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On local energy decay and propagation of regularity for dispersive models.

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“Louco não é o homem que perdeu a razão.
Louco é o homem que perdeu tudo menos a razão.“
(G. K. Chesterton)

RESUMO

A proposta desse trabalho é investigar várias questões acerca dos problemas de valores iniciais associados (PVI) associados a famílias gerais de equações dispersivas não lineares.

No Capítulo 1, nós iremos introduzir algumas notações e resultados preliminares os quais serão importantes para o desenvolvimento da teoria ao longo dessa tese. Especialmente, focando em importantes estimativas do comutador e operadores pseudo-diferenciais.

O Capítulo 2 visa estudar algumas propriedades de suavidade relativas ao problema de valor inicial para a equação dispersiva generalizada Benjamin-Ono-Zakharov-Kuznetsov.Mais precisamente, nós provaremos que as soluções desse modelo satisfazem à denominada propagação de regularidade. À grosso modo, esse princípio estabelece que se o dado inicial possue alguma suavidade adicional prescrita em uma família de semi-planos, então a regularidade é propagada com velocidade infinita. Nesse sentido, nós provamos que independente da escala medindo a regularidade extra em tal coleção de hiperplanos, então toda essa regularidade será também propagada para as soluções desse modelo. Nossa análise é baseada principalmente na dedução de fórmulas de propagação relacionando derivadas homogêneas e não homogêneas em certas regiões do plano.

O Capítulo 3 tem como principal objetivo analisar o comportamento assintótico das soluções do (PVI) associado para algumas equações dispersivas não lineares. Inicialmente, nós consideramos as dinâmicas de longa duração das soluções grandes para a equação da Benjamin-Ono. Usando técnicas viriais, nós descrevemos regiões do espaço onde toda solução em um apropriado espaço de Sobolev deve decair para zero ao longo de uma sequência de tempos. Além disso, nós estudamos esse comportamento assintótico para as soluções do (PVI) associadas a equação da Benjamin e identificamos a relação entre o operador de dispersão e sua dinâmica com o tempo. Finalmente, nós aplicamos a teoria desenvolvida previamente para uma classe de operadores mais geral de modelos dispersivos que incluem, por exemplo, o (PVI) associado as equações das Ondas longas Intermediárias, Fracionária Korteweg-de Vries and dispersiva generalizada Benjamin-Ono.

Palavras-chave: Modelos dispersivos. Propagação de regularidade. Comportamento Assintótico. Decaimento local. Espaços de Energia.

ABSTRACT

The purpose of this work is to investigate several issues regarding solutions to the initial value problem (IVP) associated to a general families of nonlinear dispersive equations.

In Chapter 1, we will introduce some notations and preliminaries results that will be important on the development of the theory and techniques throughout this thesis. Especially, focusing in important commutator estimates and pseudo-differential operators.

The Chapter 2 aims to study some smoothness properties concerning the initial value problem associated to the dispersive generalized Benjamin-Ono-Zakharov-Kuznetsov equation. More precisely, we prove that the solutions to this model satisfy the so-called propagation of regularity. Roughly speaking, this principle states that if the initial data enjoys some extra smoothness prescribed on a family of half-spaces, then the regularity is propagated with infinite speed. In this sense, we prove that regardless of the scale measuring the extra regularity in such hyperplane collection, then all this regularity is also propagated by solutions of this model. Our analysis is mainly based on the deduction of propagation formulas relating homogeneous and non-homogeneous derivatives in certain regions of the plane.

The Chapter 3 main purpose is to analyze special asymptotic behavior of solutions to the IVP associated to some nonlinear dispersive equations. First, we consider the long time dynamics of large solutions to the Benjamin-Ono equation. Using virial techniques, we describe regions of space where every solution in a suitable Sobolev space must decay to zero along sequences of times. Moreover, we study this asymptotic behavior for solutions of the IVP associated to Benjamin equation and identify the relation between the dispersive operator and the time dynamics. Finally, we apply the theory previous developed to a more general class of dispersive models that includes, for example, the IVP associated to Intermediate Long Wave, Fractionary Korteweg-de Vries and dispersion generalized Benjamin-Ono.

Keywords: Dispersive models. Propagation of regularity. Asymptotic behavior. Local decay. Energy spaces.

LIST OF FIGURES

List of abbreviations and acronyms

Bn	Benjamin
BO	Benjamin-Ono
BOZK	Benjamin-Ono-Zakharov-Kuznetsov
gBOZK	Dispersive generalized Benjamin-Ono-Zakharov-Kuznetsov
GWP	Global well-posed
IVP	Initial Value Problem
LWP	Locally well-posed
PVI	Problema de valor inicial

CONTENTS

Introduction

This work is aimed to establish several properties for solutions of different models refereed to nonlinear dispersive equations. Before stating the PDEs and the results that will be proved on this thesis, I will present some historical context and definitions involving the dispersive equations.

We consider a linear partial differential equation

$$F(\partial_x, \partial_t)u(x,t) = 0,$$

where F is a polynomial with constant coefficients in the partial derivatives. Examining the plane wave solutions that have the form $u(x,t) = Ae^{i(kx-\omega t)}$, where A, k and ω are constants representing the amplitude, the wavenumber and the frequency, respectevely. Hence u will be a solution if and only if

$$F(ik, -i\omega) = 0.$$

The last formula is called the dispersion relation. In particular, when we can write the dispersion relation as $\omega = \omega(k)$, a real function depending on k, and we define the group velocity as $c_g = \omega'(k)$. We will say that the equation is dispersive if c_g is not constant. In this context, dispersion means that waves of different wavelength propagate at different phase velocities. For more details, see [66] and references therein.

On Edinburgh-Glasgow canal in 1834, J. Scott Russell observing a motion of a boat realized the existence of a mass of water in this canal assuming a form of a large solitary elevation apparently without change of form or diminution of speed. Fascinated by this, he tried to model this phenomena of "solitary wave". Following in the attempt to model this, in the 1865, through the work [16], Boussinesq proposed the model

$$\eta_{tt} - gh\eta_{xx} - gh\left[\frac{3}{2h}\eta^2 + \frac{h^2}{3}\eta_{xx}\right]_{xx} = 0, \tag{1}$$

where η represents the elevation of the water surface, g is the gravitational acceleration and h is the depth of the channel and using this equation he obtained an explicit representation of solitary waves. Despite it does not represent the model observed by Russell, it is important to note that it was here the genesis of the theory nowadays called the theory of stability of solitary wave solutions. In 1895, D. J. Korteweg and G. de Vries [52] formulated a mathematical model which provided an explanation of the phenomenon observed by Russell, namely,

$$\eta_t - \frac{(gh)^{1/2}}{h}\left[\left(\epsilon + \frac{3}{2}\eta\right)\eta_x + \frac{h}{2}\left(\frac{h^2}{3} - \frac{T}{g\rho}\right)\eta_{xxx}\right] = 0, \tag{2}$$

where T is the surface tension, ρ is the density and for a small parameter ϵ. This is the original form of the **Korteweg-de Vries** equation. We shall call it the **KdV** equation.

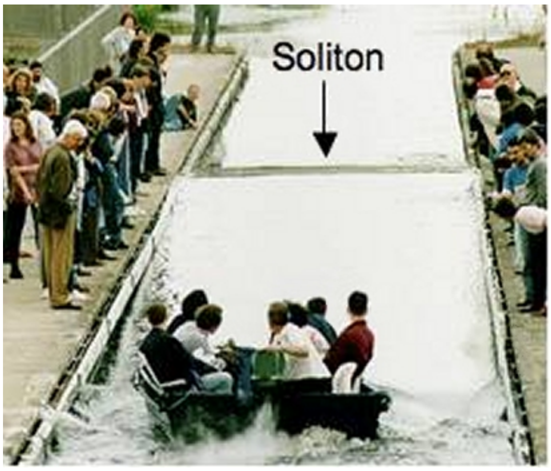

Figure 1 – Recreation of a solitary wave on the Scott Russell Aqueduct.

In this work, we will consider the **Zakharov-Kuznetsov (ZK) equation**, which is an important model of dispersive dynamics that describes the **ionic-acoustic waves** in a uniformly magnetized plasma. The **initial valued problem(IVP)** associated to the (ZK) equation is given by

$$\begin{cases} \partial_t u + \partial x_1 \Delta u + u\partial_{x_1} u = 0, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n, n \geq 2, \end{cases} \tag{3}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_1}^2 + ... + \partial_{x_n}^2$ is the n-dimensional Laplacian.

This equation was deduced by Zakharov and Kuznetsov in [100] when they where studying set of equations that describes two kinds of oscillations-ion-sound and cyclotron oscillations see Fig. 2, in dimension 3, they realized that the slow motion ion-sound oscillations is directed along the magnetic field, then they reduced this set of equations to an equation that models ion-sound waves propagating in one direction along the magnetic field. This reduced equation in the context of 1 dimension is a generalization of the (KdV) equation. Then we can treat the (ZK) equation as a generalization of (KdV) equation in higher dimensions and due to its physical relevance the (ZK) equation has been intensively studied in recent years.

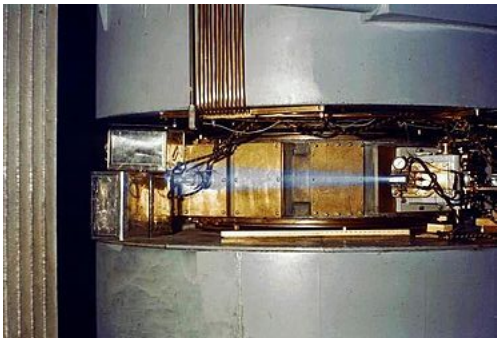

Figure 2 – Ionized air glows blue around a beam of particulate ionizing radiation from a cyclotron.

Now, we will introduce a class of equations that we will treat in the Chapter 2 named as the **dispersion generalized Benjamin-Ono-Zakharov-Kuznetsov (gBOZK) equation**, which includes equation (3),:

$$\begin{cases} \partial_t u - D_x^{\alpha+1} u_x + u_{xyy} + uu_x = 0, \quad (x,y,t) \in \mathbb{R}^3,\ 0 \leq \alpha < 1, \\ u(x,y,0) = u_0(x,y). \end{cases} \tag{gBOZK}$$

This equation arises as a mathematical model to study the effect of dispersion on the propagation direction applied to the initial value problem for the Zakharov-Kuznetsov equation. We recall that for $\alpha = 1$, (gBOZK) is the ZK equation and when $\alpha = 0$, the equation (gBOZK) coincides with the **Benjamin-Ono-Zakharov-Kuznetsov (BOZK) equation**

$$\partial_t u - \mathcal{H}_x \partial_x^2 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \tag{4}$$

that is presented in [20, 56] as a model for thin nanoconductors on a dielectric substrate.

Now, if we want to study some properties for these dispersive models, we need at least the existence of a solution for the IVP associated to each equation. Then before starting the study of properties for the nonlinear models considered here, we will define what is well-posedness for the IVP associated to these equations. This gives us in particular the kind of solutions we will deal with.

Definition 0.0.1. *The (IVP)* (gBOZK) *is said to be* ***locally well-posed (LWP)*** *in the function space X if given any datum $u_0 \in X$,*

- *there exist $T > 0$ and $u \in C([-T,T], X)$ such that u is the unique solution of the (IVP).*
- *The map-data solution $u_0 \longmapsto u$ is continuous.*

Additionally, in the case where T can be taken arbitrarily large, one says that the problem is ***globally well-posed (GWP)****.*

Assuming that the IVP (gBOZK) is (LWP) in some Sobolev space $H^s(\mathbb{R}^2)$, $s > s_\alpha$. If we suppose an additional hypotheses, to ask that the initial data has an extra regularity in some region of the real plane, *i. e*, $u_0 \in H^l(\{(x_0, \infty) \times \mathbb{R}\})$, for some $x_0 \in \mathbb{R}$ and $l > s_\alpha$, what can be said about the regularity of solution of the IVP gBOZK?

Following this stream of thought, we will discuss briefly some results in the literature that describe (LWP) and (GWP) of the (gBOZK). Concerning the case $\alpha = 2$, Faminskii [25] proved global well-posedness in $H^j(\mathbb{R}^2)$, $j \in \mathbb{Z}^+$, $j \geq 1$, later Linares and Pastor [62] proved well-posedness in $H^s(\mathbb{R}^2)$, $s > 3/4$. It was shown by Grünrock and Herr in [32] and by Molinet and Pilod in [79] (LWP) of (gBOZK) in $H^s(\mathbb{R}^2)$, $s > 1/2$. Recently, Kinoshita [50] proved (LWP) in $H^{-\frac{1}{4}+}$ that according to the scaling argument it is optimal up to the end-point. From these results one can have global well-posedness in $H^s(\mathbb{R}^2), s \geq 0$.

Cunha and Pastor [22] obtained the local well-posedness for the (IVP) of BOZK for the case $\alpha = 1$ in the Sobolev spaces $H^s(\mathbb{R}^2)$, $s > 11/8$. Their proof is based on the refined Strichartz estimates introduced by Koch and Tzvetkov [51] in the context of the Benjamin-Ono equation. Finally, Ribaud and Vento [92] obtained the (LWP) for (gBOZK) in the spaces E^s, $s > \frac{2}{\alpha} - \frac{3}{4}$, where $\|f\|_{E^s} = \| < |\xi|^\alpha + \mu^2 >^s \hat{f}\|_{L^2(\mathbb{R}^2)}$. As a consequence, they established (GWP) in the Energy space $E^{1/2}$ as soon as $\alpha > \frac{8}{5}$. More recently, Schippa in [94] shows (LWP) and (GWP) for the IVP associated to the fractional Zakharov-Kuznetsov equation in $H^s(\mathbb{R}^n)$, $s > \frac{n+3}{2} - \alpha$ and $H^s(\mathbb{R}^2)$, $\alpha > 5/3$ and $s = \alpha/2$, respectively.

Answering the question above, Isaza, Linares and Ponce [39] observed for solutions of the IVP associated to the generalized Korteweg-de Vries equation the regularity of the initial data on a family of half-spaces propagates with infinite speed as time evolves. Our main goal in the Chapter 2 is to extend this analysis to the case in that the regularity of the initial data is measured on a fractional scale and to describe the propagation of regularity property present in solutions of the IVP (gBOZK). The main difficulty regarding the gBOZK equation is the presence of the operators D_x^s, J_x^s which are not local operators. In addition, the arguments developed by Mendez in [76] does not seem to be directly applicable because the operator $D_x^s J_x^s$ is not a pseudo-differential operator.

In order to develop the techniques in the Chapter 2, we dedicated some efforts to obtain results about the interaction between differential operators and localized smoothness. After, using energy estimates and Kato's smoothing effect through a interaction process we recover the desired result for the IVP of gBOZK equation.

In this thesis we also consider solutions of the (IVP) associated to the **Benjamin-**

Ono (BO) equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{BO}$$

where $u = u(x,t)$ is a real-valued function and $\mathcal{H}$ is the Hilbert transform, defined on the line as

$$\mathcal{H}f(x) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy. \tag{5}$$

The BO equation was first deduced in the context of long internal gravity waves in a stratified fluid [10, 87]. Later, BO equation was shown to be completely integrable (see [4] and references therein).

The IVP (BO) has been extensively studied, especially, the local well-posedness (LWP) and global well-posedness (GWP) measured in the Sobolev scale $H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2}L^2(\mathbb{R})$, $s \in \mathbb{R}$. In this regard, one has the following list of works: Iorio [38], Abdelouhab, Bona, Felland, and J. Saut [1], Ponce [89], Koch and Tzvetkov [51], Kenig and Koenig [44], Tao [99], Burq and Planchon [17], Ionescu and Kenig [36], Molinet and Pilod [78] and Ifrim and Tataru [35], among others. In particular, in [36], Ionescu and Kenig proved the global well-posedness in $L^2(\mathbb{R})$ of the IVP (BO) was established. For further details and results concerning the IVP associated to the (BO) equation we refer to [93] and [77].

In [81], Molinet, Saut and Tzvetkov proved that none well-posedness for the IVP (BO) in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ can be established by an argument based only on the contraction principle argument.

We recall that the BO equation possesses traveling wave solutions (solitons) $u(x,t) = \phi(x-t)$ of the form, namely

$$\phi(x) = \frac{4}{1+x^2}, \tag{6}$$

which is smooth and exhibits a mild polynomial decay. The solitary waves are stable and unique.

We will also consider the initial value problem (IVP)

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + \partial_x^3 u + u\partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{Bn}$$

where $u = u(x,t)$ is a real-valued function.

The IVP (Bn) is associated to the **Benjamin equation**. This integro-differential equation models the unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin [11] to study gravity-capillary surface waves of solitary type on deep water under the effects of surface tensions represented by the term ∂_x^3 .

Existence and stability of solitary wave solutions as well as the Cauchy problem of the IVP (Bn) have been studied by Albert, Bona and Restrepo in [5], Angulo in [7] and [8]. Regarding the (LWP) and (GWP), by using the contraction mapping principle and the ideias of Kenig, Ponce and Vega, in [47] and [48], Linares proved the (GWP) for (Bn) equation in $L^2(\mathbb{R})$ and for the periodic case in $L^2(\mathbb{T})$. After in [54], Kozono, Ogawa and Tanisaka established the (LWP) in $H^{-\frac{3}{4}+}(\mathbb{R})$. Linares and Scialom, in [68], proved the (LWP) and (GWP) in the energy space for the solutions of the generalized Benjamin equation and generalizations of this. The (GWP) in the Sobolev space $H^{-\frac{3}{4}+}(\mathbb{R})$ was established by Li and Wu in [59]. Moreover, ill-posedness for the IVP (Bn) in $H^s(\mathbb{R})$, $s < -\frac{3}{4}$, was shown by Chen, Guo and Chiao in [19].

Breathers solutions of the dispersive problems are solutions periodic in time. Recently, Ponce and Muñoz in [83] studied the existence of this kind of solutions for the IVP associated to the generalized KdV equation, Gardner equation and a class of non-trivial pertubations of the modified KdV. They found methods to determine whether or not dispersive models present this type of solution. These authors, using the solitons associated to (BO), obtained a decay in time of the Sobolev norm for solutions of (IVP) (BO) in [84]. We devote the Chapter 3 to use the ideias developed in [60], [72], [73] and [82] specially the virial estimates, combined with conserved laws to obtain a decay in an increasing-in-time region of the space for the IVP (BO) and (Bn). Finally, we finish this chapter studying the decay of solutions to a more general class of dispersive models.

This manuscript is organized as follows: In Chapter 1 we start with some general notation and preliminaries that will be implemented to develop the theory in the next chapters. Subsequently, Chapter 2 is devoted to provide results on propagation of regularity for solutions of the IVP (gBOZK) and we proved a (LWP) for the IVP (gBOZK), a Kato's smoothing effect that characterize the spatial behavior of solutions for this family of equations and we study how the regularity behaves when we localize Sobolev norms in certain regions of the space. Finally, the Chapter 3 concerns the study of the asymptotic behavior for solutions of the some nonlinear dispersive models. First, we consider the solutions of the IVP (BO) and study decay properties. Subsequently, we analyse the effect of the dispersion operator "$\partial_x^3 - \mathcal{H}\partial_x^2$" presented in the IVP (Bn) and the action of the virial estimate in this operator producing a decay property of these solutions. We conclude establishing an asymptotic behavior of a general class of dispersive equations and discussing some particular dispersive models which we have the structure to apply the theorem.

1

Preliminaries

1.1 Introduction

Next, we will introduce some important operators and notation useful in the reading of this thesis.

1.2 Notation

Given two positive quantities a and b, $a \lesssim b$ means that there exists $c > 0$ such that $a \leq cb$. The object $[A, B]$ denotes the *commutator* between the operators A and B, that is

$$[A, B] = AB - BA.$$

We shall employ the standard multi-index notation, $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}^d$, $\partial^\gamma = \partial_{x_1}^{\gamma_1} \cdots \partial_{x_d}^{\gamma_d}$, $|\gamma| = \sum_{j=1}^{d} \gamma_j$, and $\gamma! = \gamma_1! \cdots \gamma_d!$.

Given $p \in [1, \infty]$ and $d \geq 1$ integer, $L^p(\mathbb{K})$ or simply L^p denotes the usual *Lebesgue space*, where the set $\mathbb{K}$ will be easily deduced in each context. To emphasize the dependence on the variables when $d = 2$, we will denote by $\|f\|_{L^p(\mathbb{R}^2)} = \|f\|_{L^p_{xy}(\mathbb{R}^2)}$. We denote by $C_c^\infty(\mathbb{R}^d)$ the spaces of *smooth functions with compact support* and $S(\mathbb{R}^d)$ the space of *Schwarz functions.* The Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx, \text{ for } f \in S(\mathbb{R}^d).$$

The operators $D^s = (-\Delta)^{s/2}$ (the *homogeneous derivative of order* $s \in \mathbb{R}$) and $J^s = (I - \Delta)^{s/2}$ denote the *Riesz and Bessel potentials* of order $-s$, respectively. As defined above, $D_x^s f$ and $D_y^s f$ denote the operators

$$\widehat{D_x^s f}(\xi, \eta) = |\xi|^s \widehat{f}(\xi, \eta) \text{ and } \widehat{D_y^s f}(\xi, \eta) = |\eta|^s \widehat{f}(\xi, \eta).$$

or equivalently as $D_x^s = (\mathcal{H}_x \partial_x)^s$, whence $\mathcal{H}_x$ denotes the *Hilbert transform* in the x-direction, that is,

$$(\mathcal{H}_x f)(x,y) = \frac{1}{\pi}\text{p.v.}\int \frac{f(z,y)}{x-z}dz = \Big(-i\,\text{sign}(\xi)\widehat{f}(\xi,\eta)\Big)^{\vee}(x,y).$$

and *p.v.* indicates the Cauchy Principal Value. Analogously, $J_x^s f$ and $J_y^s f$ are determined by

$$\widehat{J_x^s f}(\xi,\eta) = (1+|\xi|^2)^{s/2}\widehat{f}(\xi,\eta) \text{ and } \widehat{J_y^s f}(\xi,\eta) = (1+|\eta|^2)^{s/2}\widehat{f}(\xi,\eta).$$

1.3 Functional Spaces

Given $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ denotes the L^2-based Sobolev space with norm $\|f\|_{H^s} = \|J^s f\|_{L^2}$. If B is a space of functions on $\mathbb{R}$, $T > 0$ and $1 \le p \le \infty$, we define the spaces $L_T^p B$ and $L_t^p B$ by the norms

$$\|f\|_{L_T^p B} = \|\|f(\cdot,t)\|_B\|_{L^p([0,T])} \text{ and } \|f\|_{L_t^p B} = \|\|f(\cdot,t)\|_B\|_{L^p(\mathbb{R})}.$$

To describe our results, we shall fix properly the space solution where the property described below will take place. Following [42], we say that the initial value problem IVP associated to a dispersive equation is *locally well-posed* (LWP) in the Banach space X, if for every initial condition $u_0 \in X$, there exist $T > 0$, and a unique solution $u(t)$ satisfying:

$$u \in C([0,T] : X) \cap A_T \tag{1.1}$$

where A_T is an auxiliary function space. Moreover, the solution map $u_0 \longmapsto u$, is continuous from X into the class (1.1). If T can be taken arbitrarily large, one says that this IVP is *globally well-posed* (GWP) in the space X.

Consider a cutoff function ζ such that

$$0 \le \zeta \le 1, \quad \zeta|_{[-1,1]} = 1, \quad \zeta \in C_0^\infty(\mathbb{R}). \tag{1.2}$$

We introduce the Bourgain spaces as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norms

$$||u||_{X^{s,b}} = \left(\int_{\mathbb{R}^2} \langle \eta + |\xi|\xi\rangle^{2b}\langle\xi\rangle^{2s}|\hat{u}(\xi,\eta)|^2 d\xi d\eta\right)^{1/2} \tag{1.3}$$

$$||u||_{Z^{s,b}} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \langle \eta + |\xi|\xi\rangle^{b}\langle\xi\rangle^{s}|\hat{u}(\xi,\eta)| d\eta\right) d\xi\right)^{1/2} \tag{1.4}$$

$$||u||_{\tilde{Z}^{s,b}} = ||(1-P_{hi})u||_{Z^{s,b}} + \left(\sum_N ||P_N u||^2_{Z^{s,b}}\right)^{1/2} \tag{1.5}$$

$$||u||_{Y^s} = ||u||_{X^{s,\frac{1}{2}}} + ||u||_{\tilde{Z}^{s,0}} \tag{1.6}$$

where $\langle\xi\rangle := 1 + |\xi|$. Now, let $T > 0$, taking $B \in \{X^{s,b},\ Z^{s,b},\ \tilde{Z}^{s,b}$ and $Y^s\}$ and $u : [0,T] \times \mathbb{R} \longrightarrow \mathbb{C}$, we define

$$||u||_{B_T} := \inf\{||v||_B \,|\, v : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C},\ v|_{[0,T]\times\mathbb{R}} = u\}.$$

1.4 Commutators, interpolation and some additional estimates

To obtain estimates for the nonlinear terms, the following results will be implemented along our considerations.

Lemma 1.4.1. *If $s > 0$ and $1 < p < \infty$, then*

$$\|[J^s, f]g\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \left\|J^{s-1}g\right\|_{L^p(\mathbb{R}^d)} + \|J^s f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^\infty(\mathbb{R}^d)} . \tag{1.7}$$

Lemma 1.4.1 was first proved by Kato and Ponce in [43]. See also [12] and the references therein.

Additionally, we recall the following commutator relation for non-homogeneous derivatives.

Lemma 1.4.2. *Let $s > 0$, $1 < p < \infty$ and $1 < p_1, p_2, p_3, p_4 \leq \infty$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then,

(i) if $0 < s \leq 1$,

$$\|[D^s, f]g\|_{L^p(\mathbb{R}^d)} \lesssim \left\|D^{s-1}\nabla f\right\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}(\mathbb{R}^d)} .$$

(ii) If $s > 1$, then

$$\|[D^s, f]g\|_{L^p(\mathbb{R}^d)} \lesssim \left\|D^{s-1}\nabla f\right\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}(\mathbb{R}^d)} + \|\nabla f\|_{L^{p_3}(\mathbb{R}^d)} \left\|D^{s-1}g\right\|_{L^{p_4}(\mathbb{R}^d)} .$$

The above estimates were deduced by D. Li in [58, Corollary 5.3].

Lemma 1.4.3. *Given $s > 0$. Let $1 < p_1, p_2, q_1, q_2 \leq \infty$ with*

$$\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}.$$

Then,

$$\|D^s(fg)\|_{L^r(\mathbb{R}^d)} \lesssim \|D^s f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{q_1}(\mathbb{R}^d)} + \|f\|_{L^{p_2}(\mathbb{R}^d)} \|D^s g\|_{L^{q_2}(\mathbb{R}^d)} \tag{1.8}$$

and

$$\|J^s(fg)\|_{L^2(\mathbb{R}^d)} \lesssim \|J^s f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{L^{q_1}(\mathbb{R}^d)} + \|f\|_{L^{p_2}(\mathbb{R}^d)} \|J^s g\|_{L^{q_2}(\mathbb{R}^d)}. \tag{1.9}$$

The proof of the above estimates can be found in [31]. The case $r = p_1 = p_2 = q_1 = q_2 = \infty$ was established in [15], see also [30]. For earlier versions of this result see [43] and [48].

Lemma 1.4.4. *Let $\phi \in C^\infty(\mathbb{R})$ with $\phi' \in C_0^\infty(\mathbb{R})$. If $s \in \mathbb{R}$, then for any $l > |s-1| + 1/2$*

$$\|[J^s, \phi]f\|_{L^2(\mathbb{R})} + \|[J^{s-1}, \phi]\partial_x f\|_{L^2(\mathbb{R})} \leq c\|J^l \phi'\|_{L^2(\mathbb{R})}\|J^{s-1}f\|_{L^2(\mathbb{R})}. \tag{1.10}$$

The previous lemma was established by Kenig, Linares, Ponce and Vega in [45].

We shall employ the following generalization of Calderon's first commutator estimate in the context of the Hilbert transform deduced in [24, Lemma 3.1] (see also [58, Proposition 3.8]).

Proposition 1.4.5. *Let $1 < p < \infty$ and $l, m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \geq 1$, then*

$$\|\partial_x^l [\mathcal{H}_x, g]\partial_x^m f\|_{L^p(\mathbb{R})} \lesssim_{p,l,m} \|\partial_x^{l+m} g\|_{L^\infty(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}. \tag{1.11}$$

The next estimate is an inequality of Gagliardo-Nirenberg type whose proof can be found in [13].

Lemma 1.4.6. *There exists $C > 0$ such that for any $f \in H^{1/2}(\mathbb{R})$*

$$\|f\|_{L^3} \leq C\|f\|_{L^2}^{\frac{2}{3}}\|D^{1/2}f\|_{L^2}^{\frac{1}{3}}. \tag{1.12}$$

Finally, we consider a commutator estimate.

Lemma 1.4.7. *Let $a \in C^2(\mathbb{R})$ with a', $a'' \in L^\infty(\mathbb{R})$. There exists $c > 0$ such that for all $f \in L^2(\mathbb{R})$*

$$\|D^{1/2}\big[D^{1/2}; a\big]f\|_{L^2} \leq c\|\widehat{a'}\|_{L^1}\|f\|_{L^2} \leq c\|a'\|_{L^2}^{1/2}\|a''\|_{L^2}^{1/2}\|f\|_{L^2}. \tag{1.13}$$

Proof of Lemma 1.4.7. One sees that

$$\begin{aligned}
&\Big(D^{1/2}\widehat{\big[D^{1/2}; a\big]}f\Big)(\xi) \\
&= |\xi|^{1/2}\int |\xi|^{1/2}\Big(\widehat{a}(\xi-\eta)\widehat{f}(\eta) - \widehat{a}(\xi-\eta)|\eta|^{1/2}\widehat{f}(\eta)\Big)\,d\eta.
\end{aligned} \tag{1.14}$$

Therefore

$$\begin{aligned}&\left|\left(D^{1/2}\widehat{\left[D^{1/2};a\right]f}\right)(\xi)\right|\\&\le\int|\xi|^{1/2}\left||\xi|^{1/2}-|\eta|^{1/2}\right|\left|\widehat{a}(\xi-\eta)\widehat{f}(\eta)\right|d\eta.\end{aligned}\tag{1.15}$$

Assuming the following claim :

$$\text{There exists } c>0 \text{ such that for all } \xi,\ \eta\in\mathbb{R} \text{ implies } |\xi|^{1/2}\Big||\xi|^{1/2}-|\eta|^{1/2}\Big|\le c|\xi-\eta|,\tag{1.16}$$

we shall conclude the proof.

