4M(T - t^n_i)} K|\Pi^{(n)}|, \text{for any } i \in \{0, \dots, k_n - 1\},$$

concluding the proof. $\square$

Proof of Proposition B.1.1. 1. For any $X \in L^1(\Omega, \mathcal{F}, P)$, by Proposition 2.2.5 item 1 in Chapter 2, there exists a set $\Omega' \in \mathcal{G}$, with probability one, such that $\int_0^1 |V@R_\tau[X|\mathcal{G}]| d\tau < +\infty$ on it. Thus, we can define $AV@R_\tau[X|\mathcal{G}]$ in Ω' as in 3.3.7 and, set it 0 in $\Omega \cap (\Omega')^c$, for any $\tau \in (0,1)$. This family of random variables is, by construction and Proposition 2.2.5 item 1 in Chapter 2, in $L^1(\Omega, \mathcal{G}, P)$.

Moreover, for any $\omega \in \Omega$, the sample paths $\tau \in (0,1) \mapsto AV@R_\tau[X|\mathcal{G}](\omega) \in \mathbb{R}$ are continuous, since they are either 0 or the result of the integral a cad-lag function. It is trivial to show that, on Ω', the following holds for almost every $\tau \in (0,1)$:

$$\frac{d}{d\tau} AV@R_\tau[X|\mathcal{G}](\omega) = \frac{1}{\tau^2} \int_0^\tau (V@R_\tau[X|\mathcal{G}] - V@R_s[X|\mathcal{G}](\omega))\, ds \leqslant 0,$$

due to item 1 in Proposition 2.2.9. Then, $AV@R_\tau[X|\mathcal{G}]$ is non-increasing.

2. Due to Proposition 2.2.5 in Chapter 2.

3. By the definition of $AV@R_\tau[\cdot|\mathcal{G}]$, it is expressed as Corollary 3.3.5. Therefore, it satisfies conditional translational invariance, monotonicity, conditional coherence and conditional law-invariance.

4. Take any $X \in L^1(\Omega, \mathcal{F}, P)$ and $\Lambda \in L^\infty(\Omega, \mathcal{G}, P)$, so that $0 < \Lambda \leqslant 1$ a.s. Because the sample paths of $(AV@R_\tau[X|\mathcal{G}])_{\tau \in (0,1)}$ are continuous and this stochastic process is in $L^1(\Omega, \mathcal{G}, P)$, the composition with Λ is well-defined and in $L^0(\Omega, \mathcal{G}, P)$ – see Le Gall (2013).

5. Fix $\tau \in (0,1)$, and let $(X_n)_{n \in \mathbb{N}} \subset L^p(\Omega, \mathcal{F}, P)$, so that $\sup_{n \in \mathbb{N}} |X_n| \in L^p(\Omega, \mathcal{F}, P)$. Therefore, it follows trivially that $\sup_{n \in \mathbb{N}} |AV@R_\tau[X_n|\mathcal{G}]| \leqslant E[|\sup_{n \in \mathbb{N}} X_n||\mathcal{G}]$ a.s. by definition 3.3.7 and Proposition 2.2.9 item 6 in Chapter 2. Thus, $\liminf_{n \in \mathbb{N}} AV@R_\tau[X_n|\mathcal{G}]$ and $\limsup_{n \in \mathbb{N}} AV@R_\tau[X_n|\mathcal{G}]$ are in $L^p(\Omega, \mathcal{G}, P)$, as well as $\liminf_{n \in \mathbb{N}} X_n$ and $\limsup_{n \in \mathbb{N}} X_n$.