From (1.16) it follows that

$$\begin{aligned}E_1(\xi)\equiv&\left|\left(D^{1/2}\widehat{\left[D^{1/2};a\right]f}\right)(\xi)\right|\\&\le c\int|\xi-\eta||\widehat{a}(\xi-\eta)\widehat{f}(\eta)|d\eta=c\int|\widehat{a'}(\xi-\eta)\widehat{f}(\eta)|d\eta.\end{aligned}\tag{1.17}$$

Thus,

$$\|E_1\|_2=\|\widehat{a'}*\widehat{f}\|_{L^2}\le\|\widehat{a'}\|_{L^1}\|f\|_{L^2}.\tag{1.18}$$

Using that

$$\begin{aligned}\|\widehat{a'}\|_1&=\int_{|\xi|\le R}|\widehat{a'}|d\xi+\int_{|\xi|>R}\frac{|\xi||\widehat{a'}|}{|\xi|}\,d\xi\\&\le cR^{1/2}\|\widehat{a'}\|_{L^2}+cR^{-1/2}\|\widehat{a''}\|_{L^2}.\end{aligned}\tag{1.19}$$

Choosing $R=\|\widehat{a''}\|_{L^2}^{1/2}/\|\widehat{a'}\|_{L^2}^{1/2}$ we obtain (1.13).

It remains to prove the claim in (1.16). First, we consider the case where ξ and η have the same sign, so we assume $\xi,\ \eta>0$. In this setting one sees that for some $\theta\in(0,1)$

$$\left|\xi^{1/2}-\eta^{1/2}\right|=\frac{1}{(\theta\xi+(1-\theta)\eta)^{1/2}}\,|\xi-\eta|.\tag{1.20}$$

Thus, if $0<\xi/10<\eta$, (1.20) yields the estimate in (1.16).

If $0<\eta\le\xi/10$, one has

$$\xi^{1/2}\left(\xi^{1/2}-\eta^{1/2}\right)\le\xi\le2|\xi-\eta|.$$

In the case where ξ,η have different signs, one sees that

$$|\xi-\eta|=|\xi|+|\eta|,$$

and the estimate (1.16) holds.

□

Now, we present several commutator expansions for the operator $[-\mathcal{H}_xD_x^a,h]$ in one-dimensional variable. These results are due to Ginibre and Velo in [28, 29]. We require to introduce some additional notation. Let $a = 2\mu + 1 > 1$, n be a nonegative integer and h a smooth function with suitable decay at infinity, for instance with $h' \in C_0^\infty(\mathbb{R})$. We define the operators

$$R_n(a) := [HD_x^a; h] - \frac{1}{2}(P_n(a) - HP_n(a)H), \tag{1.21}$$

where $H = -\mathcal{H}_x$ and

$$P_n(a) := a \sum_{0 \leq j \leq n} c_{2j+1}(-1)^j D_x^{\mu-j} h^{2j+1} D_x^{\mu-j}, \tag{1.22}$$

with

$$c_1 = 1 \text{ and } c_{2j+1} = \frac{1}{(2j+1)!} \prod_{0 \leq k \leq j} (a^2 - (2k+1)^2). \tag{1.23}$$

The next proposition establish some properties for the operator $R_n(a)$.

Proposition 1.4.8. *Let n be a non-negative integer, $a \geq 1$, and $b \geq 0$ be such that*

$$2n + 1 \leq a + 2b \leq 2n + 3. \tag{1.24}$$

Then

(i) the operator $D_x^b R_n(a) D_x^a$ is bounded in $L^2(\mathbb{R})$ and satisfy

$$\left\| D_x^b R_n(a) D_x^b f \right\|_{L^2} \leq (2\pi)^{-1/2} C \left\| \widehat{D_x^{a+2b} h} \right\|_{L^1_\xi} \|f\|_{L^2}. \tag{1.25}$$

In particular if $a \geq 2n + 1$, one can take $c = 1$.

(ii) If in addition $a + 2b < 2n + 3$, then the operator $D_x^b R_n(a) D_x^a$ is compact in $L^2(\mathbb{R})$.

Proof. See Proposition 2.2 in [29]. □

1.5 Pseudo-differential Operators

To facilitate the exposition of our results, this section is intended to briefly indicate some preliminaries results concerning pseudo-differential operators, as well as, some of their consequences.

Definition 1.5.1. *Let $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying*

$$|\partial_x^\beta \partial_\xi^\gamma a(x, \xi)| \leq A_{\gamma,\beta}(1 + |\xi|)^{m-|\gamma|},$$

for some $m \in \mathbb{R}$ and for all the multi-index γ and β. This function a will be called a symbol of order m in $\mathcal{S}^m(\mathbb{R}^d \times \mathbb{R}^d)$, simplifying $\mathcal{S}^m$ will represent the set of these type of functions.

Definition 1.5.2. *A pseudo-differential operator is a mapping $f \mapsto \Psi_a f$ given by*

$$(\Psi_a f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)\widehat{f}(\xi)\, d\xi,$$

where $a(x,\xi) \in \mathcal{S}^m$ is the symbol of Ψ_a for some $m \in \mathbb{R}$.

Through this thesis Ψ_a will indicate a pseudo-differential operator with symbol $a \in \mathcal{S}^m$ for some $m \in \mathbb{R}$. After these definitions, we will need the following theorem.

Theorem 1.5.3. *Let a be a symbol of order 0, i.e., $a \in \mathcal{S}^0$. Then, the operator Ψ_a, initially defined on $S(\mathbb{R}^d)$ can be extended to a bounded operator from $L^2(\mathbb{R}^d)$ to itself.*

Proof. The proof can be consulted in [98, Chapter VI, Theorem 1]. □

For a generalization of Theorem 1.5.3, we refer to [90, Chapter 3, Theorem 3.6]. A key ingredient in our considerations is the following kernel representation for pseudo-differential operators.

Proposition 1.5.4. *Let $a \in \mathcal{S}^m$ and Ψ_a its associated pseudo-differential operator. Then, there exists a kernel $k_a \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ satisfying the following properties:*

(i) The operator admits the following representation

$$\Psi_a f(x) = \int k_a(x, x-y) f(y)\, dy,$$

for all $x \notin \operatorname{supp}(f)$,

(ii) Additionally, for all multi-indices γ and β, and all $N \geq 0$, integer, it follows

$$|\partial_x^\beta \partial_z^\gamma k_a(x,z)| \leq A_{\gamma,\beta,N}|z|^{-d-m-|\gamma|-N}, \quad z \neq 0,$$

whenever $d + m + |\gamma| + N > 0$.

The following result will be useful to approximate the composition between pseudo-differential operators.

Proposition 1.5.5. *Let a and b symbols belonging to $\mathcal{S}^{r_1}$ and $\mathcal{S}^{r_2}$ respectively. Then, there is a symbol $c \in \mathcal{S}^{r_1+r_2}$ so that*

$$\Psi_c = \Psi_a \circ \Psi_b.$$

Moreover,

$$c \sim \sum_\beta \frac{1}{(2\pi i)^{|\beta|}\beta!}(\partial_\xi^\beta a)\cdot(\partial_x^\beta b), \tag{1.26}$$

in the sense that

$$c - \sum_{|\beta|<N} \frac{(2\pi i)^{-|\beta|}}{\beta!}(\partial_\xi^\beta a)\cdot(\partial_x^\beta b) \in \mathcal{S}^{r_1+r_2-N}$$

for all $N \geq 0$.

Another important consequence regarding pseudo-differential operators is the following symbolic calculus for commutators.

Proposition 1.5.6. *For $a \in \mathcal{S}^{r_1}$ and $b \in \mathcal{S}^{r_2}$ we define the commutator $[\Psi_a, \Psi_b]$ by*

$$[\Psi_a, \Psi_b] = \Psi_a \circ \Psi_b - \Psi_b \circ \Psi_a.$$

Then, the symbol of the operator is given by

$$c = \frac{1}{2\pi i} \sum_{j=1}^{d} \left(\frac{\partial a}{\partial_{\xi_j}} \frac{\partial b}{\partial_{x_j}} - \frac{\partial a}{\partial_{x_j}} \frac{\partial b}{\partial_{\xi_j}} \right) \; mod \; \mathcal{S}^{r_1+r_2-2}. \tag{1.27}$$

2

Propagation of Regularity

2.1 Introduction

The main objective of this chapter is to determine how dispersion affects the regularity of solutions when we restrict the initial data to a certain class of subsets of the Cartesian plane. This is why, we consider a nonlinear model that represents a dispersive interpolation between the ZK equation and the BOZK equation. In fact, in many problems arising from Physics or Continuum Mechanics, these models are considered to determine competition between the nonlinearity and the dispersion. In our case, we are interested in studying the propagation of regularity of solutions of the IVP of (gBOZK). Our motivation comes from the results shown in [39], where considering suitable solutions to the IVP associated to the $k-$generalized KdV equation, it was determined propagation of regularity on the right-hand side (r.h.s) half space where behaves the extra smoothness of the initial datum for positive times. In that sense, we will show that the solutions of (gBOZK) satisfies this property when we restrict ourselves to an appropriated class of half-spaces determined by the dominant direction of the dispersion. For other studies on the propagation of regularity for dispersive models we refer to [40, 67, 82, 85, 95]. The main results in this chapter are contained in [27]

Real-valued solutions of the IVP (gBOZK) (smooth enough) formally satisfy the following conserved quantities (time invariant):

$$\mathcal{I}(u) = \int_{\mathbb{R}^2} u(x,y,t)\, dxdy \tag{2.1}$$

$$\mathcal{M}(u) = \int_{\mathbb{R}^2} u^2(x,y,t)\, dxdy, \tag{2.2}$$

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left\{ |D_x^{\frac{\alpha+1}{2}} u(x,y,t)|^2 + |\partial_y u(x,y,t)|^2 - \frac{1}{3}u^3(x,y,t) \right\} dxdy. \tag{2.3}$$

Regarding (LWP) for (gBOZK) in L^2-type Sobolev spaces, the current best known result available was determined by Ribaud and Vento [92]. They addressed this question in the space E^s defined by $\|f\|_{E^s} = \left\|(1+|\xi|^{\alpha+1}+\eta^2)^{s/2}\widehat{f}(\xi,\eta)\right\|_{L^2}$. It was established that (gBOZK) is (LWP) in E^s whenever $s > \frac{2}{\alpha+1} - \frac{3}{4}$ for $0 \leq \alpha \leq 1$, and (GWP) in the energy space $E^{1/2}$ as soon as $\alpha > 3/5$. We remark that their results are based on the short-time Bourgain spaces approach developed by Ionescu, Kenig and Tataru [37], combined with localized Strichartz estimates and a modified energy technique. Very recently, Cunha and Pastor in [23] studied the Cauchy problem (gBOZK) in weighted anisotropic Sobolev spaces as well as some unique continuation principles, which establish optimal spatial decay in the x-spatial variable.

Since our main purposes depend on techniques based on weighted energy estimates for the equation in (gBOZK), it is not clear how to address the propagation of regularity phenomena for solutions provided by the (LWP) result in [92] relaying on the short-time Bourgain spaces. Instead, we establish the following local well-posedness result which is suitable with the methods developed in this work.

Theorem 2.1.1. *Assume that $0 \leq \alpha < 1$ fixed. Let $s > s_\alpha$, where $s_\alpha := (17-2\alpha)/12$. Then, for any $u_0 \in H^s(\mathbb{R}^d)$, there exist a positive time $T = T(\|u_0\|_{H^s})$, and a unique solution u to* (gBOZK) *that belongs to*

$$C\Big([0,T]; H^s(\mathbb{R}^2)\Big) \cap L^1\Big([0,T]; W^{1,\infty}(\mathbb{R}^2)\Big). \tag{2.4}$$

Moreover, the flow map $u_0 \mapsto u(t)$ is continuous from $H^s(\mathbb{R}^2)$ to $H^s(\mathbb{R}^2)$.

Theorem 2.1.1 is deduced by means of the short-time linear Strichartz approach method developed by Koch and Tzvetkov [51], and its extension given by Kenig and König [44]. See [34, 49, 65] for applications to higher-dimensional models. Since we are more concerned with special regularity properties for (gBOZK), the proof of Theorem 2.1.1 is given in the appendix in the end of this Chapter. We shall emphasize that the Sobolev regularity attained in Theorem 2.1.1 does not yield to an improvement with respect to the conclusions in [92], and to the results in [22, 86] for $\alpha = 0$. Nevertheless, when $0 \leq \alpha < 1$, Theorem 2.1.1 states the best-known result involving solutions of (gBOZK) in the class (2.4). This conclusion is useful to deal with techniques based on energy estimates as the one we are interested in this work.

In [42], T. Kato deduced a property known as Kato's smoothing effect to the (KdV) equation. He proved that the solution u of this equation gains one local derivative in comparison to the initial data u_0 , that is,

$$\left(\int_{-T}^{T}\int_{-R}^{R} |\partial_x u(x,t)|^2 dxdt\right)^{1/2} \leq C(T,R)||u_0||_{L^2(\mathbb{R})}, \quad \textit{for any} \quad R > 0. \tag{2.5}$$

Inspired by how the regularity will propagate if you localize the smooth properties on some unbounded region of the space, Isaza, Linares and Ponce in [39] started the study of this kind of regularity properties to dispersive equations. The study for the (IVP) (BO) it was established in [40], by the same authors, here the dispersive operator is a non-local operator and Kato's smoothing effect only guarantees the gain of local $D_x^{1/2}$ derivative, then is necessary to iterate the process twice to obtain the desired result. After, Linares, Ponce and Smith in [67] showed that this fact is valid for solutions of the general quasilinear equation of KdV type. Sgata and Smith in [95] established the propagation of regularity and persistence of decay for fifth order dispersive models.

In [45], Kenig et al. studied the fractionary propagation localized in space, i. e., how $||J^s U_0||_L^2(x_0,\infty)$ propagates to the solution of the solutions to the k-generalized Korteweg-de Vries equation. Using the ideias in [45] and using a special decomposition of a commutator, Mendez in [74] and [75] proved in a iteration process the fractionary propagation of regularity for the IVP associated to generalized (BO) and fractionary (KdV) equation. To propagation of regularity of dispersive models in higher dimensions, we refer [85] and [76].

Since we have described all the requirements that our space solution has to satisfy, we present our main result that is summarized in the following theorem.

Theorem 2.1.2. *Assume* $0 \leq \alpha < 1$ *fixed. Let* $u_0 \in H^{s_\alpha^+}(\mathbb{R}^2)$ *where* $s_\alpha = (17-2\alpha)/12$. *If for some* $s \in \mathbb{R}$, $s > s_\alpha$ *and* $x_0 \in \mathbb{R}$

$$\|J_x^s u_0\|^2_{L^2_{xy}((x_0,\infty)\times\mathbb{R})} = \int_{-\infty}^{\infty}\int_{x_0}^{\infty} (J_x^s u_0)^2(x,y)\, dxdy < \infty, \tag{2.6}$$

then the corresponding solution of the IVP (gBOZK) *provided by Theorem 2.1.1 satisfies for any* $v > 0$ *and any* $\epsilon > 0$, $\tau \geq 5\epsilon$,

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{-\infty}^{\infty}\int_{x_0+\epsilon-vt}^{\infty} (J_x^r u)^2(x,y,t)\, dxdy \\ &+ \int_0^T \int_{-\infty}^{\infty}\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} \Big((D_x^{\frac{\alpha+1}{2}} J_x^s u)^2(x,y,t) + (\partial_y J_x^s u)^2(x,y,t)\Big)\, dxdydt \leq c, \end{aligned} \tag{2.7}$$

for all $r \in (0,s]$, *where* $c = c\Big(\epsilon; T; v; \|u_0\|_{H^{s_\alpha^+}}; \|J_x^s u_0\|_{L^2_{xy}((x_0,\infty)\times\mathbb{R}))}\Big) > 0$.

If in addition to (2.6),

$$\|J^{s+\frac{1-\alpha}{2}} u_0\|^2_{L^2_{xy}((x_0,\infty)\times\mathbb{R})} = \int_{-\infty}^{\infty}\int_{x_0}^{\infty} \Big(J_x^{s+\frac{1-\alpha}{2}} u_0\Big)^2 (x,y)\, dxdy < \infty, \tag{2.8}$$

then for any $v > 0$, $\epsilon > 0$ *and* $\tau \geq 5\epsilon$,

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{-\infty}^{\infty}\int_{x_0+\epsilon-vt}^{\infty} (J_x^r u)^2(x,y,t)\, dxdy \\ &+ \int_0^T \int_{-\infty}^{\infty}\int_{x_0+\epsilon-vt}^{x_0+\tau-vt} \Big((J_x^{s+1} u)^2(x,y,t) + (\partial_y J_x^{s+\frac{1-\alpha}{2}} u)^2(x,y,t)\Big)\, dxdydt \leq c, \end{aligned}$$

for all $r \in \Big(0, s+\frac{1-\alpha}{2}\Big]$, *where the constant depends on the following parameters* $c = c\Big(\epsilon; T; v; \|u_0\|_{H^{s_\alpha^+}}; \|J_x^{s+\frac{1-\alpha}{2}} u_0\|_{L^2_{xy}((x_0,\infty)\times\mathbb{R}))}\Big) > 0$.

The result in Theorem 2.1.2 is relevant to predict the behavior of the flow solution in terms of regularity just by knowing how regular the initial data is on a particular class of subsets of the plane. More precisely, for $\epsilon \geq 0$ and $v \geq 0$, we set the family of half-spaces

$$\mathfrak{H}_{\epsilon}(t) := \left\{(x,y) \in \mathbb{R}^2 \,|\, x \geq x_0 + \epsilon - vt\right\}, \quad t \geq 0.$$

The first term on the r.h.s of (2.7) describes the following behavior: The regularity in the $x-$direction of u_0 in the half space $\mathfrak{H}_{\epsilon}(0)$, that is, $J_x^s u_0 \in L^2\left(\mathfrak{H}_{\epsilon}(0)\right)$ is propagated with infinite speed to the left by the flow solution.

In other words,

$$J_x^r u(\cdot,t) \in L^2\left(\mathfrak{H}_{\epsilon}(t)\right) \subset L^2\left(\mathfrak{H}_0(0)\right) \quad \text{for all} \quad t > 0, \tag{2.9}$$

and all $r \in (0,s]$.

Furthermore, the second term on the r.h.s of (2.7) describes the extra regularity obtained in a particular class of subset of the plane, this phenomenon can be better understood by defining a new class of subsets as we did previously. More precisely, for $\epsilon > 0$ and $\tau \geq 5\epsilon$, we define the channel

$$\mathcal{Q}(t) := \left\{(x,y) \in \mathbb{R}^2 \,|\, x_0 - \epsilon + vt < x < x_0 + \tau - vt\right\} \quad \text{for all} \quad t \geq 0. \tag{2.10}$$

In this setting, the second term on the r.h.s of (2.7) describes the smoothing effect of the solution in the channel $\mathcal{Q}(t)$, for all $t > 0$. Unlike the studied for solutions of the ZK equations (cf. [76]), in our case, the solution enjoys of some "anisotropic smoothing effect" it means that u becomes smoother by one derivative in the $y-$variable when we restrict to $\mathcal{Q}(t)$ for $t > 0$. Instead, in the $x-$variable a "weaker" smoothing occurs since there is only a gain of $\frac{\alpha+1}{2}$ derivatives prescribed in the channel $\mathcal{Q}(t)$ for $t > 0$. In geometrical terms, the above dynamic can be summarized in Figure 3 below.

Additionally, it is worth emphasizing several issues that do not fall under the scope of Theorem 2.1.2. In comparison with our conclusions, we notice that for the case of the (ZK) equation ($\alpha = 1$), their solutions propagate regularity in both variables in a wider class of subsets of the plane (cf. [76]). Certainly, this contrast regarding the behavior with (ZK) could be attributed to the differences in the nature of the fractional operator involved in the dispersion in (gBOZK), which is non-local and tend to spread out all the information. We think that describing the full behavior in both variables requires an analysis that goes beyond the methods employed in this work, and it would require new tools to handle the interaction between the operators J^s (in the full variables) and $D_x^{\alpha+1}$.

The proof of Theorem 2.1.2 follows in spirit the techniques and arguments presented in [39, 74, 75, 76] regarding propagation of local derivatives, and the conclusions in [45] for the fractional setting. However, in the case of (gBOZK), we face several additional difficulties expected from the interaction between the dispersion $\partial_x D_x^{\alpha+1}$ and the

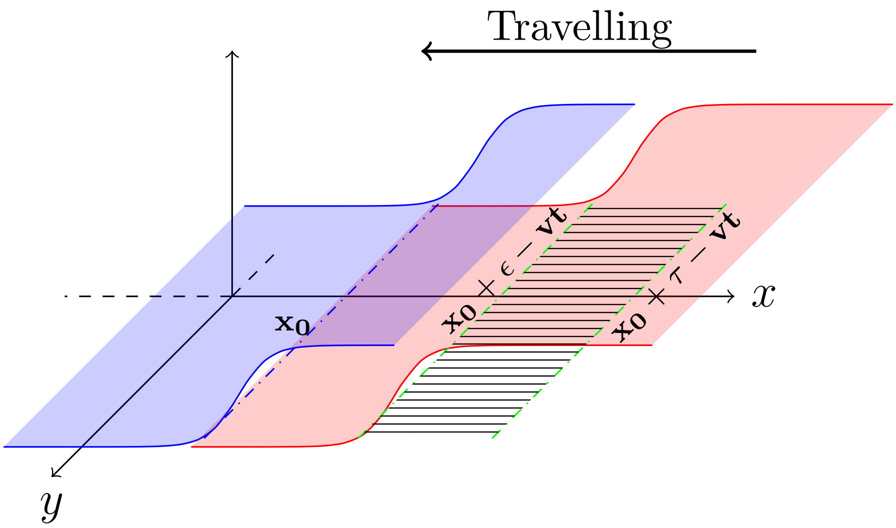

Figure 3 – Propagation of regularity from $\mathfrak{H}_\epsilon(0)$ to $\mathfrak{H}_\epsilon(t)$.

operator J_x^s. Among them, we require to deduce new localization formulas (see Lemma 2.2.5 below) relating the propagation on certain domains between homogeneous and non-homogeneous derivatives. This analysis is provided by studying the kernel determined by the difference $J_x^s - D_x^s$ as well as examining some class of pseudo-differential operators. In fact, we believe that these localization formulas are of independent interest and could certainly be applied to a wider class of equations in arbitrary spatial dimension. For an application involving differential operators and the aforementioned kernel, we refer to the work of Bourgain and Li [15].

Remark 2.1.3. *In the case of physical relevance $\alpha = 0$ in* (gBOZK)*, i.e., the BOZK equation, Theorem 2.1.2 leads to an extension to the fractional setting of the conclusions derived in [86, Theorem 1.4] concerning the propagation of regularity principle for local derivatives.*

Remark 2.1.4. *The method of proof of Theorem 2.1.2 applies for a given initial data $u_0 \in H^r(\mathbb{R}^2)$ with arbitrary regularity $r > 0$ provided that one can assure the existence of a corresponding solution $u \in C([0,T], H^r(\mathbb{R}^2))$ of* (gBOZK) *such that*

$$u, \partial_x u \in L^1([0,T]; L^\infty(\mathbb{R}^2)).$$

In particular, in the case of the BOZK equation, $\alpha = 0$ in (gBOZK)*, the LWP result in [86, Theorem 1.3] determines the validity of Theorem 2.1.2 for initial data with regularity $H^r(\mathbb{R}^2)$, $r > 5/4$.*

Remark 2.1.5. *We believe that our results can be adapted to study the propagation of regularity principle for other two-dimensional models involving non-local operators. For instance, we expect to obtain similar conclusion to that of Theorem 2.1.2 for solutions of* *the Cauchy problem associated to the fractional Kadomtsev Petviashvili-equations (KP) type equation*

$$\partial_t u - \partial_x D_x^{\alpha} u + \kappa \partial_x^{-1} \partial_y^2 u + u u_x = 0, \quad (x, y, t) \in \mathbb{R}^3,$$

where $0 < \alpha \leq 2$*, see [65]. Additionally, we expect that our considerations may work as an initial step to obtain fractional propagation of regularity for the IVP associated to the Shrira equation*

$$\partial_t u - \mathcal{H}_x \partial_x^2 u - \mathcal{H}_x \partial_y^2 u + u \partial_x u = 0, \quad (x, y, t) \in \mathbb{R}^3.$$

This equation was deduced as a simplified model to describe a two-dimensional weakly nonlinear long-wave perturbation on the background of a boundary-layer type plane-parallel shear flow (see [88]). For some references dealing with LWP issues see [18, 91].

2.2 Pseudo-differential operators and weighted functions

In this section, we will use the definitions and the results of the Section 1.5 to obtain information of the action of a pseudo-differential operator in a function which has the regularity concentrated in some region of the space.

Proposition 2.2.1. *Let* $s > 0$ *and* m, d *be positive integers such that* $m \geq \max\{s, d\}$*. Additionally, we consider* $\theta \in C^{\infty}(\mathbb{R}^d)$*,* $0 \leq \theta \leq 1$*, satisfying that* $\partial^{\gamma}\theta \in L^{\infty}(\mathbb{R}^d)$ *for any multi-index* γ*. Then, there exist some constants* c_{γ}*, pseudo-differential operators of order zero* Ψ^{γ} *for each multi-index* $0 \leq |\gamma| \leq m$*, and* K_{s-m} *of order* $s - m$ *such that*

$$J^s(\theta f) = \sum_{1 \leq |\gamma| \leq m} c_{\gamma} \partial_x^{\gamma} \theta \, \Psi^{\gamma}(J^{s-|\gamma|} f) + \theta J^s f + K_{s-m} f, \tag{2.11}$$

provided that f *is regular enough.*

Proof. In virtue of the identity

$$J^s(\theta f) = [J^s, \theta] f + \theta J^s f,$$

we are reduced to decompose the operator $[J^s, \theta]$. Thus, by employing Propositions 1.5.5 and 1.5.6, we find

$$[J^s, \phi] f(x) = \int a_s(x, \xi) \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi,$$

where

$$a_s(x, \xi) = \sum_{1 \leq |\gamma| \leq m} c_{\gamma} \partial_{\xi}^{\gamma} \langle \xi \rangle^s \partial_x^{\gamma} \theta(x) + k_{s-m}, \tag{2.12}$$

and $k_{s-m}\in\mathcal{S}^{s-m}\subset\mathcal{S}^0$. Now, since

$$\sum_{1\leq|\gamma|\leq m} c_\gamma \partial_\xi^\gamma(\langle\xi\rangle^s)\partial_x^\gamma\theta(x) \;=\; \sum_{1\leq|\gamma|\leq m} c_\gamma \frac{\partial_\xi^\gamma(\langle\xi\rangle^s)}{\langle\xi\rangle^{s-|\gamma|}}(\langle\xi\rangle^{s-|\gamma|}\partial_x^\gamma\theta(x)),$$

we are led to define the pseudo-differential operator Ψ^γ according to the symbol

$$\zeta^\gamma(\xi):=\langle\xi\rangle^{|\gamma|-s}\partial_\xi^\gamma(\langle\xi\rangle^s)\in\mathcal{S}^0,$$

for each $0\leq|\gamma|\leq m$. Gathering the preceding results, it follows

$$[J^s,\theta]\,f=\sum_{1\leq|\gamma|\leq m} c_\gamma\partial_x^\gamma\theta\Psi^\gamma(J^{s-\gamma}f)+K_{s-m}f,$$

where K_{s-m} is the pseudo-differential operator with symbol $k_{s-m}\in\mathcal{S}^{s-m}$ defined as in (2.12). This completes the proof. □

The kernel representation of pseudo-differential operators has been applied to obtain some regularity properties for the product of functions with separated supports. In this regard, the following result was deduced in [76].

Lemma 2.2.2. *Let γ be a multi-index and Ψ_a a pseudo-differential operator of order m. If $g\in L^2(\mathbb{R}^d)$ and $f\in L^p(\mathbb{R}^d)$, $p\in[2,\infty]$ with*

$$\operatorname{dist}(\operatorname{supp}(f),\operatorname{supp}(g))\geq\delta>0.$$

Then,

$$\|f\partial_x^\gamma\Psi_a g\|_{L^2}\lesssim\|f\|_{L^p}\|g\|_{L^2}.$$

We also require some fractional version of the above lemma. For that reason, we are interested in investigating some interactions between the non-local operators D^s and J^s. Broadly speaking, by taking advantage of the kernel obtained by the difference J^s-D^s, the idea is to transfer localization properties between homogeneous and non-homogeneous derivatives. We refer to [15] for an application dealing with the difference J^s-D^s.

Lemma 2.2.3. *Let $s\in\mathbb{R}$, $s_1\in(0,1)$. If $f\in L^\infty(\mathbb{R}^d)$ and $g\in L^2(\mathbb{R}^d)$ with*

$$\operatorname{dist}(\operatorname{supp}(f),\operatorname{supp}(g))\geq\delta>0.$$

Then,

$$\|fD^{s_1}J^s g\|_{L^2}\lesssim\|f\|_{L^\infty}\|g\|_{L^2}.$$

Proof. We begin by choosing an integer $M>1$, such that $M>(s+s_1)/2$. By employing the binomial expansion, we get

$$\begin{aligned}\langle\xi\rangle^s\Big(\langle\xi\rangle^{s_1}-|\xi|^{s_1}\Big)&=\langle\xi\rangle^{s+s_1}(1-(1-\langle\xi\rangle^{-2})^{s_1/2})\\&=\sum_{j=1}^{M-1}\binom{s_1/2}{j}\frac{(-1)^{j+1}\langle\xi\rangle^{s+s_1}}{\langle\xi\rangle^{2j}}+\sum_{j=M}^{\infty}\binom{s_1/2}{j}\frac{(-1)^{j+1}}{\langle\xi\rangle^{2j-s-s_1}}\\&=:k_M(\xi)+\sum_{j=M}^{\infty}\binom{s_1/2}{j}\frac{(-1)^{j+1}}{\langle\xi\rangle^{2j-s-s_1}},\end{aligned}$$

where given that $s_1 > 0$, the above series converges absolutely. Thus, we set the operator K_M by

$$K_M f(x) = \int k_M(\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi.$$

Consequently, we deduce the identity

$$fD^{s_1}J^s g = fJ^{s+s_1}g - fK_M g - \sum_{j=M}^{\infty}\binom{s_1/2}{j}(-1)^{j+1}(fJ^{s+s_1-2j}g). \tag{2.13}$$

We are led to estimate each factor on the right-hand side of the above equality. Since the Bessel potential satisfies $\|J^{-s'}g\|_{L^p} \leq \|g\|_{L^p}$, for any $1 \leq p \leq \infty$, and $s' > 0$, we deduce from our choice of M,

$$\begin{aligned}\|\sum_{j=M}^{\infty}\binom{s_1/2}{j}(-1)^{j+1}(fJ^{2j-s-s_1}g)\|_{L^2} &\leq \sum_{j=M}^{\infty}\left|\binom{s_1/2}{j}\right|\|f\|_{L^\infty}\|J^{s+s_1-2j}g\|_{L^2}\\ &\lesssim \|f\|_{L^\infty}\|g\|_{L^2}.\end{aligned}$$

Next, by the hypothesis between the supports of f and g, the estimate for $fJ^{s+s_1}g$ is a consequence of Lemma 2.2.2. Likewise, noticing that K_M defines a pseudo-differential operator of order $s+s_1-2$, the required estimate is again a consequence of Lemma 2.2.2. Summarizing,

$$\|fJ^{s+s_1}g\|_{L^2} + \|fK_M g\|_{L^2} \lesssim \|f\|_{L^\infty}\|g\|_{L^2}.$$

Going back to identity (2.13), we gather the previous results to complete the proof of the lemma. □

The same arguments in the proof of Lemma 2.2.3 provide the following generalization.

Corollary 2.2.4. *Let Ψ_a be a pseudo-differential operator with symbol $a \in \mathcal{S}^m$, for some $m \in \mathbb{R}$, and $s_1 \in [0,1)$. If $f \in L^\infty(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$ are such that*

$$\operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) \geq \delta > 0.$$

Then,

$$\|fD^{s_1}\Psi_a g\|_{L^2} \lesssim \|f\|_{L^\infty}\|g\|_{L^2}.$$

2.2.1 Localized Regularity

This subsection introduces the main tool required to deduce Theorem 2.1.2. Mainly, the idea is to provide formulas connecting the propagation of regularity effect in different domains. Estimates of this kind were previously presented in [45] and [76]. A contribution of the present work is the deduction of (I) and (II) below, which connect the operators D^s and J^s in different regions. We observe that these estimates are deduced by bounding the kernel determined by the difference $D^s - J^s$. Additionally, we emphasize that similar estimates (III) and (IV) were previously determined in the work of [76], but here, we perform some minor changes to consider functions $f \in H^{-m}(\mathbb{R}^d)$, $m \geq 0$.

Lemma 2.2.5. *Let $f \in H^{-m}(\mathbb{R}^d)$ for some integer $m \geq 0$, and $\theta_1, \theta_2 \in C^{\infty}(\mathbb{R}^d) \setminus \{0\}$ such that $0 \leq \theta_1, \theta_2 \leq 1$,*

$$\operatorname{dist}(\operatorname{supp}(1-\theta_1), \operatorname{supp}(\theta_2)) \geq \delta > 0, \tag{2.14}$$

and satisfying $\partial^{\gamma}\theta_1, \partial^{\gamma}\theta_2 \in L^{\infty}(\mathbb{R}^d)$ for all multi-index γ.

(I) If $0 \leq \beta < 2$ and $\theta_1 f, \theta_1 D^{\beta} f \in L^2(\mathbb{R}^d)$, then

$$\|\theta_2 J^{\beta} f\|_{L^2} \lesssim \|\theta_1 f\|_{L^2} + \|\theta_1 D^{\beta} f\|_{L^2} + \|J^{-m} f\|_{L^2},$$

so that $\theta_2 J^{\beta} f \in L^2(\mathbb{R}^d)$.

(II) If $0 \leq \beta < 2$, and $\theta_1 J^{\beta} f \in L^2(\mathbb{R}^d)$, then

$$\|\theta_2 f\|_{L^2} + \|\theta_2 D^{\beta} f\|_{L^2} \lesssim \|\theta_1 J^{\beta} f\|_{L^2} + \|J^{-m} f\|_{L^2},$$

and so $\theta_2 f, \theta_2 D^{\beta} f \in L^2(\mathbb{R}^d)$.

(III) If $s > 0$, $0 \leq r \leq s$, and $\theta_1 J^s f \in L^2(\mathbb{R}^d)$, then

$$\|\theta_2 J^{r} f\|_{L^2} \lesssim \|\theta_1 J^{s} f\|_{L^2} + \|J^{-m} f\|_{L^2},$$

and so $\theta_2 J^r f \in L^2(\mathbb{R}^d)$.

(IV) If $s > 0$ and $\theta_1 J^s f \in L^2(\mathbb{R}^d)$, then

$$\|J^{s}(\theta_2 f)\|_{L^2} \lesssim \|\theta_1 J^{s} f\|_{L^2} + \|J^{-m} f\|_{L^2},$$

that is, $J^s(\theta_2 f) \in L^2(\mathbb{R}^d)$.

Proof. We first deduce (I). We begin by analyzing the difference between $J^{\beta} - D^{\beta}$ as it was done in the proof of Lemma 2.2.3. For this purpose, let us consider some integer $M > 1$ fixed such that $M > 2m + \beta/2$, where m is such that $f \in H^{-m}(\mathbb{R}^d)$. By means of the binomial expansion, we have

$$\begin{aligned} \langle \xi \rangle^{\beta} - |\xi|^{\beta} &= \langle \xi \rangle^{\beta}(1 - (1 - \langle \xi \rangle^{-2})^{\beta/2}) \\ &= \sum_{j=1}^{M-1} \binom{\beta/2}{j} \frac{(-1)^{j+1}}{\langle \xi \rangle^{2j-\beta}} + \sum_{j=M}^{\infty} \binom{\beta/2}{j} \frac{(-1)^{j+1}}{\langle \xi \rangle^{2j-\beta}} \\ &=: k_{1,M}(\xi) + k_{2,M}(\xi), \end{aligned} \tag{2.15}$$

where given that $\beta > 0$, the above series converges absolutely. Thus, we define the operators $K_{j,M}$, $j = 1, 2$ by

$$K_{j,M}f(x) = \int k_{j,M}(\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \quad j = 1, 2.$$

In virtue of (2.15), we write

$$\theta_2 J^\beta f = \theta_2 D^\beta f + \theta_2 K_{1,M} f + \theta_2 K_{2,M} f.$$

Consequently, to deduce (I), we are reduced to prove $\theta_2 K_{j,M} f \in L^2(\mathbb{R}^d)$, $j = 1, 2$. To deal with the estimate concerning the first operator $K_{1,M}$, we perform the following decomposition

$$\theta_2 K_{1,M} f = \theta_2 K_{1,M}(\theta_1 f) + \theta_2 K_{1,M}((1-\theta_1)f).$$

Since $0 \leq \beta < 2$, we have that $k_{1,M} \in \mathcal{S}^0$, i.e., $K_{1,M}$ determines a pseudo-differential operator of order zero. Thus, Theorem 1.5.3 yields

$$\|\theta_2 K_{1,M}(\theta_1 f)\|_{L^2} \lesssim \|\theta_2\|_{L^\infty}\|\theta_1 f\|_{L^2}. \tag{2.16}$$

Next, denoting by $\widetilde{k}_{1,M} \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ the kernel associated to $K_{1,M}$ determined by Proposition 1.5.4, by hypothesis (2.14) and integrating by parts we find

$$\begin{aligned} &\theta_2(x)K_{1,M}((1-\theta_1)f)(x) \\ &\quad = \theta_2(x)\int \widetilde{k}_{1,M}(x, x-y)(1-\theta_1(y))J^{2m}J^{-2m}f(y)\, dy \\ &\quad = \sum_{0\leq|\gamma_1|+|\gamma_2|\leq 2m} c_{\gamma_1,\gamma_2}\theta_2(x) \\ &\qquad\quad \times \int_{|x-y|\geq\delta} (\partial^{\gamma_1}\widetilde{k}_{1,M})(x, x-y)\partial^{\gamma_2}(1-\theta_1(y))J^{-2m}f(y)\, dy, \end{aligned} \tag{2.17}$$

for some constant c_{γ_1,γ_2} with $0 \leq |\gamma_1| + |\gamma_2| \leq 2m$, which are not relevant for our considerations. The preceding estimate, Proposition 1.5.4 (ii) for some integer $N > 0$ fixed, and Young's inequality allow us to deduce

$$\begin{aligned} &\|\theta_2 K_{1,M}((1-\theta_1)f)\|_{L^2} \\ &\quad \lesssim \|\theta_2\|_{L^\infty}\Big(\sum_{0\leq|\gamma|\leq 2m}\|\partial^\gamma(1-\theta_1)\|_{L^\infty}\Big)\sum_{l=0}^{2m}\Big\|\frac{\chi_{\{|\cdot|\geq\delta\}}}{|\cdot|^{d+l+N}} * |J^{-2m}f|\Big\|_{L^2} \\ &\quad \lesssim_{\theta_1,\theta_2} \|J^{-m}f\|_{L^2}, \end{aligned} \tag{2.18}$$

where we have also employed $\|J^{-2m}f\|_{L^2} \leq \|J^{-m}f\|_{L^2}$. Collecting (2.16) and (2.18), we conclude

$$\|\theta_2 K_{1,M} f\|_{L^2} \lesssim \|\theta_1 f\|_{L^2} + \|J^{-m}f\|_{L^2}.$$

On the other hand, since $K_{2,M}$ does not determine a pseudo-differential operator, we must employ a different reasoning to bound this operator. Instead, we write

$$K_{2,M}f = \sum_{j=M}^{\infty}\binom{\beta/2}{j}(-1)^{j+1}G_{2j-\beta} * f,$$

where G_δ, $\delta > 0$, denotes the Bessel kernel (see [9, 97]) defined by

$$G_\delta(x) = c_\delta \int_0^\infty e^{-\pi|x|^2/w} e^{-w/4\pi} w^{(-d+\delta)/2} \frac{dw}{w},$$

for some constant $c_\delta > 0$. Additionally, we recall the estimate

$$|\partial^\gamma G_2(x)| \leq c_2\Big(G_2(x) + G_1(x)\Big), \tag{2.19}$$

which holds for all multi-index γ of order $|\gamma| = 1$. Now, writing $f = J^{-2m}J^{2m}f$, by properties between convolution and derivatives, it is not difficult to deduce

$$K_{2,M}f = \sum_{j=M}^{\infty} \binom{\beta/2}{j} (-1)^{j+1} \sum_{0\leq|\gamma|\leq 2m} c_\gamma\Big((\partial^\gamma G_{2j-\beta}) * J^{-2m}f\Big). \tag{2.20}$$

To estimate the above equality, we decompose each multi-index γ with $0 \leq |\gamma| \leq 2m$ as a sum of $2m$ multi-indexes of order less or equal than 1, that is, $\gamma = \sum_{l=1}^{2m} \gamma_l$, where $0 \leq |\gamma_l| \leq 1$. From this, we have

$$\partial^\gamma G_{2j-\beta} = G_{2j-4m-\beta} * \underbrace{\partial^{\gamma_1} G_2 * \cdots * \partial^{\gamma_{2m}} G_2}_{2m-\text{times}},$$

for each $j \geq M > 2m - \beta/2$, and $0 \leq |\gamma| \leq 2m$. Then, for these set of indexes, (2.19), and the fact that $\|G_\delta\|_{L^1} = 1$, for all $\delta > 0$, imply

$$\|\partial^\gamma G_{2j-\beta} * J^{-2m}f\|_{L^2} \leq (2c_2)^{2m} \|J^{-2m}f\|_{L^2}.$$

Plugging the previous estimate in (2.20) reveals

$$\|K_{2,M}f\|_{L^2} \lesssim \sum_{j=0}^{\infty} \left|\binom{\beta/2}{j}\right| \|J^{-2m}f\|_{L^2} \lesssim \|J^{-m}f\|_{L^2}. \tag{2.21}$$

In particular, this shows $\theta_2 K_{2,M}f \in L^2(\mathbb{R}^d)$, and in consequence the proof of (I) is complete.

Next, we deduce (II). Writing $f = J^{-\beta}(J^\beta f)$, we have

$$\theta_2 f = \theta_2 J^{-\beta}(J^\beta f) = \theta_2 J^{-\beta}(\theta_1 J^\beta f) + \theta_2 J^{-\beta}((1-\theta_1)J^\beta f). \tag{2.22}$$

The first term of the above equality satisfies

$$\|\theta_2 J^{-\beta}(\theta_1 J^\beta f)\|_{L^2} \lesssim \|\theta_2\|_{L^\infty} \|\theta_1 J^\beta f\|_{L^2}. \tag{2.23}$$

Now, the remaining estimate for the r.h.s of (2.22) is obtained by arguing exactly as in (2.17). Indeed, letting $\widetilde{m}$ be an integer such that $2\widetilde{m} \geq m+\beta$, and $q_\beta \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ be the kernel associated to $J^{-\beta}$, by (ii) in Proposition 1.5.4 for some integer $N > 0$, by using (2.14) and integrating by parts, we find

$$\begin{aligned}
&\|\theta_2 J^{-\beta}((1-\theta_1)J^\beta f)\|_{L^2}\\
&\quad\lesssim \sum_{0\leq|\gamma_1|+|\gamma_2|\leq 2\widetilde{m}} \|\theta_2(x)\Big(\partial^{\gamma_1} q_\beta(x,\cdot) * (\partial^{\gamma_2}(1-\theta_1)J^{-2\widetilde{m}+\beta}f)\Big)(x)\|_{L^2}\\
&\quad\lesssim \|\theta_2\|_{L^\infty}\Big(\sum_{0\leq|\gamma|\leq 2m} \|\partial^\gamma(1-\theta_1)\|_{L^\infty}\Big) \sum_{l=0}^{2m} \Big\| \frac{\chi_{\{|\cdot|\geq\delta\}}}{|\cdot|^{d+l+N}} * |J^{-2\widetilde{m}+\beta}f| \Big\|_{L^2}\\
&\quad\lesssim_{\theta_1,\theta_2} \|J^{-m}f\|_{L^2}.
\end{aligned} \tag{2.24}$$

Gathering (2.22)-(2.24), we conclude that $\theta_2 f \in L^2(\mathbb{R}^d)$.

On the other hand, following the same arguments in the proof of (I), we have

$$\theta_2 D^\beta f = \theta_2 J^\beta f - \theta_2 K_{1,M} f - \theta_2 K_{2,M} f, \tag{2.25}$$

where $M > 1$ is a fixed integer number such that $M > 2m + \beta/2$, and the operators $K_{j,M}$ are defined as above according to $k_{j,m}$ given by (2.15) for all $j = 1, 2$. Notice that (2.21) establishes the desired estimate for $\theta_2 K_{2,M} f$.

To control $\theta_2 K_{1,M} f$, once again we set $J^{-\beta} J^\beta f$, then denoting by $\widetilde{K}_{1,M}$ the pseudo-differential operator given by the composition $K_{1,M} J^{-\beta}$ (see Proposition 1.5.5), it is seen that

$$\theta_2 K_{1,M} f = \theta_2 K_{1,M} J^{-\beta}(J^\beta f) = \theta_2 \widetilde{K}_{1,M}(\theta_1 J^\beta f) + \theta_2 \widetilde{K}_{1,M}((1-\theta_1) J^\beta f).$$

Consequently, the previous equality is bounded by the same estimates concerning the r.h.s of (2.22). To avoid repetitions, we omit the details. From this, it follows $\theta_2 K_{1,M} f \in L^2(\mathbb{R}^2)$, and so, by equation (2.25), $\theta_2 D^\beta f \in L^2(\mathbb{R}^2)$ which establishes (II).

To deduce (III), we decompose

$$\theta_2 J^r f = \theta_2 J^{-(s-r)} J^s f = \theta_2 J^{-(s-r)}(\theta_1 J^s f) + \theta_2 J^{-(s-r)}((1-\theta_1) J^s f).$$

The above identity and similar considerations as in (2.22) yield the deduction of (III).

Finally, we deal with (IV). Recalling that $f \in H^{-m}(\mathbb{R}^d)$, we consider an integer $m_1 > \max\{s+m, d\}$ such that by Proposition 2.2.1 it is seen that

$$J^s(f\theta_2) = \sum_{1 \leq |\gamma| \leq m_1} c_\gamma \partial_x^\gamma \theta_2 \, \Psi^\gamma(J^{s-|\gamma|} f) + \theta_2 J^s f + K_{s-m_1} f, \tag{2.26}$$

where Ψ^γ is a given pseudo-differential operator of order zero for each $1 \leq |\gamma| \leq m_1$ and K_{s-m_1} is of order $s - m_1$. Clearly, $\|\theta_2 J^s f\|_{L^2} \lesssim \|\theta_1 J^s f\|_{L^2}$, thus we focus on the remaining parts in (2.26). We first estimate

$$\|K_{s-m_1} f\|_{L^2} = \|K_{s-m_1} J^m J^{-m} f\|_{L^2} \lesssim \|J^{-m} f\|_{L^2}.$$

Now, for each multi-index $1 \leq |\gamma| \leq m_1$, we write

$$\partial_x^\gamma \theta_2 \, \Psi^\gamma(J^{s-|\gamma|} f) = \partial_x^\gamma \theta_2 \left(\Psi^\gamma J^{-|\gamma|}\right)(\theta_1 J^s f) + \partial_x^\gamma \theta_2 \left(\Psi^\gamma J^{-|\gamma|}\right)((1-\theta_1) J^s f).$$

By recurrent arguments using that $\Psi^\gamma J^{-|\gamma|}$ is a pseudo-differential operator of order zero and the assumption on the supports, on one hand we have

$$\|\partial_x^\gamma \theta_2 \left(\Psi^\gamma J^{-|\gamma|}\right)((1-\theta_1) J^s f)\|_{L^2} \lesssim \|J^{-m} f\|_{L^2},$$

while on the other it is seen that

$$\|\partial_x^\gamma \theta_2 \left(\Psi^\gamma J^{-|\gamma|}\right)(\theta_1 J^s f)\|_{L^2} \lesssim \|\theta_1 J^s f\|_{L^2}.$$

Gathering the previous results we complete the deduction of (IV). □

2.2.2 Weighted functions

In this part, we introduce the cutoff functions to be employed in our arguments. This class of functions was first used in [39, 45]. For the sake of brevity, we will only present those properties required for our considerations. For a more detailed discussion, see Isaza, Linares and Ponce [39].

Given $\epsilon > 0$ and $b \geq 5\epsilon$, we define the family of functions

$$\chi_{\epsilon,b},\ \phi_{\epsilon,b},\ \tilde{\phi}_{\epsilon,b},\ \psi_{\epsilon},\ \eta_{\epsilon,b}\ \in\ C^{\infty}(\mathbb{R}),$$

satisfying the following properties:

(i) $\chi'_{\epsilon,b} \geq 0,$

(ii) $\chi_{\epsilon,b}(x) = \begin{cases} 0, & x \leq \epsilon \\ 1, & x \geq b, \end{cases},$

(iii) $\chi'_{\epsilon,b}(x) \geq \frac{1}{10(b-\epsilon)} \mathbb{1}_{[2\epsilon, b-2\epsilon]}(x),$

(iv) $\chi_{\epsilon,b}(x) \geq \frac{1}{2}\frac{\epsilon}{b-3\epsilon}, \quad$ whenever $x \in (3\epsilon, \infty),$

(v) $\operatorname{supp}(\chi'_{\epsilon,b}) \subset [\epsilon, b],$

(vi) $\operatorname{supp}(\phi_{\epsilon,b}), \operatorname{supp}(\widetilde{\phi}_{\epsilon,b}) \subset [\epsilon/4, b],$

(vii) $\phi_{\epsilon}(x) = \widetilde{\phi}_{\epsilon,b}(x) = 1, \quad x \in [\epsilon/2, \epsilon],$

(viii) $\operatorname{supp}(\psi_{\epsilon}) \subset (-\infty, \epsilon/2].$

(ix) Given $x \in \mathbb{R}$, we have the following partitions of unity

$$\chi_{\epsilon,b}(x) + \phi_{\epsilon,b}(x) + \psi_{\epsilon}(x) = 1 \tag{2.27}$$

and

$$\chi^2_{\epsilon,b}(x) + \widetilde{\phi_{\epsilon,b}}^{2}(x) + \psi_{\epsilon}(x) = 1. \tag{2.28}$$

By a slight abuse of notation, when it is required, we shall assume that the above functions act in two variables as follows $\chi_{\epsilon,b}(x,y) := \chi_{\epsilon,b}(x)$, similarly for the other weighted functions introduced above.

2.3 Kato's smoothing effect

We are in the condition to establish the following Kato's smoothing effect for solutions of (gBOZK).

Proposition 2.3.1. *Let $0 \leq \alpha < 1$. Consider $s > s_\alpha = (17-2\alpha)/12$, and $u_0 \in H^s(\mathbb{R}^2)$. Then the corresponding solution $u \in C([0,T]; H^s(\mathbb{R}^2))$ of the IVP* (gBOZK) *with initial data u_0 determined by Theorem 2.1.1 satisfies for any $R > 0$, $T > 0$ and $0 \leq r \leq s$ that*

$$D_x^{\frac{\alpha+1}{2}} A^r u,\ \mathcal{H}_x D_x^{\frac{\alpha+1}{2}} A^r u,\ \partial_y A^r u \in L^2((-R,R)_x \times \mathbb{R}_y \times (0,T)), \tag{2.29}$$

where A^r is any among the operators $J^r, J_x^r, J_y^r, D^r, D_x^r$ and D_y^r.

Proof. We first consider the case $A^r = J^r$ for fixed $0 \leq r \leq s$. The following computations can be justified approximating with smooth solutions of (gBOZK) and taking the limit in our estimates. Thus, we will perform our considerations assuming the required regularity on the solution. We let $\psi \in C^\infty(\mathbb{R})$ with $\psi' \geq 0$, and ψ' compact supported. Applying the operator J^r to (gBOZK), then multiplying the resulting expression by $J^r u\psi$ and integrating in space, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |J^r u|^2 \psi(x)\, dxdy - \int (\partial_x D_x^{\alpha+1} J^r u) J^r u \psi(x)\, dxdy \\ + \frac{1}{2}\int |\partial_y J^r u|^2 \psi'(x)\, dxdy + \int J^r(uu_x) J^r u\psi(x)\, dxdy = 0.\end{aligned} \tag{2.30}$$

To deal with the second term on the left-hand side of (2.30), by writing $\partial_x D_x^{\alpha+1} = -\mathcal{H}_x D_x^{\alpha+2}$, we apply the expansion (1.21) with $a = \alpha+2$, $b = 0$ and $n = 0$ to deduce

$$\begin{aligned}&-\int (\partial_x D_x^{\alpha+1} J^r u) J^r u\psi\, dxdy \\ &= \frac{1}{2}\int J^r u[-\mathcal{H}_x D_x^{\alpha+2}, \psi] J^r u\, dxdy \\ &= \frac{(\alpha+2)}{4}\int |D_x^{\frac{\alpha+1}{2}} J^r u|^2 \psi'\, dxdy + \frac{(\alpha+2)}{4}\int |\mathcal{H}_x D_x^{\frac{\alpha+1}{2}} J^r u|^2 \psi'\, dxdy \\ &\quad + \frac{1}{2}\int J^r u R_0(\alpha+2) J^r u\, dxdy,\end{aligned} \tag{2.31}$$

where $\|R_0(\alpha+2)\|_{L^2_{xy} \to L^2_{xy}} \leq c = c(\psi')$.

On the other hand, to bound the fourth term on the left-hand side of (2.30), we integrate by parts to obtain

$$\begin{aligned}&\int J^r(uu_x) J^r u\psi\, dxdy \\ &= \int [J^r, u]u_x J^r u\psi\, dxdy + \frac{1}{2}\int u\partial_x |J^r u|^2 \psi\, dxdy \\ &= \int [J^r, u]u_x J^r u\psi\, dxdy - \frac{1}{2}\int u_x |J^r u|^2 \psi\, dxdy - \frac{1}{2}\int u|J^r u|^2 \psi'\, dxdy.\end{aligned}$$

Consequently, the preceding equality, the fact that $r \leq s$ and the Kato-Ponce inequality (1.7) allow us to deduce

$$\begin{aligned}&\left|\int J^r(uu_x) J^r u\psi\, dxdy\right| \\ &\lesssim \|[J^r, u]u_x\|_{L^2_{xy}} \|J^r u\|_{L^2_{xy}} + (\|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}}) \|J^r u\|^2_{L^2_{xy}} \\ &\lesssim (\|u\|_{L^\infty_{xy}} + \|\nabla u\|_{L^\infty_{xy}}) \|J^s u\|^2_{L^2_{xy}}.\end{aligned} \tag{2.32}$$

We remark that the implicit constant above depends on $\|\psi\|_{L^\infty}$ and $\|\psi'\|_{L^\infty}$. Thus, gathering the above estimates yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |J^r u|^2\psi\, dxdy &+ \frac{(\alpha+2)}{4}\int |D_x^{\frac{\alpha+1}{2}} J^r u|^2\psi'\, dxdy\\ &+ \frac{(\alpha+2)}{4}\int |\mathcal{H}_x D_x^{\frac{\alpha+1}{2}} J^r u|^2\psi'\, dxdy\\ &+ \frac{1}{2}\int |\partial_y J^r u|^2\psi'\, dxdy\\ &\le (1+\|u\|_{L^\infty_{xy}} + \|\nabla u\|_{L^\infty_{xy}})\,\|J^s u\|^2_{L^2_{xy}}\,.\end{aligned} \tag{2.33}$$

Noticing that Theorem 2.1.1 establishes that $u \in L^1([0,T]; W^{1,\infty}(\mathbb{R}^2))$, we can apply Gronwall's lemma in (2.33), obtaining the desired conclusion for the case $A^r = J^r$.

The estimates for the remaining cases A^r follow by similar reasoning as above, the only modification required concerns the estimate for the nonlinear term, i.e., (2.33). By implementing (1.7) on each variable, we still control the cases $A^r = J^r_x, J^r_y$ (the former cases holds by the assumption $\partial_y u \in L^1([0,T]; L^\infty(\mathbb{R}^2))$). Whereas, Lemma 1.4.2 allows us to deal with $A^r = D^r, D^r_x, D^r_y$. The proof is complete. □

2.4 Proof of Theorem 2.1.2

Our analysis follows the technique introduced in [39] (see also [45, 74, 75, 76]), so that our starting point will be basically to obtain weighted energy estimate by localizing the regions in $\mathbb{R}^2$ where the information concerning the regularity is available.

By translation, we may set $x_0 = 0$. Additionally, we shall assume that the solution u of the IVP (gBOZK) has the required regularity to justify our estimates. At the end, the desired conclusion follows by a limit process employing smooth solutions and our estimates. Therefore, by applying directly the operator $J_x^{\bar{s}}$ to the equation in (gBOZK), followed by a multiplication by $J_x^{\bar{s}}u(x,y)\chi^2_{\epsilon,b}(x+vt)$, that combined with integration by parts allow us to deduce the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int (J_x^{\bar{s}}u)^2\chi^2_{\epsilon,b}\, dxdy \underbrace{-\frac{v}{2}\int (J_x^{\bar{s}}u)^2(\chi^2_{\epsilon,b})'dxdy}_{A_1(t)}\\ &\underbrace{-\int (\partial_x D_x^{1+\alpha} J_x^{\bar{s}}u) J_x^{\bar{s}}u\chi^2_{\epsilon,b}\, dxdy}_{A_2(t)} + \int \partial_x\partial_y^2 J_x^{\bar{s}}u J_x^{\bar{s}}u\chi^2_{\epsilon,b}\, dxdy\\ &\underbrace{+\int J_x^{\bar{s}}(u\partial_x u) J_x^{\bar{s}}u\chi^2_{\epsilon,b}\, dxdy}_{A_3(t)} = 0.\end{aligned} \tag{2.34}$$

We notice that integrating by part yields

$$\int \partial_x\partial_y^2 J_x^{\bar{s}}u J_x^{\bar{s}}u\chi^2_{\epsilon,b}\, dxdy = \int (\partial_y J_x^{\bar{s}}u)^2\chi_{\epsilon,b}\chi'_{\epsilon,b}\, dxdy \ge 0.$$

Consequently, we only need to estimate $A_1(t)$, $A_2(t)$ and $A_3(t)$ determined by (2.34). To simplify the exposition, the preceding differential inequality and the corresponding terms $A_j(t)$, $j=1,2,3$ will be employed for different values $\bar{s}$ previously fixed. We notice that our objective is bounding equation (2.34) corresponding to the case $\bar{s}=s$, whenever $s>s_\alpha$.

Since the estimate for $A_2(t)$ follows by rather general arguments independent of $\bar{s}$, for the sake of brevity, we develop this estimate in the following lemma.