Moreover, Proposition 2.4.4 in Chapter 2 and Fatou's lemma for integral imply that:

$$\begin{aligned} AV@R_\tau[\limsup_{n \in \mathbb{N}} X_n|\mathcal{G}] &= \frac{1}{\tau} \int_0^\tau Q_{(1-s)}[-\limsup_{n \in \mathbb{N}} X_n|\mathcal{G}] ds = \frac{1}{\tau} \int_0^\tau Q_{(1-s)}[\liminf_{n \in \mathbb{N}} (-X_n)|\mathcal{G}] ds, \\ &\leqslant \frac{1}{\tau} \int_0^\tau \liminf_{n \in \mathbb{N}} Q_{(1-s)}[-X_n|\mathcal{G}] ds \leqslant \liminf_{n \in \mathbb{N}} \frac{1}{\tau} \int_0^\tau Q_{(1-s)}[-X_n|\mathcal{G}] ds, \\ &= \liminf_{n \in \mathbb{N}} AV@R_\tau[X_n|\mathcal{G}], \text{ a.s.} \end{aligned}$$

Repeating the argument given above, we conclude that:

$$\begin{aligned} \limsup_{n \in \mathbb{N}} AV@R_\tau[X_n|\mathcal{G}] &= \limsup_{n \in \mathbb{N}} \frac{1}{\tau} \int_0^\tau Q_{(1-s)}[-X_n|\mathcal{G}] ds \leqslant \frac{1}{\tau} \int_0^\tau \limsup_{n \in \mathbb{N}} Q_{(1-s)}[-X_n|\mathcal{G}] ds, \\ &\leqslant \frac{1}{\tau} \int_0^\tau Q_{(1-s)+}[\limsup_{n \in \mathbb{N}} (-X_n)|\mathcal{G}] ds = \frac{1}{\tau+} \int_0^\tau Q_{(1-s)}[-\liminf_{n \in \mathbb{N}} X_n|\mathcal{G}] ds, \\ &= AV@R_\tau[\liminf_{n \in \mathbb{N}} X_n|\mathcal{G}], \text{ a.s.} \end{aligned}$$

6. This property follows from the linearity of the integral, Jensen's inequality for integrals and Proposition 2.4.5.

7. If $\mathsf{X} = (\mathsf{X}_1, \ldots, \mathsf{X}_n) \in \mathsf{L}^1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$ is $\mathcal{G}$-comonotonic, then Proposition 2.2.5 in Chapter 2 guarantees that there exists $\Omega' \in \mathcal{G}$, with full probability, so that $\tau \in (0,1) \mapsto \mathsf{V@R}_\tau[\mathsf{X}|\mathcal{G}](\omega)$ is continuous except in a countable set of points in $(0,1)$, for any $\omega \in \Omega'$. Thus, under the conditions of Theorem 2.3.4,

$$\mathsf{V@R}_\tau[\psi(\mathsf{X})|\mathcal{G}] = -\psi(-\mathsf{V@R}_\tau[\mathsf{X}_1|\mathcal{G}], \ldots, -\mathsf{V@R}_\tau[\mathsf{X}_n|\mathcal{G}]),$$

for every $\tau \in (0,1)$, expect countably many, and $\omega \in \Omega'$. Taking $\psi(x) = \sum_{i=1}^n x_i$, we have that:

$$\mathsf{AV@R}_\tau\left[\sum_{i=1}^n \mathsf{X}_i \middle| \mathcal{G}\right] = \sum_{i=1}^n \mathsf{AV@R}_\tau[\mathsf{X}_i|\mathcal{G}],$$

for any $\tau \in (0,1)$, except countably many, in Ω'. Since both sides are continuous functions agreeing in a dense set, they equal everywhere in $(0,1)$.

$\square$

Bibliography

Apiwatcharoenkul, W., Lake, L., Jensen, J., 2016. Uncertainty in proved reserves estimates by decline curve analysis, in: SPE/IAEE Hydrocarbon Economics and Evaluation Symposium, Society of Petroleum Engineers. doi:https://doi.org/10.2118/179992-MS.

Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999. Coherent measures of risk. Math. Finance 9, 203–228. doi:10.1111/1467-9965.00068.

Billingsley, 1968. Convergence of Probability Measures. Wiley, New York, New York.