Lemma 2.4.1. *Let $\chi_{\epsilon,b}, \phi_{\epsilon,b}$ and ψ_ϵ defined as in Subsection 2.2.2, satisfying* (2.27). *Then there exist some positive constants c_0, c_1 such that*

$$A_2(t) = Sm_1(t) + Sm_2(t) + R_{A_2}(t),$$

where

$$\begin{aligned} Sm_1(t) = \frac{\alpha+2}{2}\Big(&\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u(t)\|_{L^2_{xy}}^2 \\ &+\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\mathcal{H}_x D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u(t)\|_{L^2_{xy}}^2\Big), \\ Sm_2(t) := \frac{\alpha+2}{2}\Big(&\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)(t)\|_{L^2_{xy}}^2 \\ &+\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\mathcal{H}_x D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)(t)\|_{L^2_{xy}}^2\Big). \end{aligned}$$

and

$$\begin{aligned} |R_{A_2}(t)| \leq &\frac{1}{4}Sm_1(t) + c_0\|u\|_{H^{s_\alpha^+}}^2 \\ &+ c_1\sum_{0\leq j\leq \max\{\bar{s}-s_\alpha,0\}} \|\chi_{\epsilon,b}J_x^{\bar{s}-j}u\|_{L^2_{xy}}^2 + \|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}^2. \end{aligned} \tag{2.35}$$

Proof. By writing $\partial_x = -\mathcal{H}_x D_x$ and using (2.27), we decompose the estimate for A_2 as follows

$$\begin{aligned} A_2(t) =&\frac{1}{2}\int_{\mathbb{R}^2} J_x^{\bar{s}}u\left[D_x^{\alpha+1}\partial_x, \chi_{\epsilon,b}^2\right]J_x^{\bar{s}}u\,dxdy \\ =&\frac{1}{2}\int_{\mathbb{R}^2} J_x^{\bar{s}}\left((u\chi_{\epsilon,b})+(u\phi_{\epsilon,b})+(u\psi_\epsilon)\right) \\ &\qquad\times\left[-\mathcal{H}_x D_x^{\alpha+2}, \chi_{\epsilon,b}^2\right]J_x^{\bar{s}}\left((u\chi_{\epsilon,b})+(u\phi_{\epsilon,b})+(u\psi_\epsilon)\right)\,dx\,dy \\ =&\sigma(J_x^{\bar{s}}(u\chi_{\epsilon,b}), J_x^{\bar{s}}(u\chi_{\epsilon,b})) + 2\sigma(J_x^{\bar{s}}(u\chi_{\epsilon,b}), J_x^{\bar{s}}(u\phi_{\epsilon,b})) \\ &+\sigma(J_x^{\bar{s}}(u\phi_{\epsilon,b}), J_x^{\bar{s}}(u\phi_{\epsilon,b})) + 2\sigma(J_x^{\bar{s}}(u\chi_{\epsilon,b}), J_x^{\bar{s}}(u\psi_\epsilon)) \\ &+2\sigma(J_x^{\bar{s}}(u\phi_{\epsilon,b}), J_x^{\bar{s}}(u\psi_\epsilon)) + \sigma(J_x^{\bar{s}}(u\psi_\epsilon), J_x^{\bar{s}}(u\psi_\epsilon)) \\ =:&A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t)+A_{2,4}(t)+A_{2,5}(t)+A_{2,6}(t), \end{aligned} \tag{2.36}$$

where we have set

$$\sigma(f,g) = \frac{1}{2}\int f\left[-\mathcal{H}_x D_x^{\alpha+2}, \chi_{\epsilon,b}^2\right]g\,dxdy,$$

and we have used

$$\sigma(f,g) = \sigma(g,f).$$

To deduce a convenient factorization for the operator σ, we employ Proposition 1.4.8 with $a=\alpha+2$, $b=0$ and $n=0$ to get

$$
\begin{aligned}
\sigma(f,g)=&\frac{\alpha+2}{2}\int (D_x^{\frac{\alpha+1}{2}}f)(D_x^{\frac{\alpha+1}{2}}g)\chi_{\epsilon,b}\chi_{\epsilon,b}'\,dxdy\\
&+\frac{\alpha+2}{2}\int (\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}f)(\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}g)\chi_{\epsilon,b}\chi_{\epsilon,b}'\,dxdy\\
&+\frac{1}{2}\int fR_0(\alpha+2)g\,dxdy,
\end{aligned}
\tag{2.37}
$$

where the operator $R_0(\alpha+2)$ satisfies

$$
\|R_0(\alpha+2)\|_{L^2_{xy}\to L^2_{xy}}\lesssim \|\widehat{D_x^{\alpha+2}(\chi_{\epsilon,b}^2)}(\xi)\|_{L^1_\xi}. \tag{2.38}
$$

Consequently, we divide the analysis of (2.36) according to those cases where the above decomposition leads bounded expressions, and in those where we can estimate directly the commutator defining $\sigma(\cdot,\cdot)$. Indeed, (2.37), and the fact that $\chi_{\epsilon,b}+\phi_{\epsilon,b}=1-\psi_\epsilon$ allow us to write

$$
\begin{aligned}
&A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t)\\
&=\frac{\alpha+2}{2}\int \Big(\big(D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u\big)^2+\big(\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u\big)^2\Big)\chi_{\epsilon,b}\chi_{\epsilon,b}'\,dxdy\\
&\quad+\frac{\alpha+2}{2}\int \Big(\big(D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)\big)^2+\big(\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)\big)^2\Big)\chi_{\epsilon,b}\chi_{\epsilon,b}'\,dxdy\\
&\quad-(\alpha+2)\int \Big((D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u)(D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon))\\
&\qquad\qquad+(\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}u)(\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon))\Big)\chi_{\epsilon,b}\chi_{\epsilon,b}'\,dxdy\\
&\quad+\frac{1}{2}\int \Big(J_x^{\bar{s}}(u\chi_{\epsilon,b})+J_x^{\bar{s}}(u\phi_{\epsilon,b})\Big)R_0(\alpha+2)\Big(J_x^{\bar{s}}(u\chi_{\epsilon,b})+J_x^{\bar{s}}(u\phi_{\epsilon,b})\Big)\\
&=:Sm_1(t)+Sm_2(t)+\widetilde{A}_{2,1}(t)+\widetilde{A}_{2,2}(t).
\end{aligned}
$$

Integrating between $[0,T]$, we notice that $Sm_1(t)$ corresponds to the required smoothing effect, while $Sm_2(t)$ provides a positive quantity. On the other hand, since $R_0(\alpha+2)$ satisfies (2.38), we have

$$
\begin{aligned}
|\widetilde{A}_{2,2}(t)|&\lesssim \|J^{\bar{s}}(u\chi_{\epsilon,b})\|_{L^2_{xy}}^2+\|J^{\bar{s}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}^2\\
&=:\widetilde{A}_{2,2,1}(t)+\|J^{\bar{s}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}^2.
\end{aligned}
\tag{2.39}
$$

Let $m_1\geq \max\{2,\bar{s}\}$, by employing Proposition 2.2.1 with $\theta(x)=\chi_{\epsilon,b}(x+vt)$, it is seen that

$$
\begin{aligned}
|\widetilde{A}_{2,2,1}(t)|&\lesssim \sum_{j=1}^{m_1}\|\|\chi_{\epsilon,b}^j\Psi^{(j)}(J_x^{\bar{s}-j}u)\|_{L^2_x}\|_{L^2_y}^2+\|\|\chi_{\epsilon,b}J_x^{\bar{s}}u\|_{L^2_x}\|_{L^2_y}^2\\
&\quad+\|\|u\|_{L^2_x}\|_{L^2_y}^2\\
&\lesssim \sum_{j=1}^{m_1}\|\chi_{\epsilon,b}^j\Psi^{(j)}(J_x^{\bar{s}-j}u)\|_{L^2_{xy}}^2+\|\chi_{\epsilon,b}J_x^{\bar{s}}u\|_{L^2_{xy}}^2+\|u\|_{L^2_{xy}}^2.
\end{aligned}
\tag{2.40}
$$

Now, for each $j=1,\dots,m_1$, we employ (2.27) to deduce

$$\begin{aligned}
&\|\chi_{\epsilon,b}^{j}\Psi^{(j)}(J_x^{\bar{s}-j}u)\|_{L^2_{xy}}\\
&\quad\lesssim\|\chi_{\epsilon,b}^{j}\Psi^{(j)}(J_x^{\bar{s}-j}(u\chi_{\epsilon,b}))\|_{L^2_{xy}}+\|\chi_{\epsilon,b}^{j}\Psi^{(j)}(J_x^{\bar{s}-j}(u\phi_{\epsilon,b}))\|_{L^2_{xy}}\\
&\qquad+\|\chi_{\epsilon,b}^{j}\Psi^{(j)}(J_x^{\bar{s}-j}(u\psi_{\epsilon}))\|_{L^2_{xy}}\\
&\quad\lesssim\|J_x^{\bar{s}-j}(u\chi_{\epsilon,b})\|_{L^2_{xy}}+\|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}+\|u\|_{L^2_{xy}},
\end{aligned}\tag{2.41}$$

where we have also applied Lemma 2.2.2 to estimate

$$\|\chi_{\epsilon,b}^{j}\Psi^{(j)}(J_x^{\bar{s}-j}(u\psi_{\epsilon}))\|_{L^2_{xy}}\lesssim\|u\|_{L^2_{xy}}.\tag{2.42}$$

We emphasize that the above considerations yield the factor $\|J_x^{\bar{s}-j}(u\chi_{\epsilon,b})\|_{L^2_{xy}}$. This suggests that we can iterate the arguments in (2.40)-(2.42) decreasing the derivatives considered in each step until we arrive at

$$|\widetilde{A}_{2,2,1}(t)|\lesssim\sum_{j=0}^{m_1}\|\chi_{\epsilon,b}J_x^{\bar{s}-j}u\|_{L^2_{xy}}^2+\|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}^2+\|u\|_{H^{s_\alpha^+}}^2.$$

Since $u\in C([0,T];H^{s_\alpha^+}(\mathbb{R}^2))$, we can modify the constant in the above inequality to reduce the previous sum to integers $0\le j\le\max\{\bar{s}-s_\alpha,0\}$. Thus, we gather these conclusions to deduce

$$|\widetilde{A}_{2,2}(t)|\lesssim\sum_{0\le j\le\max\{\bar{s}-s_\alpha,0\}}\|\chi_{\epsilon,b}J_x^{\bar{s}-j}u\|_{L^2_{xy}}^2+\|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}^2+\|u\|_{H^{s_\alpha^+}}^2.$$

Now, we turn to $\widetilde{A}_{2,1}(t)$. By Young's inequality $a_1a_2\le\frac{a_1^p}{p}+\frac{a_2^{p'}}{p'}$, with $\frac{1}{p}+\frac{1}{p'}=1$, and the fact that $\chi_{\epsilon,b}\chi_{\epsilon,b}'\ge 0$, we have

$$\begin{aligned}
|\widetilde{A}_{2,1}(t)|\le&\frac{1}{4}|Sm_1(t)|+2(\alpha+2)\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\|_{L^2_{xy}}^2\\
&+2(\alpha+2)\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\|_{L^2_{xy}}^2.
\end{aligned}\tag{2.43}$$

Bearing in mind that

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}\chi_{\epsilon,b}'),\operatorname{supp}(\psi_{\epsilon}))\ge\epsilon/2,$$

an application of Lemma 2.2.3 on the x-spatial variable yields

$$\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\|_{L^2_{xy}}\lesssim\|(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\|_{L^\infty}\|u\|_{L^2_{xy}}.\tag{2.44}$$

To estimate the third term on the right-hand side of (2.43), we consider a function $\vartheta_{\epsilon,b}=\vartheta_{\epsilon,b}(x+vt)\in C_c^\infty(\mathbb{R})$ with $0\le\vartheta_{\epsilon,b}\le 1$, such that $\vartheta_{\epsilon,b}\chi_{\epsilon,b}\chi_{\epsilon,b}'=\chi_{\epsilon,b}\chi_{\epsilon,b}'$ and $\operatorname{dist}(\operatorname{supp}(\vartheta_{\epsilon,b}),\operatorname{supp}(\psi_{\epsilon}))\ge\epsilon/8$, then we write

$$\begin{aligned}
(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})=&(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\vartheta_{\epsilon,b}\mathcal{H}_xD_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\\
=&(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}[\vartheta_{\epsilon,b},\mathcal{H}_x]D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\\
&+(\chi_{\epsilon,b}\chi_{\epsilon,b}')^{1/2}\mathcal{H}_x\Big(\vartheta_{\epsilon,b}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_{\epsilon})\Big).
\end{aligned}$$

Clearly, since $\mathcal{H}_x$ is a bounded operator on $L^2(\mathbb{R}^2)$, the second term on the r.h.s of the above inequality is estimated as in (2.44). Now, let m_2 be an integer fixed such that $2m_2 > \bar{s}+\frac{\alpha+1}{2}$, then by Proposition 1.4.5 we find

$$\begin{aligned}
&\|[\vartheta_{\epsilon,b},\mathcal{H}_x]D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)\|_{L^2}\\
&\quad=\|\|[\vartheta_{\epsilon,b},\mathcal{H}_x]J_x^{2m_2}J_x^{-2m_2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)\|_{L^2_x}\|_{L^2_y}\\
&\quad\lesssim\|J_x^{2m_2}(\vartheta_{\epsilon,b})\|_{L^\infty}\|J_x^{-2m_2}D_x^{\frac{\alpha+1}{2}}J_x^{\bar{s}}(u\psi_\epsilon)\|_{L^2_{xy}}\\
&\quad\lesssim\|u\|_{L^2_{xy}}.
\end{aligned}$$

This completes the estimate for (2.43) and in turn the study of $A_{2,1}(t)+A_{2,2}(t)+A_{2,3}(t)$.

Finally, we deal with $A_{2,4}(t), A_{2,5}(t)$ and $A_{2,6}(t)$ defined in (2.36). By using again that $1=\chi_{\epsilon,b}+\phi_{\epsilon,b}+\psi_\epsilon$, we write

$$\begin{aligned}
A_{2,4}(t)+A_{2,5}(t)+A_{2,6}(t)=&2\sigma(J_x^{\bar{s}}u,J_x^{\bar{s}}(u\psi_\epsilon))\\
&-\sigma(J_x^{\bar{s}}(u\psi_\epsilon),J_x^{\bar{s}}(u\psi_\epsilon)).
\end{aligned}\tag{2.45}$$

We proceed to estimate each factor of the above identity. By opening up the commutator defining σ, using that $J^{\bar{s}}$ is a symmetry operator and that $D_x^{\alpha+1}=\mathcal{H}_xD_x^\alpha\partial_x$, it is deduced that

$$\begin{aligned}
2\sigma(J_x^{\bar{s}}u,J_x^{\bar{s}}(u\psi_\epsilon))=&\int uJ_x^{\bar{s}}\mathcal{H}_xD_x^\alpha\partial_x^2\Big(\chi_{\epsilon,b}^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\,dxdy\\
&-\int uJ_x^{\bar{s}}\Big(\chi_{\epsilon,b}^2\mathcal{H}_xD_x^\alpha\partial_x^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\,dxdy\\
=&\int uJ_x^{\bar{s}}\mathcal{H}_xD_x^\alpha\partial_x^2\Big(\chi_{\epsilon,b}^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\,dxdy\\
&-\int uJ_x^{\bar{s}}\Big([\chi_{\epsilon,b}^2,\mathcal{H}_x]D_x^\alpha\partial_x^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\,dxdy\\
&-\int uJ_x^{\bar{s}}\mathcal{H}_x\Big(\chi_{\epsilon,b}^2D_x^\alpha\partial_x^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\,dxdy\\
=:&\widetilde{A}_{2,3}(t)+\widetilde{A}_{2,4}(t)+\widetilde{A}_{2,5}(t).
\end{aligned}$$

Let m_3 be an even integer sufficiently large such that $m_3>\bar{s}+2+\alpha$, then by the embedding $H^{m_3}(\mathbb{R})\hookrightarrow H^{\bar{s}+2+\alpha}(\mathbb{R})$ on the x-variable, we find

$$\begin{aligned}
&|\widetilde{A}_{2,3}(t)|+|\widetilde{A}_{2,5}(t)|\\
&\quad\lesssim\sum_{j=0}^{m_3}\Big(\|\partial_x^j\Big(\chi_{\epsilon,b}^2J_x^{\bar{s}}(u\psi_\epsilon)\Big)\|_{L^2_{xy}}\\
&\qquad\qquad+\|\partial_x^j(\chi_{\epsilon,b}^2D_x^\alpha\partial_x^2J_x^{\bar{s}}(u\psi_\epsilon))\|_{L^2_{xy}}\Big)\|u\|_{L^2_{xy}}.
\end{aligned}\tag{2.46}$$

Given that $\partial_x^kJ_x^{\bar{s}}$ determines a pseudo-differential operator on the x-variable for all integer $k\geq 0$, and that $\chi_{\epsilon,b}$ and ψ_ϵ have separated support, by applying Corollary 2.2.4 we get

$$\begin{aligned}
&\|\|(\partial_x^k\chi_{\epsilon,b}^2)(\partial_x^{j-k}J_x^{\bar{s}})(u\psi_\epsilon)\|_{L^2_x}\|_{L^2_y}+\|\|(\partial_x^k\chi_{\epsilon,b}^2)D_x^\alpha(\partial_x^{j-k+2}J_x^{\bar{s}})(u\psi_\epsilon)\|_{L^2_x}\|_{L^2_y}\\
&\quad\lesssim\|\|\partial_x^k\chi_{\epsilon,b}^2\|_{L^\infty}\|u\psi_\epsilon\|_{L^2_x}\|_{L^2_y}\\
&\quad\lesssim\|u\|_{L^2_{xy}},
\end{aligned}$$

for each $k = 0, \ldots, j$ and $j = 0, \ldots, m_3$. Summing over these indexes, we control the right-hand side of (2.46).

Now, from our choice of m_3,

$$\begin{aligned}
|\widetilde{A}_{2,4}(t)| &\lesssim \| \| J_x^{m_3} [\mathcal{H}_x, \chi_{\epsilon,b}^2] J_x^{m_3} (J_x^{-m_3} D_x^{\alpha} \partial_x^2 J_x^{\bar{s}}(u\psi_\epsilon)) \|_{L_x^2} \|_{L_y^2} \|u\|_{L_{xy}^2} \\
&\lesssim \| J_x^{2m_3} \chi_{\epsilon,b}^2 \|_{L^\infty} \| \| J^{-m_3} D_x^{\alpha} \partial_x^2 J_x^{\bar{s}}(u\psi_\epsilon) \|_{L_x^2} \|_{L_y^2} \|u\|_{L_{xy}^2} \\
&\lesssim \|u\|_{L_{xy}^2}^2.
\end{aligned}$$

Collecting the previous estimates, we complete the analysis of $\sigma(J_x^{\bar{s}}u, J_x^{\bar{s}}(u\psi_\epsilon))$. In light of the fact that the above argument clearly applies to $\sigma(J_x^{\bar{s}}f, J_x^{\bar{s}}(u\psi_\epsilon))$ as long as $f \in L^2(\mathbb{R}^2)$, the remaining estimate in (2.45) can be controlled in a similar fashion. The proof of the lemma is complete. □

Remark 2.4.2. *It is worth emphasizing that the above considerations dealing with $\widetilde{A}_{2,2}$ in the proof of Lemma 2.4.1 provide the following inequality*

$$\|J^{\bar{s}}(u\chi_{\epsilon,b})\|_{L_{xy}^2} \lesssim \sum_{0 \leq j \leq \bar{s} - s_\alpha} \|\chi_{\epsilon,b} J_x^{\bar{s}-j} u\|_{L_{xy}^2} + \|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L_{xy}^2} + \|u\|_{H^{s_\alpha^+}},$$

whenever $\bar{s} > s_\alpha$ is fixed. This estimate will be convenient to replace the analysis of $J^{\bar{s}}(u\chi_{\epsilon,b})$ by that of $\chi_{\epsilon,b} J^{\bar{s}} u$.

Next, we proceed to deduce Theorem 2.1.2. We divide our attention into several cases determined by the values of $s > s_\alpha = (17 - 2\alpha)/12$.

2.4.1 Case: $s \in (s_\alpha, 2)$

We further divide our consideration into two main steps.

- Step 1. Study the differential equation (2.34) for $\bar{s} = s - \frac{1+\alpha}{2}$.
- Step 2. Study the differential equation (2.34) for $\bar{s} = s$.

It should be noted that Step 1 is to obtain the local smoothing effect corresponding to J_x^s derivatives of u, which is required to deal with $A_1(t)$ in (2.34) for the desired case $\bar{s} = s$. In Step 2, we prove Theorem 2.1.2 for indexes $s \in (s_\alpha, 2)$, using the derivatives and Kato's smoothing effect obtained in Step 1.

2.4.1.1 Step 1.

In this part, we deal with identity (2.34) for $\bar{s} = r + \frac{1-\alpha}{2}$, where $r = s - 1$. We separate our analysis according to the corresponding factors $A_j(t)$, $j = 1, 2, 3$.

Estimate for A_1. Since $r = s - 1 < s_\alpha$, Proposition 2.3.1 implies

$$D_x^{\frac{1+\alpha}{2}} J_x^r u(x,y,t) \in L^2((-R,R)\times\mathbb{R}\times(0,T)),$$

for all $R > 0$. Then, in view of the fact that $J_x^r u \in C([0,T], L^2(\mathbb{R}^2))$, we can apply Lemma 2.2.5 (I) to get

$$J_x^{r+\frac{1+\alpha}{2}} u(x,y,t) \in L^2((-R,R)\times\mathbb{R}\times(0,T)),$$

for all $R > 0$. The previous conclusion and Lemma 2.2.5 (III) reveal

$$J_x^{\widetilde{s}} u(x,y,t) \in L^2((-R,R)\times\mathbb{R}\times(0,T)), \tag{2.47}$$

for all $R > 0$ and all $\widetilde{s} \in [0, r + \frac{1+\alpha}{2}]$. Consequently, we set $R_1 > 0$ such that $\operatorname{supp}(\chi_{\epsilon,b}(x+vt)\chi'_{\epsilon,b}(x+vt)) \subset (-R_1, R_1)$, for all $t \in [0,T]$. Then, by noticing that $r + \frac{1-\alpha}{2} < s - 1 + \frac{1+\alpha}{2}$ with $s \in (s_\alpha, 2)$, and by (2.47), it is seen that

$$\int_0^T |A_1(t)|\, dt \lesssim \int_0^T \int_{\mathbb{R}} \int_{-R_1}^{R_1} (J_x^{r+\frac{1-\alpha}{2}} u)^2(x,y,t)\, dxdydt < \infty. \tag{2.48}$$

The analysis of $A_1(t)$ is complete.

Estimate for A_2. In virtue of Lemma 2.4.1, we just need to justify the validity of the r.h.s of (2.35) under the current restrictions. Indeed, since $r + \frac{1-\alpha}{2} - s_\alpha < 1$, we are reduced to control

$$\|\chi_{\epsilon,b} J_x^{r+\frac{1-\alpha}{2}} u\|_{L^2_{xy}} + \|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}. \tag{2.49}$$

The first term on the right-hand side of the above expression is the quantity to be estimate through (2.34) and Gronwall's lemma, while the second one is bounded by Lemma 2.2.5 (IV) and (2.47) as follows

$$\begin{aligned} \|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}} &\lesssim \|\|\phi_{R_1,x} J_x^{r+\frac{1-\alpha}{2}} u\|_{L^2_x}\|_{L^2_y} + \|\|u\|_{L^2_x}\|_{L^2_y} \\ &= \|\phi_{R_1,x} J_x^{r+\frac{1-\alpha}{2}} u\|_{L^2_{xy}} + \|u\|_{L^2_{xy}}, \end{aligned} \tag{2.50}$$

where $R_1 > 0$, is given as in (2.48) and we have set $\phi_{R_1,x} = \phi_{R_1,x}(x) \in C_c^\infty(\mathbb{R})$ such that $\phi_{R_1,x}(x) = 1$ on $[-R_1, R_1]$, and $\operatorname{dist}\Big((1-\phi_{R_1,x})(x), \phi_{\epsilon,b}(x+vt)\Big) \geq \epsilon/4$, for all $t \in [0,T]$. This completes the considerations for A_2.

Estimate for A_3. Recalling (2.27) and (2.28), we begin by writing

$$\begin{aligned} &\chi_{\epsilon,b} J_x^{r+\frac{1-\alpha}{2}}(u\partial_x u) \\ &\quad = -\frac{1}{2}[J_x^{r+\frac{1-\alpha}{2}}, \chi_{\epsilon,b}]\partial_x\Big((u\chi_{\epsilon,b})^2 + (u\widetilde{\phi}_{\epsilon,b})^2 + u^2\psi_\epsilon\Big) \\ &\qquad + [J_x^{r+\frac{1-\alpha}{2}}, u\chi_{\epsilon,b}]\partial_x\Big(u\chi_{\epsilon,b} + u\phi_{\epsilon,b} + u\psi_\epsilon\Big) + u\chi_{\epsilon,b}\partial_x J_x^{r+\frac{1-\alpha}{2}} u \\ &\quad =: A_{3,1} + A_{3,2} + A_{3,3} + A_{3,4} + A_{3,5} + A_{3,6} + A_{3,7}. \end{aligned}$$

In the first place, we obtain, after applying Lemmas 1.4.3 and 1.4.4,

$$\begin{aligned}
\|A_{3,1}\|_{L^2_{xy}} &= \frac{1}{2}\|\|[J_x^{r+\frac{1-\alpha}{2}}, \chi_{\epsilon,b}]\partial_x(u\chi_{\epsilon,b})^2\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|\|J_x^l\chi'_{\epsilon,b}\|_{L^2}\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})^2\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|\|u\chi_{\epsilon,b}\|_{L^\infty_x}\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|u\|_{L^\infty_{xy}}\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_{xy}}.
\end{aligned}$$

A similar analysis can be applied to provide upper bounds for $A_{3,2}$ as follows

$$\|A_{3,2}\|_{L^2_{xy}} \lesssim \|u\|_{L^\infty_{xy}}\|J_x^{r+\frac{1-\alpha}{2}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}.$$

Instead, for $A_{3,3}$, by opening the commutator involved, we take hand of the relationship of the weighted functions involved. More precisely, since

$$\operatorname{dist}(\operatorname{supp}(\chi_{\epsilon,b}), \operatorname{supp}(\psi_\epsilon)) \geq \frac{\epsilon}{2} > 0,$$

we obtain after applying Lemma 2.2.2 on the x-spatial variable the bound

$$\begin{aligned}
\|A_{3,3}\|_{L^2_{xy}} &= \|\|\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}(u^2\psi_\epsilon)\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|\|\chi_{\epsilon,b}\|_{L^\infty}\|u^2\psi_\epsilon\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|u\|_{L^\infty_{xy}}\|u\|_{L^2_{xy}}.
\end{aligned}$$

A similar analysis applied to $A_{3,6}$ produces

$$\|A_{3,6}\|_{L^2_{xy}} \lesssim \|u\|_{L^\infty_{xy}}\|u\|_{L^2_{xy}}.$$

Instead, the terms $A_{3,4}$ and $A_{3,5}$ require implementing more sophisticated tools. In this sense, Kato-Ponce commutator estimate and Lemma 1.4.1 guarantees that

$$\begin{aligned}
\|A_{3,4}\|_{L^2_{xy}} &= \|\|[J_x^{r+\frac{1-\alpha}{2}}, u\chi_{\epsilon,b}]\partial_x(u\chi_{\epsilon,b})\|_{L^2_x}\|_{L^2_y} \\
&\lesssim \|\|\partial_x(u\chi_{\epsilon,b})\|_{L^\infty_x}\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_x}\|_{L^2_y} \\
&\lesssim (\|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}})\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_{xy}}.
\end{aligned}$$

For $A_{3,5}$, the same combination of techniques as before allow us to obtain

$$\|A_{3,5}\|_{L^2_{xy}} \lesssim (\|u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}})\Big(\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_{xy}} + \|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}\Big).$$

Finally, the term $A_{3,7}$ can be handled by going back at the integral defining A_3 and integrating by parts. More precisely,

$$\begin{aligned}
&\int A_{3,7}\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u\,dx\,dy \\
&\quad= \int u\chi_{\epsilon,b}\partial_x J_x^{r+\frac{1-\alpha}{2}}u(\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u)\,dx\,dy \\
&\quad= -\frac{1}{2}\int \partial_x u\chi_{\epsilon,b}^2\Big(J_x^{r+\frac{1-\alpha}{2}}u\Big)^2 dx\,dy - \int u\chi_{\epsilon,b}\chi'_{\epsilon,b}\Big(J_x^{r+\frac{1-\alpha}{2}}u\Big)^2 dx\,dy \\
&\quad= A_{3,7,1}(t) + A_{3,7,2}(t).
\end{aligned}$$

Notice that the first term on the r.h.s. can be bounded by using the Strichartz estimate provided by the local theory, that is

$$|A_{3,7,1}(t)| \lesssim \|\partial_x u\|_{L^\infty_{xy}} \|\chi_{\epsilon,b} J_x^{r+\frac{1-\alpha}{2}} u\|^2_{L^2_{xy}},$$

being the last expression the one to be estimated after using Gronwall's inequality.

We emphasize that in spite of $A_{3,7,1}$ and $A_{3,7,2}$ share some similarities, the regions involved in each of these estimates are different, and so the way we can provide some upper bounds for their respective expressions. In this sense, we obtain by Sobolev embedding

$$|A_{3,7,2}(t)| \leq \|u\|_{L^\infty_{xy}} \|\chi_{\epsilon,b}\chi'_{\epsilon,b} J_x^{r+\frac{1-\alpha}{2}} u\|^2_{L^2_{xy}} \lesssim \|u\|_{H^{s_\alpha^+}} A_1(t),$$

where the term A_1 was already bounded at the beginning of this section.

In order to fully control the above estimates generated in the study of the nonlinear part, it remains to deal with the terms

$$\|J_x^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_{xy}}, \ \|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}, \ \text{and} \ \|J_x^{r+\frac{1-\alpha}{2}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}. \tag{2.51}$$

Recalling that the $\operatorname{supp}(\phi_{\epsilon,b}), \operatorname{supp}(\widetilde{\phi}_{\epsilon,b}) \subset [\epsilon/4, b]$, the same reasoning around (2.50) allow us to bound $\|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}}$ and $\|J_x^{r+\frac{1-\alpha}{2}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}$.

On the other hand, given that $r+\frac{1-\alpha}{2}-s_\alpha = s-1+\frac{1-\alpha}{2}-s_\alpha < 1$, $s \in (s_\alpha, 2)$, by Remark 2.4.2 it is seen that

$$\|J^{r+\frac{1-\alpha}{2}}(u\chi_{\epsilon,b})\|_{L^2_{xy}} \lesssim \|\chi_{\epsilon,b} J_x^{r+\frac{1-\alpha}{2}} u\|_{L^2_{xy}} + \|J_x^{r+\frac{1-\alpha}{2}}(u\phi_{\epsilon,b})\|_{L^2_{xy}} + \|u\|_{H^{s_\alpha^+}},$$

where the first term on the right-hand side of the above expression is the quantity to be estimated employing Gronwall's lemma, and the second term was considered in previous discussions. Thereby, the estimate for the factor arising from the nonlinear $A_3(t)$ is complete.

Remark 2.4.3. *The arguments in the above estimate are quite general and they can be implemented to provide a bound for the factor $A_3(t)$ in* (2.34) *for any regularity $\bar{s} > s_\alpha$. Indeed, we just need to justify the terms in* (2.51) *obtained after replacing the regularity $r+\frac{1-\alpha}{2}$ by $\bar{s}$, that is to say,*

$$\begin{aligned} \sum_{0\leq j\leq \bar{s}-s_\alpha} \|\chi_{\epsilon,b} J_x^{\bar{s}-j} u\|_{L^2_{xy}} + \|J_x^{\bar{s}-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}} & \\ + \|J_x^{\bar{s}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}} + \|u\|_{H^{s_\alpha^+}}, & \end{aligned} \tag{2.52}$$

where we have applied Remark 2.4.2 to bound $\|J^{\bar{s}}(u\chi_{\epsilon,b})\|_{L^2}$.

Finally, recalling $Sm_1(t)$ and $Sm_2(t)$ defined in Lemma 2.4.1, we gather the estimates above to conclude that there exist some constants c_0 and c_1 such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u\|_{L^2_{xy}}^2 + Sm_1(t) + Sm_2(t) + \|(\chi\chi_{\epsilon,b}')^{1/2}\partial_y J_x^{r+\frac{1-\alpha}{2}}u\|_{L^2_{xy}}^2 \\ \leq \frac{1}{4}Sm_1(t) + c_0\|u\|^2_{L^\infty_T H^{s_\alpha^+}} + (1+\|u\|_{L^\infty_T H^{s_\alpha^+}})\mathfrak{g}(t) \\ + c_1\Big(1+\|u\|_{L^\infty}+\|\partial_x u\|_{L^\infty}\Big)\|\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u\|^2_{L^2_{xy}},\end{aligned}$$

where the function $\mathfrak{g}(t)$ satisfies

$$\int_0^T \mathfrak{g}(t)dt \lesssim \int_0^T\int_{\mathbb{R}}\int_{-R_1}^{R_1}(J_x^{r+\frac{1-\alpha}{2}}u)^2(x,y,t)\,dxdydt < \infty,$$

being $R_1 > 0$ provided by (2.48). Thus, Gronwall's inequality and integration in time yield

$$\begin{aligned}\sup_{t\in[0,T]}\|\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u(t)\|^2_{L^2_{xy}} + \int_0^T Sm_1(t)\,dt \\ + \int_0^T\int_{\mathbb{R}^2}\chi_{\epsilon,b}\chi_{\epsilon,b}'(\partial_y J_x^{r+\frac{1-\alpha}{2}}u)^2(x,y,t)\,dxdydt \lesssim \widetilde{c},\end{aligned} \tag{2.53}$$

where $\widetilde{c} = \widetilde{c}(\epsilon, v, T, \|\chi_{\epsilon,b}J_x^{r+\frac{1-\alpha}{2}}u_0\|_{L^2_{xy}}, \|u\|_{L^\infty_T H^{s_\alpha^+}}, \|\partial_x u\|_{L^1_T L^\infty_{xy}})$ and we have used that

$$\int_0^T Sm_s(t)\,dt \geq 0.$$

This completes Step 1.

2.4.1.2 Step 2.

Fixing ϵ and $b \geq 5\epsilon$, this part concerns the analysis of (2.34) for $\widetilde{s} = s$, $s \in (s_\alpha, 2)$, i.e., we establish the proof of Theorem 2.1.2 for such indexes s. We shall only estimate the terms $A_j(t)$, $j = 1,2,3$. Once this has been done, following similar considerations leading to (2.53), we obtain the desired conclusion. To avoid repetition, we omit these details.

Estimate for A_1. By applying the arguments in Step 1 above with the function $\chi^2_{\epsilon/24,b+13\epsilon/12}$, and by properties (iii) and (iv) in Subsection 2.2.2, the local smoothing effect obtained in Step 1 shows

$$\int_0^T\int_{\mathbb{R}^2}\mathbb{1}_{[\epsilon/8,b+\epsilon]}(x+vt)(D_x^{\frac{1+\alpha}{2}}J_x^{s-1+\frac{1-\alpha}{2}}u)^2(x,y,t)\,dxdydt \leq c. \tag{2.54}$$

Additionally, (2.47) determines

$$\int_0^T\int_{\mathbb{R}^2}\mathbb{1}_{[\epsilon/8,b+\epsilon]}(x+vt)(J_x^{s-1+\frac{1-\alpha}{2}}u)^2(x,y,t)\,dxdydt \leq c. \tag{2.55}$$

Then, we consider functions $\theta_{1,\epsilon,b},\theta_{2,\epsilon,b},\theta_{3,\epsilon,b}\in C_c^{\infty}(\mathbb{R})$ with $0\leq \theta_{j,\epsilon,b}\leq 1$, $j=1,2,3$ such that

$$\begin{aligned}
&\theta_{1,\epsilon,b}\equiv 1 \text{ on } [5\epsilon/32, b+3\epsilon/4], && \text{and } \operatorname{supp}(\theta_{1,\epsilon,b})\subset [\epsilon/8, b+\epsilon],\\
&\theta_{2,\epsilon,b}\equiv 1 \text{ on } [7\epsilon/32, b+\epsilon/2], && \text{and } \operatorname{supp}(\theta_{2,\epsilon,b})\subset [3\epsilon/16, b+5\epsilon/8],\\
&\theta_{3,\epsilon,b}\equiv 1 \text{ on } [\epsilon, b], && \text{and } \operatorname{supp}(\theta_{3,\epsilon,b})\subset [\epsilon/4, b+\epsilon/4].
\end{aligned} \tag{2.56}$$

We shall assume that the above functions act only on the x-variable in the following manner, $\theta_{j,\epsilon,b}=\theta_{j,\epsilon,b}(x+vt)$, $j=1,2,3$. Then, from (2.54) and (2.55), we infer that $\theta_{1,\epsilon,b}J_x^{s-1+\frac{1-\alpha}{2}}u$ and $\theta_{1,\epsilon,b}D_x^{\frac{1+\alpha}{2}}J_x^{s-1+\frac{1-\alpha}{2}}u\in L^2(\mathbb{R}^2\times(0,T))$. Additionally, we have $J_x^{s-1+\frac{1-\alpha}{2}}u(\cdot,y,t)\in H^{-m}(\mathbb{R})$ for $m>s-1+\frac{1-\alpha}{2}$ for almost every y and t. Therefore, we are in condition to apply Lemma 2.2.5 (I) to obtain

$$\begin{aligned}
&\|\theta_{2,\epsilon,b}J_x^s u\|_{L^2(\mathbb{R}^2\times(0,T))}\\
&\quad=\|\|\theta_{2,\epsilon,b}J_x^s u\|_{L^2_x}\|_{L^2_{yt}(\mathbb{R}\times(0,T))}\\
&\quad\lesssim\|\|\theta_{1,\epsilon,b}J_x^{s-1+\frac{1-\alpha}{2}}u\|_{L^2_x}\|_{L^2_{yt}(\mathbb{R}\times(0,T))}\\
&\qquad+\|\|\theta_{1,\epsilon,b}D^{\frac{\alpha+1}{2}}J_x^{s-1+\frac{1-\alpha}{2}}u\|_{L^2_x}\|_{L^2_{yt}(\mathbb{R}\times(0,T))}\\
&\qquad+\|\|J_x^{-m}J_x^{s-1+\frac{1-\alpha}{2}}u\|_{L^2_x}\|_{L^2_{yt}(\mathbb{R}\times(0,T))}\\
&\quad\lesssim\|\theta_{1,\epsilon,b}J_x^{s-1+\frac{1-\alpha}{2}}u\|_{L^2_{xyt}(\mathbb{R}^2\times(0,T))}\\
&\qquad+\|\theta_{1,\epsilon,b}D^{\frac{\alpha+1}{2}}J_x^{s-1+\frac{1-\alpha}{2}}u\|_{L^2_{xyt}(\mathbb{R}^2\times(0,T))}+\|u\|_{L^2_{xyt}(\mathbb{R}\times(0,T))}.
\end{aligned} \tag{2.57}$$

We conclude that $\theta_{2,\epsilon,b}J_x^s u\in L^2(\mathbb{R}^2\times(0,T))$. From this fact, and by similar reasoning to (2.57), employing Lemma 2.2.5 (III) instead, we deduce

$$\theta_{3,\epsilon,b}J^{\widetilde{s}}u\in L^2(\mathbb{R}^2\times(0,T)),\ \text{ for any } \widetilde{s}\in(0,s]. \tag{2.58}$$

Finally, by (2.58) and support considerations, we arrive at

$$\begin{aligned}
\Big|\int_0^T A_1(t)\,dt\Big|&=|v|\int_0^T\int \chi_{\epsilon,b}\chi'_{\epsilon,b}(J^s u)^2\,dxdydt\\
&\lesssim\int_0^T\int(\theta_{3,\epsilon,b}J^s u)^2\,dxdydt<\infty.
\end{aligned}$$

The estimate for A_1 is complete.

Estimate for A_2 and A_3. To control A_2, given that $0<s-s_\alpha<1$, Lemma 2.4.1 and (2.35) reduce our efforts to estimate $\|J_x^s(u\phi_{\epsilon,b})\|_{L^2_{xy}}$. However, this estimate can be obtained as a consequence of the support of the function $\theta_{2,\epsilon,b}$ defined above and Lemma 2.2.5 (IV) as follows

$$\begin{aligned}
\|J_x^s(u\phi_{\epsilon,b})\|_{L^2_{xy}}&\lesssim\|\|\theta_{2,\epsilon,b}J_x^s u\|_{L^2_x}\|_{L^2_y}+\|\|u\|_{L^2_x}\|_{L^2_y}\\
&=\|\theta_{2,\epsilon,b}J_x^s u\|_{L^2_{xy}}+\|u\|_{L^2_{xy}}.
\end{aligned} \tag{2.59}$$

Since we already proved that $\theta_{2,\epsilon,b}J_x^s u\in L^2(\mathbb{R}^2\times(0,T))$, the estimate for A_2 is complete.

Finally, to estimate A_3, according to Remark 2.4.3, we just need to justify (2.52) for $\bar{s}=s$. However, due to the restriction $s-s_\alpha<1$, this is equivalent to study the norms $\|J_x^s(u\phi_{\epsilon,b})\|_{L^2_{xy}}$ and $\|J_x^s(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}$. Recalling that $\operatorname{supp}(\phi_{\epsilon,b}),\operatorname{supp}(\widetilde{\phi}_{\epsilon,b})\subset[\epsilon/4,b]$, these estimates follow from (2.59).

2.4.2 Case $k-(k-2)\left(\frac{1-\alpha}{2}\right)\leq s<k+1-(k-1)\left(\frac{1-\alpha}{2}\right)$, $k\geq 2$

We have developed all the set up required to provide the proof of Theorem 2.1.2 for arbitrary regularity $s>s_\alpha=(17-2\alpha)/12$. Indeed, we will proceed employing an inductive argument.

Let $\epsilon'>0$, $b'\geq 5\epsilon'$ and $k\geq 2$ be given, we shall assume by the inductive hypothesis

$$\begin{aligned}\sup_{t\in[0,T]}&\|\chi_{\epsilon',b'}J_x^{\widetilde{s}}u(t)\|_{L^2_{xy}}+\|(\chi_{\epsilon',b'}\chi'_{\epsilon',b'})^{1/2}J_x^{\widetilde{s}}u\|_{L^2(\mathbb{R}^2\times(0,T))}\\&+\|(\chi_{\epsilon',b'}\chi'_{\epsilon',b'})^{1/2}D_x^{\frac{1+\alpha}{2}}J_x^{\widetilde{s}}u\|_{L^2(\mathbb{R}^2\times(0,T))}\\&+\|(\chi_{\epsilon',b'}\chi'_{\epsilon',b'})^{1/2}\partial_yJ_x^{\widetilde{s}}u\|_{L^2(\mathbb{R}^2\times(0,T))}\\&\leq\widetilde{c}_{\widetilde{s}}(\epsilon',b',T,\|\chi_{\epsilon',b'}J_x^{\widetilde{s}}u_0\|_{L^2},\|u\|_{L^\infty_TH^{s_\alpha^+}},\|\partial_xu\|_{L^1_TL^\infty_{xy}}),\end{aligned}\tag{2.60}$$

whenever $l-(l-2)\left(\frac{1-\alpha}{2}\right)\leq\widetilde{s}<l+1-(l-1)\left(\frac{1-\alpha}{2}\right)$ if $2\leq l<k$, (which holds when $k>2$), or $\widetilde{s}<2$ when $k=2$. It is worth pointing out that the second term on the l.h.s of (2.60) corresponds to the estimate for A_1 in (2.34) for $\bar{s}=\widetilde{s}$, while the third and fourth terms are the smoothing effect granted by the dispersion $-D^{\alpha+1}\partial_x+\partial_x\partial_y^2$ in the equation in (gBOZK). In addition, we remark that by Lemma 2.2.5 (III), and the hypothesis $\mathbb{1}_{\{x>0\}}J_x^su_0\in L^2(\mathbb{R}^2)$, we have

$$\|\chi_{\epsilon',b'}J_x^{\widetilde{s}}u_0\|_{L^2_{xy}}\lesssim\|\mathbb{1}_{\{x>0\}}J_x^su_0\|_{L^2_{xy}}+\|u_0\|_{L^2_{xy}},$$

whenever $\widetilde{s}\in[0,s]$, which justify the validity of the implicit energy estimates behind the inductive hypothesis.

Setting in (2.34), $\epsilon>0$, $b\geq 5\epsilon$ and $\bar{s}=s$ with $k-(k-2)\left(\frac{1-\alpha}{2}\right)\leq s<k+1-(k-1)\left(\frac{1-\alpha}{2}\right)$, $k\geq 2$, the desired estimate is obtained after controlling the respective factors A_1,A_2 and A_3.

Estimate for A_1**.** We consider the functions $\theta_{1,\epsilon,b},\theta_{2,\epsilon,b}$ and $\theta_{3,\epsilon,b}$ determined by (2.56) with $\theta_{j,\epsilon,b}=\theta_{j,\epsilon,b}(x+vt)$, $j=1,2,3$. By assumption (2.60) with $\widetilde{s}=s-1+\frac{1-\alpha}{2}$, $\epsilon'=\epsilon/24$ and $b'=b+13\epsilon/12$, we infer

$$\theta_{1,\epsilon,b}J_x^{s-1+\frac{1-\alpha}{2}}u,\ \theta_{1,\epsilon,b}D_x^{\frac{\alpha+1}{2}}J_x^{s-1+\frac{1-\alpha}{2}}u\in L^2(\mathbb{R}^2\times(0,T)).$$

Then, by Lemma 2.2.5 (I) and similar considerations to (2.57), the above consequence establishes

$$\theta_{2,\epsilon,b}J_x^su\in L^2(\mathbb{R}^2\times(0,T)),\tag{2.61}$$

and so from Lemma 2.2.5 (III),

$$\theta_{3,\epsilon,b}J_x^{s_*}u \in L^2(\mathbb{R}^2\times(0,T)), \text{ whenever } s_*\in(0,s].$$

Since $\theta_{3,\epsilon,b}(x+vt)=1$ on $[\epsilon-vt,b-vt]$, the above display yields the desired estimate for $A_1(t)$.

Estimate for A_2 and A_3. To estimate A_2 and A_3, by previous arguments relaying on Lemma 2.4.1 and Remark 2.4.3, it is enough to control the following expression

$$\sum_{0\leq j\leq s-s_\alpha}\|\chi_{\epsilon,b}J_x^{s-j}u\|_{L^2_{xy}}+\|J_x^{s-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}+\|J_x^{\bar{s}}(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}. \tag{2.62}$$

When $j=0$, $\|\chi_{\epsilon,b}J_x^{s-j}u\|_{L^2_{xy}}$ is the quantity to be estimated by (2.34) and Gronwall's lemma. Whereas $\|\chi_{\epsilon,b}J_x^{s-j}u\|_{L^2_{xy}}$ for $1<j\leq s-s_\alpha$ are controlled by the inductive hypothesis (2.60).

On the other hand, by support consideration, there exists $\widetilde{\theta}_{2,\epsilon,b}\in C_c^\infty(\mathbb{R})$ with $0\leq\widetilde{\theta}_{2,\epsilon,b}\leq 1$ such that $\text{dist}(1-\theta_{2,\epsilon,b},\widetilde{\theta}_{2,\epsilon,b})\geq\epsilon/32$ and $\text{dist}(1-\widetilde{\theta}_{2,\epsilon,b},\phi_{\epsilon,b})\geq\epsilon/32$. Thus, we apply Lemma 2.2.5 (IV) followed by part (III) to obtain

$$\begin{aligned}\|J_x^{s-j}(u\phi_{\epsilon,b})\|_{L^2_{xy}}&\lesssim\|\widetilde{\theta}_{2,\epsilon,b}J_x^{s-j}u\|_{L^2_{xy}}+\|u\|_{L^2_{xy}}\\&\lesssim\|\theta_{2,\epsilon,b}J_x^{s}u\|_{L^2_{xy}}+\|u\|_{L^2_{xy}},\end{aligned}$$

which is controlled by (2.61) for all integer $0\leq j\leq s-s_\alpha$. Noticing that

$$\text{supp}(\widetilde{\phi}_{\epsilon,b}),\ \text{supp}(\phi_{\epsilon,b})\ \subset\ [\epsilon/4,b],$$

the estimate for $\|J_x^s(u\widetilde{\phi}_{\epsilon,b})\|_{L^2_{xy}}$ is obtained by the same argument above. These comments provide a control to (2.62), and so the estimates for A_2 and A_3 are complete.

Finally, gathering the previous results and applying Gronwall's inequality, we deduce (2.60) for $s\in[k-(k-2)\big(\frac{1-\alpha}{2}\big),k+1-(k-1)\big(\frac{1-\alpha}{2}\big))$, $k\geq 2$. This completes the inductive step and in consequence the proof of Theorem 2.1.2.

2.5 Appendix

This section is aimed to prove Theorem 2.1.1. Since our arguments follow similar considerations employed in [31, 34, 49], we will state the main ingredients and differences needed to implement these ideas for the IVP (gBOZK). Let us first introduce some notation to be employed along with our arguments.

Consider $\varrho\in C_c^\infty(\mathbb{R})$ such that

$$0\leq\varrho\leq 1,\ \ \varrho(\xi)=1 \text{ for } |\xi|\leq 1,\ \ \varrho(\xi)=0 \text{ for } |\xi|\geq 2,$$

and $\varrho_0(\xi)=\varrho(\xi)-\varrho(2\xi)$ which is supported on $1/2\leq|\xi|\leq 2$. For any $f\in S(\mathbb{R}^2)$ and $j\in\mathbb{Z}$, we define the Littlewood-Paley projection operators

$$\widehat{P_j^x f}(\xi,\eta)=\varrho_0(2^{-j}\xi)\widehat{f}(\xi,\eta). \tag{2.63}$$

2.5.1 Strichartz estimates

The homogeneous problem associated to (gBOZK) is given by

$$\begin{cases} \partial_t u - D_x^{\alpha+1} u_x + u_{xyy} = 0, \quad (x,y,t) \in \mathbb{R}^3,\ 0 \le \alpha \le 1, \\ u(x,y,0) = \phi(x,y). \end{cases} \tag{2.64}$$

Sufficiently regular solutions of (2.64) will be denoted as

$$S(t)\phi(x,y) := \int_{\mathbb{R}^2} \widehat{\phi}(\xi,\eta) e^{i\xi|\xi|^{\alpha+1}t + i\xi\eta^2 t + ix\xi + iy\eta}\, d\xi d\eta,$$

for all $t \in \mathbb{R}$. The result in [92, Proposition 3] establishes the following estimate for solutions of (2.64)

$$\left\| P_j^x S(t)\phi \right\|_{L^p_{xy}} \lesssim \frac{2^{-\frac{j}{3}(\alpha+\frac{1}{2})(1-\frac{2}{p})}}{|t|^{\frac{5}{6}(1-\frac{2}{p})}} \|\phi\|_{L^{p'}_{xy}}, \text{ for all } j \in \mathbb{Z}, \tag{2.65}$$

whenever $1/p + 1/p' = 1$ with $p \ge 2$. Hence, an application of the TT^*-argument yields the estimate.

Lemma 2.5.1. *Let $0 \le \alpha < 1$, $1 < p < \infty$ and $j \in \mathbb{Z}$. Then the following estimate holds*

$$\left\| P_j^x S(t)\phi \right\|_{L^q_t L^p_{xy}} \lesssim 2^{-\frac{j}{6}(\alpha+\frac{1}{2})(1-\frac{2}{p})} \|\phi\|_{L^2_{xy}}, \tag{2.66}$$

for all $\frac{2}{q} + \frac{5}{3p} = \frac{5}{6}$.

The same argument in the proof of [92, Proposition 3] establishes the following *rough* dispersive estimate,

$$\left\| P_j^x S(t)\phi \right\|_{L^p_{xy}} \lesssim \frac{2^{\frac{j}{2}(1-\frac{2}{p})}}{|t|^{\frac{1}{2}(1-\frac{2}{p})}} \|\phi\|_{L^{p'}_{xy}}, \tag{2.67}$$

for all $j \in \mathbb{Z}$, $1/p + 1/p' = 1$ with $p \ge 2$. This result is convenient for our purposes to avoid the negative derivative carried by the right-hand side of (2.66) near the origin in the frequency domain. Consequently, (2.67) provides the following estimate.

Lemma 2.5.2. *Let $0 \le \alpha < 1$, $1 < p < \infty$ and $j \in \mathbb{Z}$. Then it holds*

$$\left\| P_j^x S(t)\phi \right\|_{L^q_t L^p_{xy}} \lesssim 2^{\frac{j}{4}(1-\frac{2}{p})} \|\phi\|_{L^2_{xy}}, \tag{2.68}$$

whenever $\frac{2}{q} + \frac{1}{p} = \frac{1}{2}$.

Notice that the case $(q,p) = (2,\infty)$ is not part of the conclusions in Lemmas 2.5.1 and 2.5.2. Accordingly, we require some additional regularity to control this norm.

Corollary 2.5.3. *Let $0 \le \alpha < 1$ and $j \in \mathbb{Z}$. For each $T > 0$, $0 < \delta < 1$ and $\theta \in [0,1]$, there exists $\widetilde{k}_{\delta,\theta} \in (0, 1/2)$ such that*

$$\left\|P_j^x S(t) f\right\|_{L^2_T L^\infty_{xy}} \le c_\delta T^{\widetilde{k}_{\delta,\theta}} 2^{\max\{j(\frac{1}{4}-\frac{\theta}{3}(1+\frac{\alpha}{2})), j(\frac{1}{4}-\frac{\theta}{3}(1+\frac{\alpha}{2}))(1-\delta)\}} \left\|J^\delta f\right\|_{L^2_{xy}}. \tag{2.69}$$

Proof. Let us consider $1 < p < \infty$ sufficiently large to assure that $\delta > \frac{2}{p}$. Combining Sobolev's embedding and (2.66), we get

$$\begin{aligned} &\|P_j^x S(t) f\|_{L^2_T L^\infty_{xy}} \\ &\quad\lesssim_\delta T^{\frac{q-2}{2q}} \left\|P_j^x S(t) J^\delta f\right\|_{L^q_T L^p_{xy}} \lesssim_\delta T^{\frac{q-2}{2q}} 2^{-\frac{j}{6}(\alpha+\frac{1}{2})(1-\frac{2}{p})} \left\|J^\delta f\right\|_{L^2_{xy}}, \end{aligned} \tag{2.70}$$

and using (2.68) instead,

$$\begin{aligned} &\|P_j^x S(t) f\|_{L^2_T L^\infty_{xy}} \\ &\quad\lesssim_\delta T^{\frac{\tilde{q}-2}{2q}} \left\|P_j^x S(t) J^\delta f\right\|_{L^{\tilde{q}}_T L^p_{xy}} \lesssim_\delta T^{\frac{\tilde{q}-2}{2\tilde{q}}} 2^{\frac{j}{4}(1-\frac{2}{p})} \left\|J^\delta f\right\|_{L^2_{xy}}, \end{aligned} \tag{2.71}$$

where $\frac{2}{\tilde{q}} + \frac{5}{3p} = \frac{5}{6}$ and $\frac{2}{\tilde{q}} + \frac{1}{p} = \frac{1}{2}$. Interpolating (2.70) and (2.71) yields (2.69). $\square$

The preceding conclusion is essential to derive the following Strichartz estimate, which is proved in much the same way as in [49].

Lemma 2.5.4. *Assume $0 \le \alpha < 1$, $0 < \delta < 1$, $T > 0$ and $s > s_\alpha - 1 = (5 - 2\alpha)/12$. Then, there exists $k_\delta \in (\frac{1}{2}, 1)$ such that*

$$\|w\|_{L^1_T L^\infty_{xy}} \lesssim_\delta T^{k_\delta}\Big(\sup_{t\in[0,T]} \left\|J_x^{s+\delta} J^\delta w(t)\right\|_{L^2_{xy}} + \int_0^T \left\|J_x^{s-1+\delta} J^\delta F(\cdot,\tau)\right\|_{L^2_{xy}} d\tau\Big), \tag{2.72}$$

whenever w is a solution of $\partial_t w - D_x^{\alpha+1} w_x + w_{xyy} = F$.

Proof. Recalling the projectors introduced in (2.63), an application of the triangle inequality reduces our considerations to control the r.h.s of the following expression

$$\|w\|_{L^1_T L^\infty_{xy}} \lesssim \sum_j \left\|P_j^x w\right\|_{L^1_T L^\infty_{xy}}. \tag{2.73}$$

Since $P_j^x w$ satisfies the integral equation

$$P_j^x w(t) = S(t) P_j^x w(0) + \int_0^t S(t-\tau) P_j^x F(\cdot,\tau)\, d\tau,$$

by writing $P_j^x = \widetilde{P}_j^x P_j^x$ for some adapted projection $\widetilde{P}_j^x$, we first apply Hölder's inequality and then Corollary 2.5.3 with $\theta = 0$ to get

$$\begin{aligned} \left\|P_j^x w\right\|_{L^1_T L^\infty_{xy}} &\lesssim T^{1/2+\widetilde{k}_{\delta,0}} 2^{\frac{j}{4}(1-\delta)}\Big(\left\|J^\delta P_j^x w(0)\right\|_{L^2_{xy}} + \int_0^T \left\|J^\delta P_j^x F(\tau)\right\|_{L^2_{xy}} d\tau\Big) \\ &\lesssim T^{1/2+\widetilde{k}_{\delta,0}} 2^{\frac{j}{4}(1-\delta)}\Big(\sup_{t\in[0,T]} \left\|J^\delta w(t)\right\|_{L^2_{xy}} + \int_0^T \left\|J^\delta F(\tau)\right\|_{L^2_{xy}} d\tau\Big), \end{aligned}$$

for each $j \leq 0$. Adding the above expression over $j \leq 0$, we derived the desired estimate for these indexes.

To bound the remaining sum on the right-hand side of (2.73), let us consider $j > 0$ and we split the interval $[0,T] = \bigcup_m I_m$, where $I_m = [a_m, b_m]$ and $(b_m - a_m) = cT/2^j$. As a consequence

$$\left\|P_j^x w\right\|_{L^1_T L^\infty_{xy}} \lesssim \frac{T^{1/2}}{2^{j/2}} \sum_m \left\|P_j^x w\right\|_{L^2_{I_m} L^\infty_{xy}}. \tag{2.74}$$

By employing Duhamel's formula on each I_m, we obtain

$$P_j^x w(t) = S(t-a_m) P_j^x w(\cdot, a_m) + \int_{a_k}^t S(t-\tau) P_j^x F(\cdot,\tau)\, d\tau,$$

whenever $t \in I_m$. Then, (2.74) and Corollary 2.5.3 with $\theta = 1$ show

$$\begin{aligned} &\|P_j^x w\|_{L^1_T L^\infty_x} \\ &\quad\lesssim T^{1/2+\widetilde{k}_{\delta,1}} 2^{-\frac{j}{2}-\frac{j}{6}(\alpha+\frac{1}{2})(1-\delta)} \sum_m \Big(\left\|J^\delta P_j^x w(a_m)\right\|_{L^2_{xy}} \\ &\qquad\qquad + \int_{I_m} \left\|J^\delta P_j^x F(\tau)\right\|_{L^2_{xy}} d\tau \Big) \\ &\quad\lesssim T^{1/2+\widetilde{k}_{\delta,1}} 2^{-\frac{j\delta}{2}} \Big(\sup_{t\in[0,T]} \left\|J_x^{\frac{1}{2}-\frac{1}{6}(\alpha+\frac{1}{2})(1-\delta)+\delta/2} J^\delta w(t)\right\|_{L^2_{xy}} \\ &\qquad\qquad + \int_0^T \left\|J_x^{-\frac{1}{2}-\frac{1}{6}(\alpha+\frac{1}{2})(1-\delta)+\delta/2} J^\delta F(\tau)\right\|_{L^2_{xy}} d\tau \Big). \end{aligned} \tag{2.75}$$

Summing the above expression over $j > 0$, using that $s > (17-2\alpha)/12 - 1 = 1/2 - 1/6(\alpha+1/2)$ and that $\frac{1}{6}(\alpha+\frac{1}{2})\delta < \frac{\delta}{2}$, we complete the proof. □

As a further consequence of Lemma 2.5.4, for $0 \leq \alpha < 1$, $0 < \delta \leq 1$, $T > 0$ and $s > s_\alpha$, we find that

$$\begin{aligned} &\|\partial_x w\|_{L^1_T L^\infty_{xy}} \\ &\quad\lesssim_\delta T^{k_\delta} \Big(\sup_{t\in[0,T]} \left\|J_x^{s+\delta} J^\delta w(t)\right\|_{L^2} + \int_0^T \left\|J_x^{s-1+\delta} J^\delta F(\cdot,\tau)\right\|_{L^2_{xy}} d\tau \Big), \end{aligned} \tag{2.76}$$

and

$$\begin{aligned} \|\partial_y w\|_{L^1_T L^\infty_{xy}} \quad &\lesssim_\delta T^{k_\delta} \Big(\sup_{t\in[0,T]} \left\|J_x^{s-1+\delta} J^{1+\delta} w(t)\right\|_{L^2_{xy}} \\ &\qquad + \int_0^T \left\|J_x^{s-2+\delta} J^{1+\delta} F(\cdot,\tau)\right\|_{L^2_{xy}} d\tau \Big), \end{aligned} \tag{2.77}$$

for some $k_\delta \in (\frac{1}{2}, 1)$ and where w solves the equation $\partial_t w - D_x^{\alpha+1} w_x + w_{xyy} = F$.

2.5.2 Energy Estimates

Whenever $s > 2$, Theorem 2.1.1 follows by a parabolic regularization argument on (gBOZK). Roughly speaking, an additional term $-\mu\Delta u$ is added to the equation, after which the limit $\mu \to 0$ is taken. These results follow the same arguments in [21, 38], so we omit its proof.

Lemma 2.5.5. *Let $s > 2$ and $0 \leq \alpha \leq 1$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{R}^d))$ of the IVP* (gBOZK)*. Additionally, the flow-map $u_0 \mapsto u(t)$ is continuous in the H^s-norm.*

For simplicity, we shall take $s = 3$ in the above lemma. Therefore, the proof of Lemma 2.5.5 also provides existence of smooth solutions. More specifically, given $u_0 \in H^\infty(\mathbb{R}^2) = \bigcap_{m\geq 0} H^m(\mathbb{R}^2)$ there exist $T(\|u\|_{H^3}) > 0$, and a unique solution u of (gBOZK) in the class $C([0,T]; H^\infty(\mathbb{R}^2))$. Additionally, Lemma 2.5.5 yields the following conclusion.