Bouchard, B., Touzi, N., 2004. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. Stochastic Processes Appl. 111, 175–206. doi:10.1016/j.spa.2004.01.001.

Chambers, C.P., 2009. An axiomatization of quantiles on the domain of distribution functions. Mathematical Finance 19, 335–342.

Cheridito, P., Kupper, M., 2011. Composition of time-consistent dynamic monetary risk measures in discrete time. Int. J. Theor. Appl. Finance 14, 137–162. doi:10.1142/S0219024911006292.

Cheung, K.C., 2007. Characterizations of conditional comonotonicity. Journal of Applied Probability 44. doi:10.1239/jap/1189717532.

Dela Vega, E.J.C., Elliott, R.J., 2021. Conditional coherent risk measures and regime-switching conic pricing. Probability, Uncertainty and Quantitative Risk 6, 267–300.

Delbaen, F., 2002. Coherent risk measures on general probability spaces, in: Advances in finance and stochastics. Essays in honour of Dieter Sondermann. Berlin: Springer, pp. 1–37.

Detlefsen, K., Scandolo, G., 2005. Conditional and dynamic convex risk measures. Finance Stoch. 9, 539–561. doi:10.1007/s00780-005-0159-6.

Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vyncke, D., 2002. The concept of comonotonicity in actuarial science and finance: Theory. Insurance: Mathematics and Economics 31, 3–33.

Duffie, D., Pan, J., 1997. An overview of value at risk. Journal of Derivatives 4, 7–49.

Durrett, R., 2019. Probability. Theory and examples. volume 49. Cambridge: Cambridge University Press.

Elliott, R.J., Siu, T.K., Cohen, S.N., 2015. Backward stochastic difference equations for dynamic convex risk measures on a binomial tree. J. Appl. Probab. 52, 771–785. doi:10.1239/jap/1445543845.

Embrechts, P., Höing, A., Juri, A., 2003. Using copulae to bound the value-at-risk for functions of dependent risks. Finance and Stochastics 7, 145–167. doi:10.1007/s007800200085.

Fanchi, J.R., Christiansen, R.L., 2017. Introduction to Petroleum Engineering. Wiley Online Library.

Ferguson, T.S., 1996. A Course in Large Sample Theory. Chapman and Hall, London.

Föllmer, H., Penner, I., 2006. Convex risk measures and the dynamics of their penalty functions. Stat. Decis. 24, 61–96. doi:10.1524/stnd.2006.24.1.61.

Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance and Stochastics 6, 429–447. doi:10.1007/s007800200072.

Föllmer, H., Schied, A., 2002. Stochastic finance. An introduction in discrete time. volume 27. Berlin: de Gruyter.

Frittelli, M., Rosazza Gianin, E., 2005. Law Invariant Convex Risk Measures. volume 7. pp. 33–46. doi:10.1007/4-431-27233-X_2.

Hardy, M.R., Wirch, J.L., 2004. The iterated CTE: a dynamic risk measure. N. Am. Actuar. J. 8, 62–75. doi:10.1080/10920277.2004.10596171.

Jorion, P., 2007. Value at Risk. The New Benchmark for Managing Financial Risk. volume 81. McGraw-Hill. 3rd Edition.

Jouini, E., Napp, C., 2004. Conditional comonotonicity. Decisions in Economics and Finance 27, 153–166.

Klöppel, S., Schweizer, M., 2007. Dynamic indifference valuation via convex risk measures. Math. Finance 17, 599–627. doi:10.1111/j.1467-9965.2007.00317.x.

Koenker, R., 2005. Quantile Regression. Cambridge University Press, New York, New York.

Koenker, R., Bassett, G.W., 1978. Regression quantiles. Econometrica 46, 33–49.

Koenker, R., Bassett, G.W., 2010. March madness, quantile regression bracketology and the hayek hypothesis. Journal of Business and Economic Statistics 28, 26–35.