Blow-up criteria. Let $u_0 \in H^\infty(\mathbb{R}^2)$ there exist $T^* > T(\|u_0\|_{H^3}) > 0$ and a unique maximal solution u of (gBOZK) in $C([0,T^*); H^\infty(\mathbb{R}^d))$. Moreover, if the maximal time of existence T^* is finite

$$\lim_{t\to T^*} \|u(t)\|_{H^3} = \infty. \tag{2.78}$$

Next, we deduce some estimates involving smooth solutions of (gBOZK). By recurrent arguments using Lemma 1.4.1 to control the nonlinear term in (gBOZK), it follows:

Lemma 2.5.6. *Let $T > 0$ and $u \in C([0,T]; H^\infty(\mathbb{R}^d))$ solution of the IVP associated to* (gBOZK)*. Then, there exists a positive constant c_0 such that*

$$\|u\|^2_{L^\infty_T H^s} \leq \|u_0\|^2_{H^s} + c_0 \|\nabla u\|_{L^1_T L^\infty_x} \|u\|^2_{L^\infty_T H^s} \tag{2.79}$$

for any $s > 0$.

In addition, we require further *a priori* estimates for the $L^1([0,T]; W^{1,\infty}(\mathbb{R}^2))$-norm of smooth solutions of the IVP (gBOZK).

Lemma 2.5.7. *Consider $0 \leq \alpha < 1$. Let $u \in C([0,T]; H^\infty(\mathbb{R}^d))$ be a solution of the IVP* (gBOZK)*. Then, for any $s > s_\alpha$ there exist $k_\delta \in (\frac{1}{2}, 1)$ and $c_s > 0$ such that*

$$f(T) := \|u\|_{L^1_T L^\infty_{xy}} + \|\nabla u\|_{L^1_T L^\infty_{xy}}.$$

satisfies

$$f(T) \leq c_s T^{k_\delta}(1 + f(T)) \|u\|_{L^\infty_T H^s}. \tag{2.80}$$

Proof. Let $\widetilde{s} \in (s_\alpha, s)$ fixed and $0 < \delta < \min\{(s-\widetilde{s})/2, 1\}$. From (2.76) with $F = -u\partial_x u$, we get

$$\begin{aligned} &\|\partial_x u\|_{L^1_T L^\infty_{xy}} \\ &\quad \lesssim T^{\kappa_\delta}\Big(\sup_{[0,T]} \|J^{\widetilde{s}+2\delta} u(t)\|_{L^2_{xy}} + \int_0^T \|J^{\widetilde{s}-1+2\delta}(u\partial_x u)(\tau)\|_{L^2_{xy}}\, d\tau\Big), \end{aligned} \tag{2.81}$$

for some $k_\delta \in (1/2, 1)$. Our choice of δ then shows that

$$\sup_{t\in[0,T]} \|J^{\widetilde{s}+2\delta} u(t)\|_{L^2_{xy}} \leq \|u\|_{L^\infty_T H^s}, \tag{2.82}$$

and Lemma 1.4.3 gives

$$\begin{aligned} &\|J^{\widetilde{s}-1+2\delta}(u\partial_x u)\|_{L^2_{xy}} \\ &\qquad \lesssim \|u\|_{L^\infty}\|J^{\widetilde{s}-1+2\delta}\partial_x u\|_{L^2_{xy}} + \|J^{\widetilde{s}-1+2\delta}u\|_{L^2_{xy}}\|\partial_x u\|_{L^\infty} \\ &\qquad \lesssim (\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty})\|u\|_{L^\infty_T H^s}. \end{aligned} \tag{2.83}$$

Plugging the above estimates in (2.81), we arrive at

$$\|\partial_x u\|_{L^1_T L^\infty_{xy}} \lesssim_s T^{\kappa_\delta}(1+f(T))\|u\|_{L^\infty_T H^s_x}. \tag{2.84}$$

On the other hand, (2.77) yields

$$\begin{aligned} &\|\partial_y u\|_{L^1_T L^\infty_{xy}} \\ &\quad \lesssim T^{\kappa_\delta}\Big(\sup_{[0,T]}\|J^{\widetilde{s}+2\delta}u(t)\|_{L^2_{xy}} + \int_0^T \|J_x^{\widetilde{s}-2+\delta}J^{1+\delta}(u\partial_x u)(\tau)\|_{L^2_{xy}}\,d\tau\Big) \\ &\quad \lesssim T^{\kappa_\delta}\Big(\sup_{[0,T]}\|J^{\widetilde{s}+2\delta}u(t)\|_{L^2_{xy}} + \int_0^T \|J^{\widetilde{s}+2\delta}(u^2)(\tau)\|_{L^2_{xy}}\,d\tau\Big). \end{aligned} \tag{2.85}$$

And so the above inequality allows us to argue as in (2.82) and (2.83) to obtain

$$\|\partial_y u\|_{L^1_T L^\infty_{xy}} \lesssim_s T^{\kappa_\delta}(1+f(T))\|u\|_{L^\infty_T H^s}.$$

Now, since $\widetilde{s}-1>0$, one can use (2.72) and the arguments in (2.85) to obtain the desired estimate for $\|u\|_{L^1_T L^\infty_{xy}}$. □

2.5.3 A priori Estimates

We require some additional *a priori* estimates.

Lemma 2.5.8. *Let $0 \le \alpha < 1$, $s \in (s_\alpha, 3]$. Then there exists $A_s > 0$, such that for all $u_0 \in H^\infty(\mathbb{R}^2)$, there is a solution $u \in C([0,T^*);H^\infty(\mathbb{R}^2))$ of* (gBOZK) *where $T^* = T^*(\|u_0\|_{H^3}) > (1+A_s\|u_0\|_{H^s})^{-2}$. Moreover, there exists a constant $K_0>0$ such that*

$$\|u\|_{L^\infty_T H^s} \le 2\|u_0\|_{H^s},$$

and

$$f(T) = \|u\|_{L^1_T L^\infty_{xy}} + \|\nabla u\|_{L^1_T L^\infty_{xy}} \le K_0,$$

whenever $T \le (1+A_s\|u_0\|_{H^s})^{-2}$.

Proof. In view of the Lemmas 2.5.6, 2.5.7 and the blow-up criteria (2.78) applied to the H^3-norm, the proof follows from the same arguments in [65, Lemma 5.3]. □

2.5.4 Proof of Theorem 2.1.1

According to Lemma 2.5.5, we shall assume that $(17-2\alpha)/12 = s_\alpha < s \leq 3$. Let us consider $u_0 \in H^s(\mathbb{R}^2)$ fixed. The existence part is deduced employing the Bona-Smith argument [14]. More specifically, we regularize the initial data by choosing $\rho \in C_c^\infty(\mathbb{R}^2)$ radial with $0 \leq \rho \leq 1$, $\rho(\xi,\eta) = 1$ for $|(\xi,\eta)| \leq 1$ and $\rho(\xi,\eta) = 0$ for $|(\xi,\eta)| > 2$, we set then

$$u_{0,n} := \{\rho(\xi/n, \eta/n)\widehat{u}_0(\xi,\eta)\}^\vee$$

for any integer $n \geq 1$. Now, by employing Plancherel's identity and Lebesgue dominated convergence theorem, it is not difficult to see that for $m \geq n \geq 1$,

$$n^\sigma \|J^{s-\sigma}(u_{0,n} - u_{0,m})\|_{L^2_{xy}} \underset{n\to\infty}{\to} 0, \tag{2.86}$$

whenever $0 \leq \sigma \leq s$.

Consequently, since $\|u_{0,n}\|_{H^s} \leq \|u_0\|_{H^s}$, Lemma 2.5.8 establishes the existence of regularized solutions $u_n \in C([0,T]; H^\infty(\mathbb{R}^2))$ emanating from $u_{0,n}$ $n \geq 1$, sharing the same existence time

$$0 < T \leq (1 + A_s \|u_0\|_{H^s})^{-2}, \tag{2.87}$$

satisfying

$$\|u_n\|_{L^\infty_T H^s} \leq 2\|u_0\|_{H^s}, \tag{2.88}$$

and

$$K = \sup_{n\geq 1}\left\{\|u_n\|_{L^1_T L^\infty_{xy}} + \|\nabla u_n\|_{L^1_T L^\infty_{xy}}\right\} < \infty. \tag{2.89}$$

Therefore, setting $v_{n,m} = u_n - u_m$, we find

$$\partial_t v_{n,m} - D_x^{\alpha+1}\partial_x v_{n,m} + \partial_x\partial_y^2 v_{n,m} + \frac{1}{2}\partial_x((u_n + u_m)v_{n,m}) = 0, \tag{2.90}$$

with initial condition $v_{n,m}(0) = u_{0,n} - u_{0,m}$.

Thus, by employing recurrent energy estimates, (2.89) and (2.86) we find

$$n^{s-\sigma}\|J^\sigma(u_n - u_m)\|_{L^\infty_T L^2_{xy}} \lesssim n^{s-\sigma}e^{cK}\|J^\sigma(u_{0,n} - u_{0,m})\|_{L^\infty_T L^2_{xy}} \underset{n\to\infty}{\to} 0, \tag{2.91}$$

$m \geq n \geq 1$, whenever $0 \leq \sigma < s$.

Proposition 2.5.9. *Let $m \geq n \geq 1$, then*

$$n\|u_n - u_m\|_{L^1_T L^\infty_{xy}} + \|\nabla(u_n - u_m)\|_{L^1_T L^\infty_{xy}} \underset{n\to\infty}{\to} 0, \tag{2.92}$$

provided that A_s in (2.87) *is taken large enough. Moreover,*

$$\|u_n - u_m\|_{L^\infty_T H^s} \underset{n\to\infty}{\to} 0. \tag{2.93}$$

Proof. We begin deducing the first estimate on the l.h.s of (2.92). Let $\widetilde{s} \in (s_\alpha, s)$ and $0 < \delta < \min\{1, (s-\widetilde{s})/2\}$ fixed. An application of Lemma 2.5.4 with equation (2.90) yields

$$\begin{aligned}
&\|v_{n,n}\|_{L^1_T L^\infty_{xy}} \\
&\lesssim T^{1/2}\Big(\|J^{\widetilde{s}-1+\delta} v_{n,m}\|_{L^\infty_T L^2_{xy}} + \int_0^T \|J^{\widetilde{s}-1+2\delta}((u_n+u_m)v_{n,m})(\tau)\|_{L^2_{xy}}\, d\tau\Big) \\
&\lesssim T^{1/2}\Big(\|J^{\widetilde{s}-1+\delta} v_{n,m}\|_{L^\infty_T L^2_{xy}} \\
&\qquad + \int_0^T \Big(\|J^{\widetilde{s}-1+2\delta}(u_n+u_m)(\tau)\|_{L^2_{xy}} \|v_{n,m}(\tau)\|_{L^\infty_{xy}} \\
&\qquad\qquad + \|J^{\widetilde{s}-1+2\delta} v_{n,m}(\tau)\|_{L^2_{xy}} \|(u_n+u_m)(\tau)\|_{L^\infty_{xy}}\Big)\, d\tau\Big),
\end{aligned} \tag{2.94}$$

where we have employed Lemma 1.4.3. Notice that our choice of δ and (2.91) give

$$\|J^{\widetilde{s}-1+\delta} v_{n,m}\|_{L^\infty_T L^2_{xy}} = o(n^{-1}),$$

so that (2.94), (2.89) and (2.92) imply

$$\|v_{n,n}\|_{L^1_T L^\infty_{xy}} = o(n^{-1}) + O(T^{1/2}\|v_{n,n}\|_{L^1_T L^\infty_{xy}}).$$

Hence, taking $0 < T < 1$ small with respect to the above constant (that is, A_s large in (2.87)), we find

$$\|v_{n,n}\|_{L^1_T L^\infty_{xy}} = o(n^{-1}). \tag{2.95}$$

On the other hand, by a similar reasoning dealing with (2.94), employing (2.76) and (2.77) with equation (2.90), we obtain

$$\begin{aligned}
\|\nabla v_{n,m}\|_{L^1_T L^\infty_{xy}} &\lesssim T^{1/2}\Big(\|J^{\widetilde{s}+\delta} v_{n,m}\|_{L^\infty_T L^2_{xy}} + \|J^{\widetilde{s}+2\delta}(u_n+u_m)\|_{L^\infty_T L^2_{xy}} \|v_{n,m}\|_{L^1_T L^\infty_{xy}} \\
&\qquad + \|J^{\widetilde{s}+2\delta} v_{n,m}\|_{L^\infty_T L^2_{xy}} \|v_n+v_m\|_{L^1_T L^\infty_{xy}}\Big) \\
&= o(1) + O(\|v_{n,m}\|_{L^1_T L^\infty_{xy}}),
\end{aligned}$$

where we have employed (2.88) and (2.89). Then, (2.95) completes the deduction of (2.92).

Next, we deduce (2.93). Applying J^s to (2.90), multiplying then by $v_{n,m}$, integrating in space shows

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|J^s v_{n,m}(t)\|^2_{L^2_{xy}} &= -\int J^s(v_{n,m}\partial_x u_n) J^s v_{n,m}\, dxdy \\
&\quad - \int J^s(u_m \partial_x v_{n,m}) J^s v_{n,m}\, dxdy \\
&=: A_1 + A_2.
\end{aligned} \tag{2.96}$$

Applying the Cauchy-Schwarz inequality and the commutator estimate (1.4.1),

$$\begin{aligned}
|A_1| &= \Big|\int [J^s, v_{n,m}]\partial_x u_n J^s v_{n,m}\, dxdy + \int v_{n,m}\partial_x J^s u_n J^s v_{n,m}\, dxdy\Big| \\
&\lesssim \|[J^s, v_{n,m}]\partial_x u_n\|_{L^2_{xy}} \|J^s v_{n,m}\|_{L^2_{xy}} + \|v_{n,m}\|_{L^\infty_{xy}} \|\partial_x J^s u_n\|_{L^2_{xy}} \|J^s v_{n,m}\|_{L^2_{xy}} \\
&\lesssim \|\nabla v_{n,m}\|_{L^\infty_{xy}} \|J^s u_n\|_{L^2_{xy}} \|J^s v_{n,m}\|_{L^2_{xy}} + \|\partial_x u_n\|_{L^\infty_{xy}} \|J^s v_{n,m}\|^2_{L^2_{xy}} \\
&\qquad + \|v_{n,m}\|_{L^\infty_{xy}} \|\partial_x J^s u_n\|_{L^2_{xy}} \|J^s v_{n,m}\|_{L^2_{xy}}.
\end{aligned}$$

Now, employing energy estimates with the equation in (gBOZK) and using (2.88) and (2.89), we get

$$\|\partial_x J^s u_n\|_{L^2_{xy}} \lesssim e^{cK}\|\partial_x J^s u_{0,n}\|_{L^2_{xy}} \lesssim n.$$

Gathering the previous estimates, we arrive at

$$\begin{aligned}|A_1| \lesssim \|\nabla v_{n,m}\|_{L^\infty_{xy}}\|J^s v_{n,m}\|_{L^2_{xy}} + \|\nabla u_n\|_{L^\infty_{xy}}\|J^s v_{n,m}\|^2_{L^2_{xy}}\\ + (n\|v_{n,m}\|_{L^\infty_{xy}})\|J^s v_{n,m}\|_{L^2_{xy}}.\end{aligned}$$

On the other hand, integrating by parts

$$\begin{aligned}|A_2| &= \Big|\int [J^s, u_m]\partial_x v_{n,m} J^s v_{n,m}\, dxdy - \frac{1}{2}\int \partial_x u_m (J^s v_{n,m})^2\, dxdy\Big|\\ &\lesssim \|[J^s, u_m]\partial_x v_{n,m}\|_{L^2_{xy}}\|J^s v_{n,m}\|_{L^2_{xy}} + \|\partial_x u_m\|_{L^\infty_{xy}}\|J^s v_{n,m}\|^2_{L^2_{xy}}\\ &\lesssim \|\nabla u_m\|_{L^\infty_{xy}}\|J^s v_{n,m}\|^2_{L^2_{xy}} + \|J^s u_m\|_{L^2_{xy}}\|\partial_x v_{n,m}\|_{L^\infty_{xy}}\|J^s v_{n,m}\|_{L^2_{xy}}.\end{aligned}$$

Inserting the estimates for A_1 and A_2 in (2.96), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J^s v_{n,m}(t)\|^2_{L^2_{xy}} \lesssim &(n\|v_{n,m}\|_{L^\infty_{xy}} + \|\nabla v_{n,m}\|_{L^\infty_{xy}})\|J^s v_{n,m}\|_{L^2_{xy}}\\ &+ (\|\nabla u_n\|_{L^\infty_{xy}} + \|\nabla u_m\|_{L^\infty_{xy}})\|J^s v_{n,m}\|^2_{L^2_{xy}}.\end{aligned}$$

Hence, applying Gronwall's inequality to the above expression, and recalling (2.89), we find that there exists $c>0$ such that

$$\begin{aligned}\|J^s(u_n - u_m)\|_{L^\infty_T L^2_{xy}} \lesssim &\Big(\|J^s(u_{0,n} - u_{0,m})\|_{L^\infty_T L^2_{xy}}\\ &+ (n\|v_{n,m}\|_{L^1_T L^\infty_{xy}} + \|\nabla v_{n,m}\|_{L^1_T L^\infty_{xy}})\Big)e^{cK}.\end{aligned}$$

Now, (2.93) is a consequence of (2.86) and (2.92). □

We deduce from Proposition 2.5.9 that u_n converges to a function u in

$$C([0,T]; H^s(\mathbb{R}^2) \cap L^1([0,T]; W^{1,\infty}(\mathbb{R}^2)).$$

Therefore, since u_n solves the integral equation

$$u_n(t) = S(t)u_{0,n} - \frac{1}{2}\int_0^t S(t-\tau)\partial_x u_n^2(\tau)\, d\tau,$$

letting $n \to \infty$ in the sense of $C([0,T]; H^{s-1}(\mathbb{R}^2))$, we conclude that u also solves the integral equation associated to (gBOZK). This completes the existence part of Theorem 2.1.1. Uniqueness is derived by using a similar energy estimate to (2.79) for the difference of two solutions, and then applying Gronwall's lemma. Finally, continuous dependence is extended by approximation with the sequence of smooth solutions $\{u_n\}$ and employing this same property from Lemma 2.5.5. We refer to [34, 65] for an explicit prove of these results.

3

On Local Decay Properties

3.1 Introduction

In this chapter, we study special asymptotic behavior of solutions to the IVP associated to some dispersive equations introduced in the Chapter 1. We will start studying these properties for solutions of the IVP BO equation

$$\begin{cases} \partial_t u - \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{3.1}$$

First we will introduce some notation and definitions that will be used for each IVP that we will work in this section.

Let ϕ be a smooth even and positive function such that

$$\begin{cases} \text{i)} & \phi'(x) \leq 0, \text{ for } x \geq 0, \\ \text{ii)} & \phi(x) \equiv 1, \text{ for } 0 \leq x \leq 1, \\ \text{iii)} & \phi(x) = e^{-x} \text{ for } x \geq 2, \\ \text{iv)} & e^{-x} \leq \phi(x) \leq 3e^{-x} \text{ for } x \geq 0, \\ \text{v)} & |\phi'(x)| \leq c\,\phi(x) \text{ and } |\phi''(x)| \leq c\,\phi(x) \\ & \quad \text{for some positive constant } c. \end{cases} \tag{3.2}$$

Let $\psi(x) = \int_0^x \phi(s)ds$. In particular, $|\psi(x)| \leq 1 + 3\int_1^\infty e^{-t}dt < \infty$.

Now, we consider the smooth cut-off functions $\zeta, \zeta_{B_0} : \mathbb{R} \to \mathbb{R}$ such that

$$\zeta \equiv 1 \text{ in } [0,1], \quad 0 \leq \zeta \leq 1 \quad \text{and} \quad \zeta \equiv 0 \text{ in } (\infty, -1] \cup [2, \infty), \tag{3.3}$$

$$\zeta_{B_0} \equiv 1 \text{ in } [-1,1], \quad 0 \leq \zeta_{B_0} \leq 1 \quad \text{and} \quad \zeta \equiv 0 \text{ in } (\infty, -|\xi_0|] \cup [|\xi_0|, \infty), \tag{3.4}$$

and we define $\zeta_n(x) := \zeta(x-n)$.

For the parameters $\delta, \sigma \in \mathbb{R}^+$, we define

$$\phi_\delta = \delta\phi\left(\frac{x}{\delta}\right) \quad \text{and} \quad \psi_\sigma(x) = \sigma\psi\left(\frac{x}{\sigma}\right).$$

First, we start by considering some useful parameters involved in our argument of proof.

$$\rho(t) = \pm t^m, \quad \mu_1(t) = \frac{t^b}{\log t} \quad \text{and} \quad \mu(t) = t^{(1-b)}\log^2 t, \tag{3.5}$$

where m and b are positive constants satisfying the relations

$$0 \le m \le 1 - \frac{b}{2} \quad \text{and} \quad 0 < b \le \min\left\{\frac{2}{3}, \frac{2}{2+q}\right\}, \qquad q > 0. \tag{3.6}$$

Since

$$\frac{\mu_1'(t)}{\mu_1(t)} = \frac{b}{t} - \frac{1}{t\log t} \quad \text{and} \quad \frac{\mu'(t)}{\mu(t)} = \frac{(1-b)}{t} + \frac{2}{t\log t}$$

it readily follows that

$$\frac{\mu_1'(t)}{\mu_1(t)} \sim \frac{\mu'(t)}{\mu(t)} = O\left(\frac{1}{t}\right), \quad \text{for} \quad t \gg 1 \tag{3.7}$$

where $t \gg 1$ means the values of t such that $\mu_1'(t)$ is positive. In particular, $[10, +\infty) \subset \{t \gg 1\}$.

3.2 Benjamin-Ono equation

In Chapter 1, we gave an historical overview of the developments made regarding well-posedness for the IVP (BO). The BO equation possesses an infinite number of conservation laws, being the first three the following :

$$\begin{aligned} I_1(u(\cdot,t)) &= \int_{\mathbb{R}} u(\cdot,t)\,dx, \\ I_2(u(\cdot,t)) &= M(u(\cdot,t)) = \int_{\mathbb{R}} u^2(\cdot,t)\,dx, \\ I_3(u(\cdot,t)) &= E(u(\cdot,t)) = \int_{\mathbb{R}} \left(\frac{1}{2}|D^{1/2}u(\cdot,t)|^2 + \frac{1}{6}u^3(\cdot,t)\right)dx, \end{aligned} \tag{3.8}$$

where $\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$.

The k-conservation law, $I_k(\cdot)$, $k \ge 2$, provides a global in time *a priori* estimate of the norm $\|D^{(k-2)/2}u(t)\|_{L^2}$ of the solution $u = u(x,t)$ of the (BO).

In this section, we are interested in the study of the asymptotic behavior of solutions to the IVP (BO). In fact, we shall deduce some decay properties for solutions of (BO) as time evolves. We shall mention recent progress made in this direction which have inspired our work. In [70], Martel and Merle studied the phenomenon of asymptotic stability of solitons in the energy space H1 for the solutions of the gKdv equation, because the same authors in [69] have been built blowing up solutions for this model, that guarantees the blow up in finite time (or formation of singularity in finite time) of solutions of the critical generalized KdV equation, . Kowalczyk, Martel, and Muñoz [53] showed the asymptotic stability of the kink of the ϕ^4 model with respect to odd perturbations in the energy space. In the same spirit, Muñoz and Ponce [83], [84] introduced a novel approach to study long time behavior for solutions of the KdV and BO equations. Then in [61] Linares, Mendez and Ponce extended this approach to deal with the dispersion generalized BO equation. In [72], [73] new techniques were introduced by Mendez, Munoz, Poblete and Pozo to study the KP and ZK equations which are high dimensional models, in particular, they deduced the virial estimates used in this chapter. This technique was proved useful to analyze one dimensional systems as was done by Linares and Mendez [60] for the solutions of the Schrodinger-Korteweg-de Vries system. The main results in this section are contained in [26].

The following theorem of well-posedness of the IVP (BO) can be found [78] and it guarantees the existence of a global solution which is necessary to obtain the results of this section. The definition of the spaces can be found in Chapter 1.

Theorem 3.2.1. *Let $s \geq 0$ be given. For all $u_0 \in H^s(\mathbb{R})$ and all $T > 0$, there exists a solution*

$$u \in C([0,T]; H^s(\mathbb{R})) \cap X_T^{s-1,1} \cap L_T^4 W_x^{s,4} \tag{3.9}$$

of (BO) such that

$$w = \partial_x P_{+hi}(e^{-\frac{i}{2}F[u]} \in Y_T^s).$$

This solution is unique in the following classes:

(i) $u \in L^\infty((0,T); L^2(\mathbb{R})) \cap L^4((0,T) \times \mathbb{R})$ *and* $w \in X_T^{0,1/2}$,

(ii) $u \in L^\infty((0,T); H^s(\mathbb{R})) \cap L_T^4 W_x^{s,4}$ *whenever* $s > \frac{1}{4}$,

(iii) $u \in L^\infty((0,T); H^s(\mathbb{R}))$ *whenever* $s > \frac{1}{4}$.

Moreover, $u \in C_b(\mathbb{R}; L^2(\mathbb{R}))$, *and the flow-map data solution* $u_0 \mapsto u$ *is continuous from* $H^s(\mathbb{R})$ *into* $C([0,T]; H^s(\mathbb{R}))$.

Now, we have all te conditions to establish and prove the results of this section. Our main results in this section are the following:

Theorem 3.2.2. *Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the global in time solution of the IVP* (BO) *such that $u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$. Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(0)} u^2(x,t)\,dx = 0, \tag{3.10}$$

where $B_{t^b}(0)$ denotes the ball centered in the origin with radius t^b,

$$B_{t^b}(0) := \{x \in \mathbb{R} : |x| < t^b\} \quad \text{with} \quad 0 < b < \frac{2}{3}. \tag{3.11}$$

Moreover, there exist a constant $C > 0$ and an increasing sequence of times $t_n \to \infty$ such that

$$\int_{B_{t_n^b}(0)} u^2(x,t_n)\,dx \le \frac{C}{\log^{\frac{(1-b)}{b}}(t_n)}. \tag{3.12}$$

Figure 4 – Interior Region

As a consequence of the proof of this theorem we have:

Corollary 3.2.3. *Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the global in time solution of the IVP* (BO) *such that $u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$. Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(t^m)} u^2(x,t)\,dx = 0, \tag{3.13}$$

where

$$B_{t^b}(t^m) := \{x \in \mathbb{R} : |x - t^m| < t^b\}, \tag{3.14}$$

with

$$0 < b < \frac{2}{3} \quad \text{and} \quad 0 \le m < 1 - \frac{3}{2}b. \tag{3.15}$$

Remark 3.2.4. *Under the additional hypothesis:*

$$\begin{gathered}\text{There exist } a \in [0,1/2) \text{ and } c_0 > 0 \text{ such that for all } T > 0\\ \sup_{t\in[0,T]} \int_{-\infty}^{\infty} |u(x,t)|\,dx \le c_0\,(1+T^2)^{a/2},\end{gathered} \tag{3.16}$$

a related result to those in Theorem 3.2.2 and Corollary 3.2.3 was established in [84]. The argument of the proof in [84] was based on virial identities (or weighted energy estimate) first appearing in [83] in the study of the long time behavior of solution of the generalized

Korteweg-de Vries (KdV) equation. In [82] and [61] this was extended, adapted and generalized to others one dimensional dispersive nonlinear systems under an assumption similar to that in (3.16).

In [73] a key idea was introduced to remove the hypothesis (3.16) *and to extend the argument to higher dimensional dispersive model. This approach was further implemented in [72] and [60] for systems.*

Next, we present a result concerning the decay of solutions in the energy space :

Theorem 3.2.5. *Let* $u_0 \in H^{1/2}(\mathbb{R})$ *and* $u = u(x,t)$ *be the global in time solution of the IVP* (BO) *such that*

$$u \in C(\mathbb{R} : H^{1/2}(\mathbb{R})) \cap L^{\infty}(\mathbb{R} : H^{1/2}(\mathbb{R})).$$

Then

$$\liminf_{t\to\infty} \int_{B_{t^b}(0)} \left(u^2(x,t) + |D_x^{1/2}u(x,t)|^2 \right) dx = 0, \quad 0 < b < \frac{2}{3}. \tag{3.17}$$

3.2.1 Proof of Theorem 3.2.2 and Corollary 3.2.3

The proof of Theorem 3.2.2 will be deduced as a consequence of the following lemmas, which we shall prove below.

For $u = u(x,t)$ a solution of the IVP (BO) we consider the functional

$$\mathcal{I}(t) := \frac{1}{\mu(t)} \int_{\mathbb{R}} u(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx, \tag{3.18}$$

for $q > 1$.

Lemma 3.2.6. *Let* $u(\cdot,t) \in L^2(\mathbb{R})$, $t \gg 1$. *The functional* $\mathcal{I}(t)$ *is well defined and bounded in time.*

Proof. The Cauchy-Schwarz inequality and the definition of the functions $\mu(t)$ and $\mu_1(t)$ imply that

$$\begin{aligned}
|\mathcal{I}(t)| &\le \frac{1}{\mu(t)}\|u(t)\|_{L^2}\left\|\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right\|_{L^\infty}\left\|\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right\|_{L^2}\\
&= \frac{\mu_1^{q/2}(t)}{\mu(t)}\|u(t)\|_{L^2}\|\psi_\sigma\|_{L^\infty}\|\phi_\delta\|_{L^2}\\
&\lesssim_{\sigma,\delta} \frac{1}{t^{(2-2b-bq)/2}}\frac{1}{\log^{(4+q)/2}(t)}\|u_0\|_{L^2}.
\end{aligned} \tag{3.19}$$

Since b satisfies the condition (3.6) we have that

$$\sup_{t\gg1}|\mathcal{I}(t)| < \infty.$$

□

In the next lemma, we will obtain a virial estimate to the solution of IVP BO equation involving the derivative in time of the functional introduced above and a term that belongs to $L^1(t \gg 1)$.

Lemma 3.2.7. *For any $t \gg 1$, it holds that*

$$\frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u^2(x,t)\, \psi'_\sigma\left(\frac{x}{\mu_1(t)}\right) \phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \leq 4\frac{d}{dt}\mathcal{I}(t) + h(t), \tag{3.20}$$

where $h \in L^1(t \gg 1)$.