Koenker, R., Chernozhukov, V., He, X., Peng, L., 2017. Handbook of Quantile Regression. Chapman and Hall/CRC, New York, New York.

Kupper, M., Schachermayer, W., 2009. Representation results for law invariant time consistent functions. Math. Financ. Econ. 2, 189–210. doi:10.1007/s11579-009-0019-9.

Kusuoka, S., 2001. On law invariant coherent risk measures. RIMS Kokyuroku 1215, 158–168.

Le Gall, J.F., 2006. Intégration, probabilités et processus aléatories. Mimeo, Département Mathématiques et Applications Ecole normale supérieure de Paris.

Le Gall, J.F., 2013. Brownian Motion, Martingales, and Stochastic Calculus. Springer-Verlag Berlin Heidelberg.

Madan, D., Pistorius, M., Stadje, M., 2017. On dynamic spectral risk measures, a limit theorem and optimal portfolio allocation. Finance Stoch. 21, 1073–1102. doi:10.1007/s00780-017-0339-1.

Major, P., 1978. On the invariance principle for sums of independent identically distributed random variables. Journal of Multivariate Analysis 8, 487–517.

Manski, C., 1988. Ordinal utility models of decision making under uncertainty. Theory and Decision 25, 79–104.

McNeil, A.J., Frey, R., Embrechts, P., 2005. Quantitative risk management. Concepts, techniques, and tools. Princeton, NJ: Princeton University Press.

Mendelson, H., 1987. Quantile-preserving spread. Journal of Economic Theory 42, 334–351.

Merkle, M., 2005. Jensen's inequality for medians. Statistics & Probability Letters 71, 277–281.

Molchanov, I., 2017. Theory of random sets. 2nd edition. volume 87. 2nd edition ed., London: Springer.

Pardoux, E., Peng, S.G., 1990. Adapted solution of a backward stochastic differential equation. Syst. Control Lett. 14, 55–61.

Peskir, G., Shiryaev, A., 2006. Optimal stopping and free-boundary problems. Basel: Birkhäuser.

Reed, M., Simon, B., 1972. Methods of modern mathematical physics. 1: Functional analysis. New York-London: Academic Press, Inc. XVII.

Rosazza Gianin, E., 2006. Risk measures via g-expectations. Insurance: Mathematics and Economics 39, 19–34.

Rostek, M., 2010. Quantile maximization in decision theory. Review of Economic Studies 77, 339–371.

Ruszczynski, A., Shapiro, A., 2006. Conditional risk mappings. Math. Oper. Res. 31, 544–561. doi:10.1287/moor.1060.0204.

Schmeidler, D., 1986. Integral representation without additivity. Proceedings of the American Mathematical Society 97, 255–261.

Shreve, S.E., 2004. Stochastic calculus for finance. I: The binomial asset pricing model. New York, NY: Springer.

Stadje, M., 2010. Extending dynamic convex risk measures from discrete time to continuous time: a convergence approach. Insur. Math. Econ. 47, 391–404. doi:10.1016/j.insmatheco.2010.08.005.

Sweeting, T.J., 1989. On conditional weak convergence. Journal of Theoretical Probability 2, 461.

van der Vaart, A., 1998. Asymptotic Statistics. Cambridge University Press, New York, New York.

Valadier, M., 1984. La multi-application médianes conditionnelles. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 67, 279–282.

Valadier, M., 2014. Quantiles as minimizers. Mimeo, arXiv:1411.2732.

Weber, S., 2006. Distribution-invariant risk measures, information, and dynamic consistency. Math. Finance 16, 419–441. doi:10.1111/j.1467-9965.2006.00277.x.

Williams, D., 1991. Probability with martingales. Cambridge, Cambridge University Press.

Zhao, M., Yang, X., He, Q., Zhou, Z., Ge, X., 2021. Quantile jensen's inequalities. Journal of Inequalities and Applications 86, 1–8.