Proof. Since

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) &= \frac{1}{\mu(t)} \int_{\mathbb{R}} \partial_t \left(u\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)\right) dx \\ &\quad - \frac{\mu'(t)}{\mu^2(t)} \int_{\mathbb{R}} u\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &= A(t) + B(t). \end{aligned} \tag{3.21}$$

Cauchy-Schwarz inequality and the conservation of mass, I_2 in (3.8), yield

$$\begin{aligned} |B(t)| &\leq \left|\frac{\mu'(t)}{\mu^2(t)}\right| \|u(t)\|_{L^2} \left\|\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right\|_{L^\infty} \left\|\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right\|_{L^2} \\ &\lesssim_{\sigma,\delta} \frac{1}{t^{(4-2b-bq)/2}} \frac{1}{\log^{(4+q)/2} t} \|u_0\|_{L^2}. \end{aligned} \tag{3.22}$$

Hence $B(t) \in L^1(\{t \gg 1\})$ whenever $b \leq \frac{2}{2+q}$.

To estimate A we first differentiate in time to write

$$\begin{aligned} A(t) &= \frac{1}{\mu(t)} \int_{\mathbb{R}} u_t(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &\quad - \frac{\mu_1'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u(x,t)\left(\frac{x}{\mu_1(t)}\right)\psi'_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &\quad - \frac{q\mu_1'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\left(\frac{x}{\mu_1^q(t)}\right)\phi'_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &= A_1(t) + A_2(t) + A_3(t). \end{aligned} \tag{3.23}$$

Using the equation in (BO) and integrating by parts yield

$$\begin{aligned} A_1(t) &= \frac{1}{\mu(t)} \int_{\mathbb{R}} \mathcal{H}u(x,t)\, \partial_x^2\left(\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)\right) dx \\ &\quad + \frac{1}{2\mu(t)\mu_1(t)} \int_{\mathbb{R}} u^2(x,t)\psi'_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &\quad + \frac{1}{2\mu(t)\mu_1^q(t)} \int_{\mathbb{R}} u^2(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi'_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &=: A_{1,1}(t) + A_{1,2}(t) + A_{1,3}(t). \end{aligned} \tag{3.24}$$

We remark that $A_{1,2}(t)$ is the term we want to estimate in (3.24). Then we need to show that the reminder terms are in $L^1(\{t \gg 1\})$.

Differentiating with respect to x, using the Cauchy-Schwarz inequality, Hilbert's transform properties, the conservation of mass, and the definition of $\mu(t)$ and $\mu_1(t)$ we deduce that

$$\begin{aligned}|A_{1,1}(t)| &\le \frac{1}{\mu(t)\mu_1^{3/2}(t)}\|u(t)\|_{L^2}\|\psi_\sigma''\|_{L^2}\|\phi_\delta\|_{L^\infty}\\ &\quad+\frac{1}{\mu(t)\mu_1^{(1+2q)/2}(t)}\|u(t)\|_{L^2}\|\psi_\sigma'\|_{L^2}\|\phi_\delta'\|_{L^\infty}\\ &\quad+\frac{1}{\mu(t)\mu_1^{3q/2}(t)}\|u(t)\|_{L^2}\|\psi_\sigma\|_{L^\infty}\|\phi_\delta''\|_{L^2}\\ &\lesssim_{\sigma,\delta}\frac{\|u_0\|_{L^2}}{t^{(2+b)/2}\log^{1/2}t}+\frac{\|u_0\|_{L^2}}{t^{(2-b+2bq)/2}\log^{(\frac{3}{2}-q)}t}\\ &\quad+\frac{\|u_0\|_{L^2}}{t^{(2-2b+3bq)/2}\log^{(4-3q)/2}t}.\end{aligned}\tag{3.25}$$

Since $q > 1$ and $b > 0$ it follows that $A_{1,1} \in L^1(\{t \gg 1\})$.

The term $A_{1,3}$ can be bounded by employing the conservation of mass, and the definition of $\mu(t)$ and $\mu_1(t)$.

$$\begin{aligned}|A_{1,3}| &\le \frac{\|u(t)\|_{L^2}^2}{2|\mu(t)\mu_1^q(t)|}\left\|\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\right\|_{L^\infty}\left\|\phi_\delta'\left(\frac{x}{\mu_1^q(t)}\right)\right\|_{L^\infty}\\ &\lesssim_{\sigma,\delta}\frac{\|u_0\|_{L^2}^2}{t^{1-b+bq}\log^{(2-q)}t},\end{aligned}\tag{3.26}$$

Because of $q > 1$ one has that $A_{1,3} \in L^1(\{t \gg 1\})$.

Next we turn our attention to the other terms of (3.23). First, by means of Young's inequality, we have for $\epsilon > 0$,

$$\begin{aligned}|A_2(t)| &\le \left|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\right|\int_{\mathbb{R}}\left|\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\right|\left[\frac{u^2}{4\epsilon}+4\epsilon\left|\frac{x}{\mu_1(t)}\right|^2\right]dx\\ &\le \frac{1}{4\epsilon}\left|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\right|\int_{\mathbb{R}}u^2(x,t)\psi_\sigma'\Big(\frac{x}{\mu_1(t)}\Big)\phi_\delta\Big(\frac{x}{\mu_1^q(t)}\Big)\,dx\\ &\quad+4\epsilon\left|\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\right|\,\|\phi_\delta\Big(\frac{\cdot}{\mu_1^q(t)}\Big)\|_{L^\infty}\|\Big(\frac{\cdot}{\mu_1(t)}\Big)^2\psi_\sigma'\Big(\frac{\cdot}{\mu_1(t)}\Big)\|_{L^1}.\end{aligned}$$

Then, taking $\epsilon = \mu_1'(t)$, which is positive for $t > 1$, we get

$$\begin{aligned}|A_2(t)| &\le \left|\frac{1}{4\mu_1(t)\mu(t)}\right|\int_{\mathbb{R}}u^2\psi_\sigma'\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx\\ &\quad+C_{\delta,\sigma}\frac{(b\log t-1)^2}{t^{3-3b}\log^6 t}\\ &=\frac{1}{2}A_{1,2}(t)+C_{\delta,\sigma}\frac{(b\log t-1)^2}{t^{3-3b}\log^6 t},\end{aligned}\tag{3.27}$$

where $C_{\sigma,\delta}$ is a constant depending on σ and δ.

Notice that the last term in the last inequality of (3.27) is integrable for $t \gg 1$ since $b < \frac{2}{3}$.

Finally, we consider the term A_3. Young's inequality and the conservation of the L^2-mass implies

$$\begin{aligned} |A_3(t)| \leq & \left|\frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\right| \|\psi_\sigma\|_{L^\infty} \int_{\mathbb{R}} t^{1-b} u^2(x,t)\,dx \\ & + \left|\frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\right| \|\psi_\sigma\|_{L^\infty} \int_{\mathbb{R}} \frac{1}{t^{1-b}} \left[\left(\frac{x}{\mu_1^q(t)}\right)\phi_\delta'\left(\frac{x}{\mu_1^q(t)}\right)\right]^2 dx \\ \lesssim_{\sigma,\delta} & \left|\frac{qt^{1-b}\mu_1'(t)}{\mu_1(t)\mu(t)}\right| + \left|\frac{q\mu_1'(t)\mu_1^q(t)}{t^{1-b}\mu_1(t)\mu(t)}\right|. \end{aligned} \tag{3.28}$$

Hence, the conditions on (3.7) imply, for $t \gg 1$,

$$|A_3(t)| \lesssim_{\sigma,\delta} \frac{1}{t\log^2 t} + \frac{1}{t^{3-b(2+q)}\log^{2+q} t}. \tag{3.29}$$

Since $b \leq \frac{2}{2+q}$, $A_3(t) \in L^1(\{t \gg 1\})$.

Gathering the information in (3.21)-(3.29) together we conclude that

$$\frac{1}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} u^2(x,t)\psi_\sigma'\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \leq 4\frac{d}{dt}\mathcal{I}(t) + h(t) \tag{3.30}$$

where $h \in L^1(\{t \gg 1\})$, as desired. □

The next lemma will give us a key bound in our analysis.

Lemma 3.2.8. *Assume that* $u_0 \in L^2(\mathbb{R})$. *Let* $u \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$ *be the solution of the IVP* (BO). *Then, there exists a constant* $C > 0$, *such that*

$$\int_{\{t\gg 1\}} \frac{1}{t\log t} \int_{B_{t^b}} u^2(x,t)\,dxdt \leq C. \tag{3.31}$$

Proof. From the definition, $\mu(t)\mu_1(t) = t\log t$ and a straightforward computation involving the properties of the function ϕ, it follows that

$$\frac{1}{\mu_1(t)\mu(t)} \int_{B_{t^b}} u^2(x,t)\,dx \leq \frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u^2 \psi_\sigma'\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx,$$

for suitable σ and δ, whenever $q > 1$ is chosen sufficiently close to 1 and b slightly smaller if necessary. Lemma 3.2.7 implies that

$$\int_{\{t\gg 1\}} \frac{1}{\mu_1(t)\mu(t)} \int_{B_{t^b}} u^2(x,t)\,dxdt \leq 2\|\mathcal{I}\|_{L^\infty} + \int_{\{t\gg 1\}} |h(t)|\,dt. \tag{3.32}$$

The first term on the right hand side of inequality (3.32) is bounded because of $b \leq \frac{2}{2+q} < \frac{2}{3}$ and the last one is bounded by the proof of Lemma 3.2.7. This completes the proof of the lemma.

□

Now we are ready to prove Theorem 3.2.2.

3.2.1.1 Proof of Theorem 3.2.2

Since the function $\frac{1}{t\log t} \notin L^1(B_r^c(1))$, from the previous lemma, we can ensure that there exists a sequence $(t_n) \to \infty$, such that

$$\lim_{n\to\infty} \int_{B^b_{(t_n)}} u^2(x,t_n)\,dx = 0.$$

Therefore, 0 is an accumulation point and using that $u^2 \geq 0$ we can conclude the result.

To end this section we will give a sketch of the proof of Corollary 3.2.3.

3.2.1.2 Proof of Corollary 3.2.3

The proof of this result follows the same argument as the proof given to prove Theorem 3.2.2. Hence we will present the new details introduced in the proof. We consider the functional

$$\mathcal{I}_\rho(t) = \frac{1}{\mu(t)} \int u(x,t)\,\psi_\sigma\left(\frac{x-\rho(t)}{\mu_1(t)}\right)\phi_\sigma\left(\frac{x-\rho(t)}{\mu_1^q(t)}\right)\,dx$$

where $\rho(t) = \pm t^m$, m as in the statement of the corollary, $\mu(t)$ and $\mu_1(t)$ defined as in (3.5), and ψ_σ and ϕ_δ defined as above.

As in Lemma 3.2.6 we have that

$$\sup_{t\gg 1} |\mathcal{I}_\rho(t)| < \infty.$$

We also obtain a similar inequality as (3.20) in Lemma 3.2.7, i.e.

$$\begin{aligned}\frac{1}{\mu_1(t)\mu(t)} &\int u^2(x,t)\,\psi'_\sigma\left(\frac{x-\rho(t)}{\mu_1(t)}\right)\phi_\sigma\left(\frac{x-\rho(t)}{\mu_1^q(t)}\right)\,dx\\ &\leq 4\frac{d}{dt}\mathcal{I}_\rho(t) + h_\rho(t),\end{aligned}$$

where $h_\rho \in L^1(\{t \gg 1\})$. Besides the terms previously handle in the proof of Lemma 3.2.7. The difference will appears in (3.23), where the two additional terms above will be established

$$-\frac{\rho'(t)}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\psi'_\sigma\left(\frac{x-\rho(t)}{\mu_1(t)}\right)\phi_\delta\left(\frac{x-\rho(t)}{\mu_1^q(t)}\right)\,dx = A_4(t)$$

and

$$-\frac{\rho'(t)}{\mu_1^q(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\psi_\sigma\left(\frac{x-\rho(t)}{\mu_1(t)}\right)\phi'_\delta\left(\frac{x-\rho(t)}{\mu_1^q(t)}\right)\,dx = A_5(t).$$

The Cauchy-Schwarz inequality and the mass conservation yield

$$\begin{aligned}|A_4(t)+A_5(t)| &\leq \left|\frac{\rho'(t)}{\mu_1^{1/2}(t)\mu(t)}\right| \|u_0\|_{L^2}\|\psi'_\sigma\|_{L^2}\|\phi_\delta\|_{L^\infty}\\ &\quad+\left|\frac{\rho'(t)}{\mu_1^{q}(t)\mu(t)}\right| \|u_0\|_{L^2}\|\psi_\sigma\|_{L^\infty}\|\phi'_\delta\|_{L^2}\\ &\lesssim_{\sigma,\delta,m} \frac{1}{t^{(4-2m-b)/2}\log^{3/2}t}+\frac{1}{t^{(4-2b-2m+bq)/2}\log^{(4-q)/2}t}.\end{aligned}$$

We observe that the first term in the r.h.s. above belongs to $L^1(\{t\gg 1\})$ since $m\leq 1-\frac{b}{2}$. Similarly, the last term in the last inequality belongs to $L^1(\{t\gg 1\})$ since $m\leq 1-\frac{b}{2}<1-b\left(\frac{1-q}{2}\right)$.

From this point on the argument of proof to establish Theorem 3.2.2 can be applied to end the proof of Corollary 3.2.3.

3.2.2 Proof of Thm. 3.2.5 (Asymp. Behavior in $H^{\frac{1}{2}}(\mathbb{R})$)

This section contains the proof of Theorem 3.2.5. The argument follows closely what we did in the previous section. Thus, we will give only the main new ingredients in the proof.

Lemma 3.2.9. *Let $u\in C(\mathbb{R}:H^{\frac{1}{2}}(\mathbb{R}))\cap L^\infty(\mathbb{R}:H^{\frac{1}{2}}(\mathbb{R}))$ the solution of the IVP* (BO). *Then, there exists a constant $C>0$ such that*

$$\int_{\{t\gg 1\}}\frac{1}{t\log t}\int_{B_{t^b}(0)}\left|D_x^{1/2}u(x,t)\right|^2\,dxdt\leq C. \tag{3.33}$$

Proof. Consider the functional

$$\mathcal{J}(t):=\frac{1}{\mu(t)}\int_{\mathbb{R}}u^2(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\,dx. \tag{3.34}$$

where $\mu(t)$ and $\mu_1(t)$ were defined in (3.5).

Differentiating (3.34) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(t) &= -\frac{\mu'(t)}{\mu^2(t)}\int_{\mathbb{R}}u^2(x,t)\,\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\,dx\\ &\quad+\frac{2}{\mu(t)}\int_{\mathbb{R}}u(x,t)\partial_t u(x,t)\,\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\,dx\\ &\quad-\frac{\mu_1'(t)}{\mu(t)\mu_1(t)}\int_{\mathbb{R}}u^2(x,t)\,\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)\left(\frac{x}{\mu_1(t)}\right)\,dx\\ &= C_1(t)+C_2(t)+C_3(t).\end{aligned} \tag{3.35}$$

Combining the properties $\mu(t)$ and $\mu_1(t)$, the conservation of the L^2-mass, and using (3.5), it follows that

$$|C_1(t)| + |C_3(t)| \lesssim_\sigma \frac{\|u_0\|_{L^2}^2}{t^{2-b}\log^2 t}. \tag{3.36}$$

Thus, the terms $C_1(t)$, $C_3(t)$ are integrable in $\{t \gg 1\}$.

Regarding $C_2(t)$, we use the equation in (BO) and integrate by parts to write

$$\begin{aligned} C_2(t) = &-\frac{2}{\mu(t)}\int_{\mathbb{R}} \partial_x u \mathcal{H}\partial_x u\, \psi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ &- \frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}} u\mathcal{H}\partial_x u\, \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ &+ \frac{2}{3\mu(t)\mu_1(t)}\int_{\mathbb{R}} u^3\phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ =& C_{2,1}(t) + C_{2,2}(t) + C_{2,3}(t). \end{aligned} \tag{3.37}$$

By Hilbert's transform properties, integrating by parts and Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |C_{2,1}(t)| =& \left| -\frac{1}{\mu(t)}\int_{\mathbb{R}} u\partial_x \left[\mathcal{H}, \psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right] \partial_x u\, dx \right| \\ \leq& \frac{1}{\mu(t)}\|u\|_{L^2} \left\| \partial_x \left[\mathcal{H}, \psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right]\partial_x u \right\|_{L^2}. \end{aligned} \tag{3.38}$$

Lemma 1.4.5 gives us

$$|C_{2,1}(t)| \leq \frac{1}{\mu(t)}\|u\|_{L^2}^2 \left\| \partial_x^2 \psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right\|_{L^\infty} \lesssim_\sigma \frac{1}{t^{1+b}}, \tag{3.39}$$

which belongs to $L^1(\{t \gg 1\})$.

To estimate $C_{2,2}$, we apply Parseval's identity to obtain

$$\begin{aligned} C_{2,2}(t) = & \frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} uD^{1/2}\left[D^{1/2}, \phi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right] u\, dx \\ &- \frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} \left(D^{1/2}u\right)^2 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ =& C_{2,2,1}(t) + C_{2,2,2}. \end{aligned} \tag{3.40}$$

Notice that $C_{2,2,2}$ is the term we want to estimate.

To bound the term $C_{2,2,1}$ we use Cauchy-Schwarz's inequality, the conservation of mass, (1.13) and properties of the Fourier transform to deduce that

$$\begin{aligned}|C_{2,2,2}(t)| &\leq \left|\frac{2}{\mu_1(t)\mu(t)}\right| \|u_0\|_{L^2} \left\|\left(\partial_x\phi_\sigma \widehat{\left(\frac{\cdot}{\mu_1(t)}\right)}\right)\right\|_{L^1} \\ &\lesssim_\sigma \left|\frac{2}{\mu_1^2(t)\mu(t)}\right| \|u_0\|_{L^2} \left\|\widehat{(\partial_x\phi_\sigma)}\right\|_{L^1} \\ &\lesssim_\sigma \frac{1}{t^{1+b}} \in L^1(\{t \gg 1\}).\end{aligned} \tag{3.41}$$

Finally, notice that by Lemma 1.4.6, here we are following the arguments developed in [46],

$$\begin{aligned}&\int_{\mathbb{R}} |u|^3 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ &\leq \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} (|u|\,\zeta_n)^3 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \\ &\leq \sum_{n\in\mathbb{Z}} \|u\,\zeta_n\|_{L^3}^3 \left(\sup_{x\in[n,n+1]} \phi_\sigma\left(\frac{x}{\mu_1(t)}\right)\right) \\ &\lesssim \sum_{n\in\mathbb{Z}} \|u\,\zeta_n\|_{L^2}^2 \|D^{1/2}(u\,\zeta_n)\|_{L^2} \left(\sup_{x\in[n,n+1]} \phi_\sigma\left(\frac{x}{\mu_1(t)}\right)\right).\end{aligned} \tag{3.42}$$

Moreover, by Lemma 1.4.3 and hypothesis,

$$\begin{aligned}\|D_x^{1/2}(u(t)\zeta_n)\|_{L^2} &\lesssim \left\|D^{1/2}u(t)\right\|_{L^2} \|\zeta_n\|_{L^\infty} + \|u(t)\|_{L^2}\|D^{1/2}\zeta_n\|_{L^\infty} \\ &\lesssim \|u(t)\|_{H^{1/2}(\mathbb{R})} \lesssim \|u\|_{L_t^\infty H^{1/2}}.\end{aligned} \tag{3.43}$$

Combining these estimates we deduce that

$$\int_{\mathbb{R}} |u(x,t)|^3 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \lesssim \sum_{n\in\mathbb{Z}} \|u\,\zeta_n\|_{L^2}^2 \left(\sup_{x\in[n,n+1]} \phi_\sigma\left(\frac{x}{\mu_1(t)}\right)\right).$$

A similar analysis to that given in Lemma 4.1 in [73] (see also [46]) yields

$$\int_{\mathbb{R}} |u(x,t)|^3 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \lesssim \int_{\mathbb{R}} |u(x,t)|^2 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx. \tag{3.44}$$

Using the properties of the function ϕ in (3.2) for suitable δ and σ we can apply Lemma 3.2.8 to deduce that $C_{2,3} \in L^1(\{t \gg 1\})$.

Collection the information in (3.35), (3.36), (3.39), (3.41) and (3.44) we deduce that

$$\frac{1}{t\log t} \int_{B_{t^b}} |D_x^{1/2} u(x,t)|^2 \, dxdt \leq -\frac{d}{dt}\mathcal{J}(t) + g(t),$$

where $\mathcal{J}$ is bounded and $g \in L^1(\{t \gg 1\})$.

A similar analysis as the one implemented in the proof of Theorem 3.2.2 yields the desired result.

□

3.3 Benjamin equation

Now, we are interested in the investigation of the asymptotic behavior of the solutions of the IVP (Bn). Specially, how the interaction of the difference of differential operators, $i.e$, the sum of the terms $\partial_x^3 u$ and $-\mathcal{H}\partial_x^2 u$, will describe the region of the space that the Sobolev norms would be concentrated and the evolution of this information as time evolves. Real-valued solutions of the IVP (Bn) (smooth enough) formally satisfy the following conserved quantities (time invariant):

$$\begin{aligned} I_1(u) &= M(u) = \int_{\mathbb{R}} u^2\,dx, \\ I_2(u) &= E(u) = \int_{\mathbb{R}} \left(\frac{1}{2}(-(\partial_x^2 u)^2 - |D^{1/2}u|^2) - \frac{1}{3}u^3\right)dx, \end{aligned} \tag{3.45}$$

where $\widehat{D^s f}(\xi) = |\xi|^s\,\widehat{f}(\xi)$.

As we saw in the Introduction, we have an extensive theory of well-posedness for the solutions of the IVP (Bn). Then the following result allows us to work with the evolution in time of the solutions of the Bn equation.

Firstly, consider the IVP

$$\begin{cases} \partial_t u + \nu\mathcal{H}\partial_x^2 u + \mu\partial_x^3 u + u\partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{3.46}$$

where $u = u(x,t)$ is a real-valued function and $\nu, \mu \in \mathbb{R}$ and $\nu, \mu \neq 0$. Then, due to Chen, Guo and Xiao in [19],

Theorem 3.3.1. *The IVP* (3.46) *is globally well-posed for* $u_0 \in H^s(\mathbb{R})$ *with* $s \in [-\frac{3}{4}, \infty)$.

Now, using the conservation of the mass in (3.45) and the theorem 3.3.1, we have all the conditions to work with the asymptotic behavior for global solutions of (Bn) equation. Assuming these results, we can enunciate the main results of this section.

Theorem 3.3.2. *Suppose* $v_0 \in L^2(\mathbb{R})$ *and let* $v = v(t,x)$ *be a bounded in time solution to* (Bn) *equation such that* $v \in C(\mathbb{R} : L^2(\mathbb{R}))$. *Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}} v^2(x,t)dx = 0. \tag{3.47}$$

Moreover, there exist constant $C > 0$ *and an increasing sequence of times* $t_n \to \infty$ *such that*

$$\int_{B_{(t_n^b)}} v^2(x,t)dx \leq \frac{C}{\log^{(1-b)/b}(t_n)}. \tag{3.48}$$

Theorem 3.3.3. *Suppose* $v_0 \in H^1(\mathbb{R})$ *and let* $v = v(t,x)$ *be the bounded in time solution to* (Bn) *equation such that* $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap L^\infty(\mathbb{R} : H^1(\mathbb{R}))$. *Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}} \left(|D_x^{1/2} v(x,t)|^2 + |\partial_x v(x,t)|^2\right)dx = 0. \tag{3.49}$$

Remark 3.3.4. *1. It is worth noticing that Theorems 3.3.2 and 3.3.3 can be extended to the non-centered case if we make some straightforward modifications.*

2. *Theorem 3.3.2 can be extended directly to the IVP 3.46, using the conservation laws, the GWP and the virial estimates for this case.*

3. *If in addition, we suppose $\mu > 0$, in IVP 3.46, we can recover the result in Theorem 3.3.3 to this class of equations.*

The key ingredients to deduce these results are mainly based on the virial techniques established in the Section 3.2. Then, we shall focus on presenting where the differences appears and applying similar arguments to gathering the information and obtain the desired result.

3.3.1 Proof Theorem3.3.2

For v a solution of IVP (Bn) equation, we consider the functional

$$\Upsilon(t) := \frac{1}{\mu(t)} \int_{\mathbb{R}} v(t,x)\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right) dx, \tag{3.50}$$

where $\tilde{x} := x - \rho(t)$ and $q > 1$.

Lemma 3.3.5. *For v a solution of IVP* (Bn) *equation, the functional Υ is well defined and bounded in time.*

Proof. The proof is analogous to the one of the Lemma 3.2.6. □

Lemma 3.3.6. *For any $t \gg 1$, one has the bound*

$$\frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} v^2(x,t)\psi'_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right) dx \leq 4\frac{d\Upsilon}{dt}(t) + \Upsilon_0(t), \tag{3.51}$$

where $\Upsilon_0(t)$ are terms that belong to $L^1(\{t \gg 1\})$.

Proof. We compute

$$\begin{aligned} \frac{d}{dt}\Upsilon(t) &= \frac{1}{\mu(t)} \int_{\mathbb{R}} \partial_t\left(v\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)\right) dx \\ &\quad - \frac{\mu'(t)}{\mu^2(t)} \int_{\mathbb{R}} v\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &=: \tilde{A}(t) + \tilde{B}(t). \end{aligned} \tag{3.52}$$

Now,

$$
\begin{aligned}
\tilde{A}(t) =& \frac{1}{\mu(t)} \int_{\mathbb{R}} v_t(x,t)\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&- \frac{\mu_1'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} v(x,t)\left(\frac{\tilde{x}}{\mu_1(t)}\right)\psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&- \frac{\rho'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} v(x,t)\psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&- \frac{q\mu_1'(t)}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} v(x,t)\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)\phi_\delta'\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&- \frac{\rho'(t)}{\mu_1^q(t)\mu(t)} \int_{\mathbb{R}} v(x,t)\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta'\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
=:& \tilde{A}_1(t) + \tilde{A}_2(t) + \tilde{A}_3(t) + \tilde{A}_4(t) + \tilde{A}_5(t).
\end{aligned}
\tag{3.53}
$$

Using the same accounts that we did for the terms B, A_3, A_4, A_5, we can prove that the terms $\tilde{B}, \tilde{A}_4, \tilde{A}_3, \tilde{A}_5 \in L^1(\{t \gg 1\})$, respectively.

Concerning $\tilde{A}_1$, after integrating by parts, we obtain

$$
\begin{aligned}
\tilde{A}_1(t) =& -\frac{1}{\mu(t)} \int_{\mathbb{R}} \partial_x(-\mathcal{H}\partial_x v + \partial_x^2 v + \frac{v^2}{2})\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx \\
=& -\frac{1}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} \mathcal{H}\partial_x v\psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&- \frac{1}{\mu(t)\mu_1^q(t)} \int_{\mathbb{R}} \mathcal{H}\partial_x v\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta'\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&+ \frac{1}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} \partial_x^2 v\psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&+ \frac{1}{\mu(t)\mu_1^q(t)} \int_{\mathbb{R}} \partial_x^2 v\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta'\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&+ \frac{1}{2\mu(t)\mu_1(t)} \int_{\mathbb{R}} v^2\psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
&+ \frac{1}{2\mu(t)\mu_1^q(t)} \int_{\mathbb{R}} v^2\psi_\sigma\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta'\left(\frac{\tilde{x}}{\mu_1^q(t)}\right)dx \\
=:& \tilde{A}_{1,1}(t) + \tilde{A}_{1,2}(t) + \tilde{A}_{1,3}(t) + \tilde{A}_{1,4}(t) + \tilde{A}_{1,5}(t) + \tilde{A}_{1,6}(t).
\end{aligned}
\tag{3.54}
$$

We handle $\tilde{A}_{1,1}, \tilde{A}_{1,2}$ and $\tilde{A}_{1,6}$ analogously as we did with the terms $A_{1,1}$ and $A_{1,3}$, previously. To control the terms $\tilde{A}_{1,3}$ and $\tilde{A}_{1,4}$, whose are the terms given by the part

"$\partial_x^3 u$" in the IVP (Bn), we integrate by parts and Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
|\tilde{A}_{1,3}(t)+\tilde{A}_{1,4}(t)| \le & \left|\frac{1}{\mu(t)\mu_1^3(t)}\right| \|v\|_{L^2} \left\|\psi_\sigma'''\left(\frac{\tilde{\cdot}}{\mu_1(t)}\right)\right\|_{L^2} \|\phi_\delta\|_{L^\infty} \\
& + \left|\frac{3}{\mu(t)\mu_1^2(t)\mu_1^q(t)}\right| \|v\|_{L^2}\|\psi_\sigma''\|_{L^\infty} \left\|\phi_\delta'\left(\frac{\tilde{\cdot}}{\mu_1^q(t)}\right)\right\|_{L^2} \\
& + \left|\frac{3}{\mu(t)\mu_1(t)\mu_1^{2q}(t)}\right| \|v\|_{L^2}\|\psi_\sigma'\|_{L^\infty} \left\|\phi_\delta''\left(\frac{\tilde{\cdot}}{\mu_1^q(t)}\right)\right\|_{L^2} \\
& + \left|\frac{1}{\mu(t)\mu_1^{3q}(t)}\right| \|v\|_{L^2}\|\psi_\sigma\|_{L^\infty} \left\|\phi_\delta'''\left(\frac{\tilde{\cdot}}{\mu_1^q(t)}\right)\right\|_{L^2} \\
\lesssim & \quad \frac{1}{t^{1+2b}\log^{-1} t} + \frac{1}{t^{(2+2b+bq)/2}\log^{-q/2} t} + \frac{1}{t^{(2+3bq)/2}\log^{(2-3q)/2} t} \\
& + \frac{1}{t^{(2-2b+5bq)/2}\log^{(4-5q)/2} t}, \quad \text{for } t \gg 1.
\end{aligned}
\tag{3.55}
$$

Since $b > 0$ and $q > 1$, the above inequality yield $\tilde{A}_{1,3}, \tilde{A}_{1,4} \in L^1(\{t \gg 1\})$. Now, $2\tilde{A}_{1,5}$ is the term to be estimated after integrating in time.

Finally, in regards to $\tilde{A}_2$, it can be estimated using Young's inequality. Setting $\epsilon = \mu_1'(t) > 0$. So that,

$$
\begin{aligned}
|\tilde{A}_2(t)| \le & \left|\frac{1}{4\mu_1(t)\mu(t)}\right| \int_{\mathbb{R}} v^2 \psi_\sigma'\left(\frac{\tilde{x}}{\mu_1(t)}\right)\phi_\delta\left(\frac{\tilde{x}}{\mu_1^q(t)}\right) dx + C_{\delta,\sigma}\frac{(b\log t - 1)^2}{t^{3-3b}\log^6 t} \\
= & \frac{1}{2}\tilde{A}_{1,5}(t) + \tilde{A}_2^*(t),
\end{aligned}
\tag{3.56}
$$

where $\tilde{A}_{1,2}^* \in L^1(\{t \gg 1\})$ since $b < \frac{2}{3}$.

Collecting all the estimates corresponding to this lemma, we obtain that

$$
\tilde{A}_{1,5}(t) \le 4\frac{d}{dt}\Upsilon(t) + \Upsilon_0(t), \tag{3.57}
$$

where

$$
\begin{aligned}
\frac{1}{4}\Upsilon_0(t) := & -\tilde{B}(t) - \tilde{A}_{1,1}(t) - \tilde{A}_{1,2}(t) - \tilde{A}_{1,3}(t) - \tilde{A}_{1,4}(t) \\
& - \tilde{A}_{1,6}(t) - \tilde{A}_3(t) - \tilde{A}_4(t) - \tilde{A}_5(t) + \tilde{A}_2^*(t),
\end{aligned}
\tag{3.58}
$$

belongs to $L^1(\{t \gg 1\})$, which concludes the proof. □

Lemma 3.3.7. *Assume that $v_0 \in L^2(\mathbb{R})$. Let $v \in C(\mathbb{R} : L^2(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$ be the solution of the IVP* (Bn). *Then, there exists a constant $\tilde{C} > 0$, such that*

$$
\int_{\{t \gg 1\}} \frac{1}{t\log t}\int_{B_{t^b}} u^2(x,t)\,dxdt \le \tilde{C}. \tag{3.59}
$$

Proof. The proof follows the same ideas of the proof Lemma 3.2.8. □

Finally, using the same argument to prove Theorem 3.2.2, we can deduce Theorem 3.3.2.

3.3.2 Asymptotic Behavior H^1 of Benjamin equation

Our aim here is to study how the concentration of these Sobolev norms will evolve with time. Concerning Kato's smoothing effect, which we saw in the Chapter 2, we can realize that we have the local gain of the derivatives $D_x^{1/2}$ and ∂_x given by the dispersion terms $\mathcal{H}\partial_x^2$ and ∂_x^3, respectively, presented in the linear part. Then to study the concentration of these quantities in these special regions of the real line allow us to understand the decay properties of the solution of the IVP (Bn) equation. Then, following the ideas in Section 3.2.2, we prove Theorem 3.3.3.

Lemma 3.3.8. *Let* $v \in (C(\mathbb{R} : H^1(\mathbb{R}))) \cap (L^\infty(\mathbb{R} : H^1(\mathbb{R})))$ *the corresponding solution of (Bn) with initial data* $v(0,x) = v_0(x) \in H^1(\mathbb{R})$*. Then, there exists a constant* $C_2 > 0$ *such that*

$$\int_{\{t\gg 1\}} \frac{1}{t\log t} \int_{B_{t^b}} \left(|D_x^{1/2} v(x,t)|^2 + |\partial_x v(x,t)|^2\right) dxdt \leq C_2 \tag{3.60}$$

Proof. Consider the functional

$$\tilde{\Upsilon}(t) := \frac{1}{\mu(t)} \int_{\mathbb{R}} v^2(t,x)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx \tag{3.61}$$

We compute

$$\begin{aligned}\frac{d}{dt}\tilde{\Upsilon}(t) = & \underbrace{-\frac{\mu'(t)}{\mu^2(t)} \int_{\mathbb{R}} v^2 \psi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx}_{\tilde{C}_1(t)} + \underbrace{\frac{2}{\mu(t)} \int_{\mathbb{R}} v v_t \psi_\sigma\left(\frac{x}{\mu_1(t)}\right) dx}_{\tilde{C}_2(t)} \\ & \underbrace{-\frac{\mu_1'(t)}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} v^2 \phi_\sigma\left(\frac{x}{\mu_1(t)}\right)\left(\frac{x}{\mu_1(t)}\right) dx}_{\tilde{C}_3(t)} .\end{aligned} \tag{3.62}$$

First, we deal with the terms $\tilde{C}_1$ and $\tilde{C}_3$ in the same way as we did in Section 3.2.2 with the terms C_1 and C_3, respectively.

Next, we focus our attention into $\tilde{C}_2$, after integrate by parts,

$$\begin{aligned}\tilde{C}_2(t) =& \frac{2}{\mu(t)}\int_{\mathbb{R}} v\partial_x\left(\mathcal{H}\partial_x v - \partial_x^2 v - \frac{v^2}{2}\right)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ =& -\frac{2}{\mu(t)}\int_{\mathbb{R}}\partial_x v\mathcal{H}\partial_x v\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ &-\frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}} v\mathcal{H}\partial_x v\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ &+\frac{2}{\mu(t)}\int_{\mathbb{R}}\partial_x v\partial_x^2 v\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ &+\frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}} v\partial_x^2 v\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ &+\frac{2}{3\mu(t)\mu_1(t)}\int_{\mathbb{R}} v^3\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ =:& \tilde{C}_{2,1}(t)+\tilde{C}_{2,2}(t)+\tilde{C}_{2,3}(t)+\tilde{C}_{2,4}(t)+\tilde{C}_{2,5}(t).\end{aligned} \tag{3.63}$$

We estimate the terms $\tilde{C}_{2,1}, \tilde{C}_{2,2}$, and $\tilde{C}_{2,5}$ in the same way as we did in Lemma 3.2.9 ($C_{2,1}, C_{2,2}$ and $C_{2,3}$). In particular, $\tilde{C}_{2,1}, \tilde{C}_{2,3} \in L^1(\{t \gg 1\})$ and

$$\begin{aligned}\tilde{C}_{2,2}(t) =& \frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} vD^{1/2}\left[D^{1/2}, \phi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right]vdx\\ &-\frac{2}{\mu_1(t)\mu(t)}\int_{\mathbb{R}}\left(D^{1/2}v\right)^2\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ =& \tilde{C}_{2,2,1}(t)+\tilde{C}_{2,2,2}(t),\end{aligned} \tag{3.64}$$

where $\tilde{C}_{2,,2,2}$ is the term that we want to estimate. Using Cauchy-Schwarz's inequality, the conservation of mass, (1.13) and properties of the Fourier transform allow us to deduce that $\tilde{C}_{2,,2,1} \in L^1(\{t \gg 1\})$.

Next, we consider the remaining terms. After integrating by parts,

$$\begin{aligned}\tilde{C}_{2,3}(t) =& \frac{1}{\mu(t)}\int_{\mathbb{R}}\partial_x(\partial_x v)^2\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ =& -\frac{1}{\mu(t)\mu_1(t)}\int_{\mathbb{R}}(\partial_x v)^2\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\end{aligned} \tag{3.65}$$

is the quantity to be estimated after integrating by parts.

Finally, we have

$$\begin{aligned}\tilde{C}_{2,4}(t) =& \frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}} v\partial_x^2 v\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ =& -\frac{2}{\mu(t)\mu_1(t)}\int_{\mathbb{R}}(\partial_x v)^2\phi_\sigma\left(\frac{x}{\mu_1(t)}\right)dx\\ &\frac{1}{\mu(t)\mu_1^3(t)}\int_{\mathbb{R}}(v)^2\phi_\sigma''\left(\frac{x}{\mu_1(t)}\right)dx\\ =& \tilde{C}_{2,4,1}(t)+\tilde{C}_{2,4,2}(t).\end{aligned} \tag{3.66}$$

The $\frac{1}{2}\tilde{C}_{2,4,1}$ is the term that we want to estimate and the Cauchy-Schwarz inequality and the conservation of the mass imply $\tilde{C}_{2,4,2} \in L^1(\{t \gg 1\})$.

After gathering the equalities in this step, we obtain

$$\begin{aligned} \frac{d}{dt}\tilde{\Upsilon}(t) = &- \frac{3}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} (\partial_x v)^2 \phi_\sigma \left(\frac{x}{\mu_1(t)}\right) dx \\ &- \frac{2}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} \left(D^{1/2} v\right)^2 \phi_\sigma \left(\frac{x}{\mu_1(t)}\right) dx \\ &+ \tilde{C}_1(t) + \tilde{C}_{2,1}(t) + \tilde{C}_{2,4,2}(t) + \tilde{C}_{2,5}(t) \\ &+ \tilde{C}_{2,6}(t) + \tilde{C}_3(t). \end{aligned} \tag{3.67}$$

After integration in time, we obtain the desired result.

□

A similar analysis as the one implemented in the proof of Theorem 3.2.2 yields the proof of the Theorem 3.3.3.

3.4 General Dispersive Models

As we observed in the Benjamin equation, these virial identities (or weighted energy estimate) allow us to work with different dispersive operators to obtain decay properties in the energy space. Throughout this section we consider the class of dispersive equations

$$\begin{cases} \partial_t u + L_\alpha u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \end{cases} \tag{3.68}$$

where $u = u(x,t)$ is a real-valued function and $\alpha > 0$. The operator L_α is the Fourier multiplier operator by $i\omega_{\alpha+1}$ and $\omega_{\alpha+1}$ is a real-valued odd function belonging to $(C^1(\mathbb{R}) \cap C^\infty(R^*))$ satisfying:

There exists $\xi_0 > 0$ such that for any $\xi \geq \xi_0$, it holds

$$|\partial^\beta \omega_{\alpha+1}(\xi)| \sim |\xi|^{\alpha+1-\beta}, \quad \beta \in \{0,1,2\} \tag{3.69}$$

and

$$|\partial^\beta \omega_{\alpha+1}(\xi)| \leq |\xi|^{\alpha+1-\beta}, \quad \beta \geq 3. \tag{3.70}$$

We will use the notation $B_0 := \{|x| \leq \xi_0\}$.

Remark 3.4.1. *In particular, following dispersive operators belong to this class*

(1) The Generalized (BO) operator $-D_x^\alpha \partial_x$ *satisfies these conditions for* $1 < \alpha < 2$.

(2) The Intermediate Long Wave operator $\partial_x D_x coth(D_x)$ for $\alpha = 1$.

(3) The Fractionary (KdV) operator $-D_x^{\alpha}\partial_x$ satisfies these conditions for $0 < \alpha < 1$.

It is clear that the study of the asymptotic behavior of solutions for this general class of models is meaningful if (GWP) has been previously established. That is the case for the former two examples: BO and Bn equations. Another important tool used in these two cases was the conservation of the $||\cdot||_{L^2}$ norm. Regarding to IVP (3.68), in general, we do not have the conservation of mass and/or neither (GWP) theory available yet. So, to avoid working in which case we can apply the developed techniques, we prove a more general version of the theorem and, as theorems emerge in the literature that guarantee these properties to a subclass of this model, we shall be capable of applying the theorem to these cases and obtain the result in question.

Finally, we will present some particular cases that we can apply the theorem, showing that the set of dispersive equations satisfy the hypothesis is not a null set. Then with appropriate structure we have the following answer to the local energy decay of the solutions of the IVP (3.68)

Theorem 3.4.2. *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 0$, and $u = u(x,t)$ be the global in time solution of the IVP (3.68), with $\alpha > \frac{1}{2}$, such that $u \in C(\mathbb{R} : H^s(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$. Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(0)} u^2(x,t)\,dx = 0. \tag{3.71}$$

Moreover, there exist a constant $C > 0$ and an increasing sequence of times $t_n \to \infty$ such that

$$\int_{B_{t_n^b}(0)} u^2(x,t_n)\,dx \leq \frac{C}{\log^{\frac{(1-b)}{b}}(t_n)}. \tag{3.72}$$

As a consequence of the proof of this theorem we have:

Corollary 3.4.3. *Let $u_0 \in L^2(\mathbb{R})$ and $u = u(x,t)$ be the global in time solution of the IVP (3.68) such that $u \in C(\mathbb{R} : H^s(\mathbb{R})) \cap L^\infty(\mathbb{R} : L^2(\mathbb{R}))$. Then*

$$\liminf_{t\to\infty} \int_{B_{t^b}(t^m)} u^2(x,t)\,dx = 0, \tag{3.73}$$

with

$$0 < b < \frac{2}{3} \quad \text{and} \quad 0 \leq m < 1 - \frac{3}{2}b. \tag{3.74}$$

Theorem 3.4.2 shows us that we advance one more step in the direction to understand the long time dynamics for dispersive nonlinear models and the relation between the evolution for L^2-norm (mass) of solutions for the IVP and the behavior of the soliton solutions associated to the initial equation, *i. e*, we obtain a result that follows same direction indicated for the soliton resolution conjecture. Before starting the proof of this theorem, we present some examples for which the (GWP) is known and we have the structure aimed to apply the theorem in the IVP (3.68).

Remark 3.4.4. *(1) Considering the IVP associated to the* ***Intermediate Long Wave (ILW) equation***

$$\begin{cases} \partial_t u + \mathcal{T}_\delta \partial_x^2 u + \frac{1}{\delta}\partial_x u + u\partial_x u = 0, & x,t\in\mathbb{R},\\ u(x,0)=u_0(x) \end{cases} \tag{ILW}$$

where $u=u(x,t)\in\mathbb{R}$ is a real-valued function, and

$$\mathcal{T}_\delta f(x) := -\frac{1}{2\delta} p.v. \int_{\mathbb{R}} coth\left(\frac{\pi(x-y)}{2\delta}\right) f(y)dy.$$

The Intermediate Long Wave equation was introduced by Kubota, Ko and Dobbs in [55] to describe the propagation of a long weakly nonlinear internal wave in a stratified medium of finite total depth. The ILW equation reduces formally to BO equation when the depth of the layer tends to infinity. The ILW equation has been proven to be complete integrable see [57] and [71]. In fact the Inverse Scattering formalism was given in [2] and [3]. Real solutions of IVP (ILW) *satisfy many conservation laws as*

$$\mathcal{I}(u) = \int_{\mathbb{R}} u(x,t)\,dx, \tag{3.75}$$

$$\mathcal{M}(u) = \int_{\mathbb{R}} u^2(x,t)\,dx, \tag{3.76}$$

$$\mathcal{E}(u) = \int_{\mathbb{R}} \left\{ u\mathcal{T}_\delta\partial_x u(x,t) + \frac{1}{\delta}u^2(x,t) - \frac{1}{3}u^3(x,t)\right\} dx. \tag{3.77}$$

In particular, (3.76) *is that we need to guarantee that this solution satisfy the hypothesis of Theorem 3.4.2. It is well-known that* (ILW) *possesses soliton (or solitary wave) solutions of the form*

$$u(t,x) = Q_{\delta,c}(x-ct), \quad c > \frac{1}{\delta}, \tag{3.78}$$

where $Q_{\delta,c}$ solves

$$\partial_x\mathcal{T}_\delta Q_{\delta,c} + \left(\frac{1}{\delta} - c\right) Q_{\delta,c} + \frac{1}{2}Q_{\delta,c}^2 = 0,$$

due to Joseph in [41] and it is given by the formula, see [6] for more details,

$$Q_{\delta,c}(x) := \frac{b(c)\sin(a(c)\delta)}{\cos(a(c)\delta)+\cosh(a(c)s)}, \quad a,b \text{ depending on } c. \tag{3.79}$$

The following result, in [1], established the GWP necessary to apply the Theorem 3.4.2 to the IVP associated to (ILW).

Theorem 3.4.5 ([1]). *For any initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, for any $T > 0$ and δ the IVP* (ILW) *has a unique solutions:*

$$u \in C([0,T] : H^s(\mathbb{R})) \cap C^1([0,T] : H^{s-2}(\mathbb{R})) \cap L^\infty([0,T] : H^{s+1/2}_{loc}(\mathbb{R})). \tag{3.80}$$

Moreover, the map data $\longmapsto$ solution is locally continuous from $H^s(\mathbb{R})$ to the class defined in (3.80) *and $L^2(\mathbb{R})$ norm of $u(\cdot,t)$ are uniformly bounded in time.*

For a more in-depth study on local energy decay, propagation of regularity and other properties, we refer [82] and references therein. Finally, using that

$$\mathcal{T}_\delta \partial_x^2 + \frac{\partial_x}{\delta} = \partial_x^2 \Psi_\delta(\partial_x),$$

where the symbol $i\Psi(\xi) = 1/\xi - coth(\delta\xi)$ is $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, odd, real valued and of order zero, we can see that solutions of the IVP associated to (ILW) *satisfy all the requirements of Theorem 3.4.2 .*

*(2) The results in Theorem 3.4.2 also hold for solutions of the **generalized dispersive Benjamin-Ono (gBO) equation***

$$\begin{cases} \partial_t u - D_x^{\alpha+1}\partial_x u + u\partial_x u = 0, \quad x,t \in \mathbb{R},\ 0 < \alpha < 1, \\ u(x,0) = u_0(x). \end{cases} \tag{gBO}$$

where $u = u(x,t) \in \mathbb{R}$ is a real-valued function.

This model is not complete integrable, but it possesses three conserved quantities (see [96]), that is,

$$\mathcal{I}(u) = \int_{\mathbb{R}} u(x,t)\,dx, \tag{3.81}$$

$$\mathcal{M}(u) = \int_{\mathbb{R}} u^2(x,t)\,dx, \tag{3.82}$$

$$\mathcal{E}(u) = \int_{\mathbb{R}} \left(\frac{1}{2}\left|D^{\frac{1+\alpha}{2}}u(x,t)\right|^2 - \frac{1}{6}u^3(x,t)\right)\,dx, \tag{3.83}$$

are satisfied at least for smooth solutions. The equations in (3.68) *model vorticity waves in the coastal zone. Using a renormalization method to control the strong low-high frequency interactions, Herr, Ionescu, Kenig and Koch in [33] show that the IVP* (gBO) *is globally well-posed in the space of the real-valued $L^2(\mathbb{R})$. This GWP and conservation laws allow us to apply Theorem 3.4.2 to this family of equations.*

*(3) The last IVP considered in this thesis is the one associated to the **fractional Korteweg-de Vries (fKdV) equation**, i.e.,*

$$\begin{cases} \partial_t u - D_x^{\alpha}\partial_x u + u\partial_x u = 0, \quad x,t \in \mathbb{R},\ \frac{1}{2} < \alpha < 1, \\ u(x,0) = u_0(x) \end{cases} \tag{fKdV}$$

where $u = u(x,t) \in \mathbb{R}$ is a real-valued function. This model can be seen as a "dispersive" interpolation between Burger equation ($\alpha = 0$) and the Benjamin-Ono equation ($\alpha = 1$). The case $\alpha = \frac{1}{2}$ is somewhat reminiscent of the linear dispersion of finite depth water waves with surface tension. For this dispersive family the quantities (3.81) *and* (3.83) *are conserved by the flow associated to* (fKdV). *For more details we refer [63], [64] and references therein.*

By using standard compactness methods, one can prove that the Cauchy problem associated to (fKdV) *is well-posed in $H^(\mathbb{R})$ for $s > \frac{3}{2}$. Linares, Pilod and Saut showed that the (IVP) possesses a global weak solution in $L^\infty([0,\infty); L^2(\mathbb{R})) \cap L^2_{loc}(\mathbb{R}; H^{\frac{\alpha}{2}}(\mathbb{R}))$ and Molinet, Pilod and Vento in [80] obtain the global well posedness for the solution of the (IVP) associated to* (fKdV) *in the energy space $H^{\alpha/2}(\mathbb{R})$, whenever $\alpha > \frac{6}{7}$. Then under the conditions of the well-posedness theory in [80], we have the conservated law and the (GWP) necessary to apply the Theorem 3.4.2 to this dispersive family.*

Now, proceding in the proof of the Theorem 3.4.2. Using the same virial estimate as we did to the cases before, we can obtain the result

Lemma 3.4.6. *For any $t \gg 1$, it holds that*

$$\frac{1}{\mu_1(t)\mu(t)} \int_{\mathbb{R}} u^2(x,t)\, \psi'_\sigma\left(\frac{x}{\mu_1(t)}\right) \phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \leq 4\frac{d}{dt}\Im(t) + h(t), \tag{3.84}$$

where $h \in L^1(t \gg 1)$.

Proof. We compute

$$\begin{aligned} \frac{d}{dt}\Im(t) &= \frac{1}{\mu(t)} \int_{\mathbb{R}} \partial_t \left(u\psi_\sigma\left(\frac{x}{\mu_1(t)}\right) \phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)\right) dx \\ &\quad - \frac{\mu'(t)}{\mu^2(t)} \int_{\mathbb{R}} u\psi_\sigma\left(\frac{x}{\mu_1(t)}\right) \phi_\delta\left(\frac{x}{\mu_1^q(t)}\right) dx \\ &=: \bar{A}(t) + \bar{B}(t). \end{aligned} \tag{3.85}$$

By Cauchy-Schwarz inequality and $u \in L^\infty(\mathbb{R}; L^2(\mathbb{R}))$, yield

$$\begin{aligned} |\bar{B}(t)| &\leq \left|\frac{\mu'(t)}{\mu^2(t)}\right| \|u(t)\|_{L^2} \left\|\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right\|_{L^\infty} \left\|\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right\|_{L^2} \\ &\lesssim_{\sigma,\delta} \frac{1}{t^{(4-2b-bq)/2}} \frac{1}{\log^{(4+q)/2} t} \|u_0\|_{L^2}, \quad \text{for } t \gg 1. \end{aligned} \tag{3.86}$$

Hence $\bar{B} \in L^1(\{t \gg 1\})$ whenever $b \leq \frac{2}{2+q}$. We remark that this term is bounded in $\{t \gg 1\}$.

Now,

$$\begin{aligned}\bar{A}(t) =& \frac{1}{\mu(t)}\int_{\mathbb{R}} u_t(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx\\ &-\frac{\mu_1'(t)}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\left(\frac{x}{\mu_1(t)}\right)\psi_\sigma'\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx\\ &-\frac{q\mu_1'(t)}{\mu_1(t)\mu(t)}\int_{\mathbb{R}} u(x,t)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\left(\frac{x}{\mu_1^q(t)}\right)\phi_\delta'\left(\frac{x}{\mu_1^q(t)}\right)dx\\ =:&\ \bar{A}_1(t)+\bar{A}_2(t)+\bar{A}_3(t).\end{aligned} \tag{3.87}$$

Next, integrating by parts,

$$\begin{aligned}\bar{A}_1(t) =& -\frac{1}{\mu(t)}\int_{\mathbb{R}}\left(L_\alpha u+\partial_x(\frac{u^2}{2})\right)\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx\\ =& -\frac{1}{\mu(t)}\int_{\mathbb{R}} L_\alpha u\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\lambda_1^q(t)}\right)dx\\ &+\frac{1}{2\mu(t)\mu_1(t)}\int_{\mathbb{R}} u^2\psi_\sigma'\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx\\ &+\frac{1}{2\mu(t)\mu_1^q(t)}\int_{\mathbb{R}} u^2\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta'\left(\frac{x}{\mu_1^q(t)}\right)dx\\ =:&\ \bar{A}_{1,1}(t)+\bar{A}_{1,2}(t)+\bar{A}_{1,3}(t).\end{aligned} \tag{3.88}$$

Proceeding as we did in Lemma 3.2.7, we obtain that $\bar{A}_3 \in L^1(\{t \gg 1\})$, since $b \leq \frac{2}{2+q}$,

$$|\bar{A}_2(t)| \leq \frac{1}{2}\tilde{A}_{1,2}(t) + C_{\delta,\sigma}\frac{(b\log t-1)^2}{t^{3-3b}\log^6 t}, \quad \text{for } t \gg 1. \tag{3.89}$$

Notice that the last term in the r.h.s. of (3.89) is integrable in $\{t \gg 1\}$ since $b < \frac{2}{3}$ and $\bar{A}_{1,2}$ is the term we want to estimate. Since $q > 1$ one has that $\bar{A}_{1,3} \in L^1(\{t \gg 1\})$. Therefore, proving that $\bar{A}_{1,1} \in L^1(\{t \gg 1\})$ we will obtain the required estimate to proof the asymptotic behavior of the solution of this equation.

$$\begin{aligned}\bar{A}_{1,1}(t) =& -\frac{1}{\mu(t)}\int_{\mathbb{R}} L_\alpha u\left(\psi_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)\right)dx\\ =& -\frac{1}{\mu(t)}\int_{\mathbb{R}}\mathcal{F}^{-1}\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)D_x^\beta\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)dx\\ &-\frac{1}{\mu(t)}\int_{\mathbb{R}}\mathcal{F}^{-1}\left(\frac{(1-\zeta_{B_0})\omega_\alpha}{|\xi|^{\alpha+1}}\hat{u}\right)D_x^{\alpha+1}\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)dx\\ =:&\ \bar{A}_{1,1,1}(t)+\bar{A}_{1,1,2}(t),\end{aligned} \tag{3.90}$$

where $\beta > 1$.

Now, applying Cauchy-Schwarz's inequality and Lemma 1.4.3 we obtain

$$
\begin{aligned}
|\bar{A}_{1,1,2}(t)| &\lesssim \frac{1}{|\mu(t)|}\|u\|_{L^2}\left\|D_x^{\alpha+1}\left(\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)\right\|_{L^2}\left\|\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right\|_{L^\infty} \\
&\quad + \frac{1}{|\mu(t)|}\|u\|_{L^2}\left\|\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right\|_{L^\infty}\left\|D_x^{\alpha+1}\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right)\right\|_{L^2} \\
&\lesssim_{\sigma,\delta} \frac{1}{|\mu(t)\mu_1^{\alpha q+q/2}(t)|}\|D^{\alpha+1}\phi_\delta\|_{L^2} \\
&\quad + \frac{1}{|\mu(t)\mu_1^{\alpha+1/2}(t)|}\|D^{\alpha+1}\psi_\sigma\|_{L^2} \\
&\lesssim_{\sigma,\delta} \frac{1}{t^{(2-2b+2\alpha bq+bq)/2}\log^{(4-\alpha q-q)/2}(t)} + \frac{1}{t^{(2-b+2\alpha b)/2}\log^{(3-2\alpha)/2} t} \quad \text{for } t \gg 1.
\end{aligned}
\tag{3.91}
$$

Since $q > 1$ and $\alpha > 1/2$, then $\bar{A}_{1,1,2} \in L^1(\{t \gg 1\})$.

Now, using Hölder's inequality and Lemma 1.4.3

$$
\begin{aligned}
|\bar{A}_{1,1,1}(t)| &\lesssim \frac{1}{|\mu(t)|}\int_{\mathbb{R}}\left|\mathcal{F}^{-1}\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)D_x^\beta\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)\right|dx \\
&\lesssim \frac{1}{|\mu(t)|}\left\|\mathcal{F}^{-1}\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^{p'}}\left\|D_x^\beta\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)\right\|_{L^p} \\
&\lesssim \frac{1}{|\mu(t)|}\left\|\mathcal{F}^{-1}\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^{p'}}\|\psi_\sigma\|_{L^\infty}\left\|D_x^\beta\left(\phi_\delta\left(\frac{\cdot}{\mu_1^q(t)}\right)\right)\right\|_{L^p} \\
&\quad + \frac{1}{|\mu(t)|}\left\|\mathcal{F}^{-1}\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^{p'}}\|\phi_\delta\|_{L^\infty}\left\|D_x^\beta\left(\psi_\sigma\left(\frac{\cdot}{\mu_1(t)}\right)\right)\right\|_{L^p},
\end{aligned}
\tag{3.92}
$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $1<p<2$.

Applying Hausdorff-Young's inequality, we obtain

$$
\begin{aligned}
|\bar{A}_{1,1,1}(t)| &\lesssim \frac{1}{|\mu(t)\mu_1^{q\beta-q/p}|}\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p}\|D^\beta\phi_\delta\|_{L^p} \\
&\quad + \frac{1}{|\mu(t)\mu_1^{\beta-1/p}|}\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p}\|D^\beta\psi_\sigma\|_{L^p} \\
&\lesssim_{\sigma,\delta} \frac{1}{t^{1-b+qb\beta-bq/p}\log^{2-q\beta-q/p} t}\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p} \\
&\quad + \frac{1}{t^{1-b+b\beta-b/p}\log^{2-\beta-1/p} t}\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p}.
\end{aligned}
\tag{3.93}
$$

To deal with $\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p}$, we need to use Hölder's inequality and Plancherel's Theorem to deduce that

$$
\begin{aligned}
\left\|\left(\frac{\zeta_{B_0}\omega_\alpha}{|\xi|^\beta}\hat{u}\right)\right\|_{L^p} &\le \|\hat{u}\|_{L^2(B_0)}\left\|\frac{\omega_\alpha}{|\xi|^\beta}\right\|_{L^\gamma(B_0)} \\
&\le \|u\|_{L^2}\left\|\frac{1}{|\xi|^{\beta-1-\alpha}}\right\|_{L^\gamma(B_0)},
\end{aligned}
\tag{3.94}
$$

where $\gamma = \frac{2p}{2-p}$.

The inequalities (3.93) and (3.94) lead us to

$$1 + \frac{1}{p} < \beta < \frac{2-p}{2p} + \alpha + 1 \tag{3.95}$$

In particular, if we take $\alpha > \frac{1}{2}$, we always can find a pair (β, p) satisfying (3.95), for $1 < p < 2$. Gathering this information and choosing a pair (β, p) under the conditions described above we should obtain $\bar{A}_{1,1,1}$ in $L^1(\{t \gg 1\})$.

We emphasize that the term

$$2\bar{A}_{1,2}(t) = \frac{1}{\mu(t)\mu_1(t)} \int_{\mathbb{R}} u^2(x,t)\psi'_\sigma\left(\frac{x}{\mu_1(t)}\right)\phi_\delta\left(\frac{x}{\mu_1^q(t)}\right)dx$$

is the term to be estimated after integrating in time.

Analogously for $A_{1,3}$ in Lemma 3.2.7, we deduce,

$$|\bar{A}_{1,3}(t)| \lesssim_{\sigma,\delta} \frac{\|u_0\|_{L^2}^2}{t^{1-b+bq}\log^{(2-q)} t}. \tag{3.96}$$

Because of $q > 1$ one has that $\bar{A}_{1,3}(t) \in L^1(\{t \gg 1\})$.

Gathering all the relations above together we obtain

$$2\bar{A}_{1,2}(t) \leq 4\frac{d}{dt}\Im(t) - \Im_0(t),$$

where $\Im_0 \in L^1(\{t \gg 1\})$.

□

Finally, using the same argument to prove Theorem 3.2.2, we can deduce Theorem 3.4.2.

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