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FACTORIZING GERBES AND DILOGARITHMS

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Abstract

In this thesis we study *factorization gerbes* over the Picard scheme of smooth algebraic curves. We classify them in terms of combinatorial data generalizing that of Beilinson and Drinfeld's *theta datum* and produce the first known non-trivial example using Deligne's construction of bimultiplicative characters with values on line bundles. These line bundles, that have the classical dilogarithm functions as natural trivializing sections, appear as transition line bundles of our constructed gerbes. The dilogarithmic pentagonal identity is directly interpreted as the cocycle conditions for the factorization structure.

Key words: gerbes, fatorization, dilogarithm

Resumo

Na presente tese estudamos gerbes de fatorização sobre o esquema de Picard de uma curva algébrica suave. Apresentamos sua classificação em termos combinatórios que generalizam os "theta datum" de Beilinson e Drinfeld e produzimos o primeiro exemplo não trivial utilizando a construção de Deligne de caracteres bimultiplicativos com valores em fibrados de linhas. Estes fibrados, que admitem as funções dilogarítmicas clássicas como seções trivializantes, têm o papel de fibrados de transição dos nossos gerbes. A identidade pentagonal dos dilogaritmos é diretamente interpretada como a condição de cociclo para as estruturas de fatorização.

Palavras chave: gerbes, fatorização, dilogaritmo

Contents

1 Introduction

1.1. This thesis lies in the boundary between representation theory of infinite dimensional Lie algebras, algebraic geometry of algebraic curves and principal bundles over them and higher categorical analogs like gerbes or more general stacks. The common subject that intersects these topics is that of *factorization structures*. Factorization algebras were originally defined by Beilinson and Drinfeld in [4] and have since permeated the literature around the geometric Langlands program, especially in the works of Dennis Gaitsgory, Edward Frenkel, Jacob Lurie and their collaborators [1, 5, 13–16]. In this introduction we will describe, in a relatively informal setting, the notion of factorization in §§1.2–1.13. We then mention the connection to Deligne's work on gerbes on curves in §§1.14–1.15. We briefly describe the main result of this thesis in §1.16, and finally to Aldi and Heluani's work on quantization of the sigma model in §§1.17–1.20.

We defer a formal discussion on the topic of factorization until §3, in the meantime, it is instructive to have an informal definition of what a *factorization structure* is. We choose a category $\mathcal{C}$ that has some notion of a product. It could be topological spaces, differentiable manifolds, smooth schemes, etc, with the cartesian product. It could be vector spaces, group representations, etc, with the tensor product. Or it could even be *categories* with the product of categories. Let us think for the sake of argument, that objects of $\mathcal{C}$ are *spaces*.

Roughly speaking, a *factorization space* $\mathcal{Y}$ over a topological space X consists of a family of objects $Y_x \in \mathcal{C}$ for each point $x \in X$, objects $Y_{x,y}$ for each pair of points $x, y \in X^2$, objects $Y_{x,y,z} \in \mathcal{C}$ for each triple $x, y, z \in X^3$ and so on. These data are equipped with natural isomorphisms, for pairs of points these are:

$$Y_{x,x} \simeq Y_x, \qquad Y_{x,y} \simeq Y_x \times Y_y, \qquad x \neq y. \tag{1.1}$$

For triples of points (here x, y, z are distinct points of X) we have

$$Y_{x,x,x} \simeq Y_x, \qquad Y_{x,x,y} \simeq Y_{x,y,y} \simeq Y_x \times Y_y, \qquad Y_{x,y,z} \simeq Y_x \times Y_y \times Y_z. \tag{1.2}$$

And so forth for any finite collection of points in X. Equivalently, for each finite subset $S \subset X$ we have Y_S, and for two disjoint subsets we have

$$Y_{S \amalg T} \simeq Y_S \times Y_T. \tag{1.3}$$

These *spaces* are required to vary continuously with $S \subset X$. The notion of *varying continuously* depends on the category $\mathcal{C}$. For differentiable manifolds for example we would require that the spaces Y_x assemble into a smooth manifold $Y^{(1)}$ together with a smooth map $Y^{(1)} \to X$, its fiber over a point $x \in X$ being Y_x. For $\mathcal{C}$ being vector spaces, we will require that the vector spaces Y_x assemble into a vector bundle $Y^{(1)} \to X$, its fiber over $x \in X$ being Y_x.

Similarly, we would require that the families $Y_{x,y}$ arise as fibers of a smooth fibration or a smooth vector bundle $Y^{(2)} \to X^2$, etc. The isomorphisms (1.1) can be written in terms of these global objects as

$$\Delta^* Y^{(2)} \simeq Y^{(1)}, \qquad j^* Y^{(2)} \simeq j^* \Big(Y^{(1)} \times Y^{(1)} \Big), \tag{1.4}$$

where $\Delta : X \to X^2$ is the diagonal embedding, and $j : X^2 \setminus \Delta \hookrightarrow X^2$ is its complement.

1.2. One readily sees that there are no non-trivial solutions to (1.4), that is, there is no family of smooth manifolds $Y_{x,y}$ parametrized by pairs of points, such that restricted to the diagonal Δ they give rise to a smooth family Y_x of manifolds over X, and away from it they give rise to the product of this family with itself. If the dimension of the fibers Y_x were $n > 0$ we would have $\dim Y_{x,y} = 2n > \dim Y_{x,x} = n$, contradicting the upper semicontinuity of the dimension function. Similarly in the category of vector spaces, we would be led to $\dim Y_x = 0$. If Y_x is then a finite set with more than one element, say n, the cardinality of the fibers will lead to the same inequality above. Thus we are lead to $Y^{(n)} = X^n$ being the only solution to (1.4). There is one instance in which $n^2 \leq n$, thus not contradicting upper semicontinuity, and this is the case $n = \infty$, thus one is led immediately to study families of infinite dimensional manifolds or schemes over X. One has the following example, known as the *Beilinson-Drinfeld* Grassmanian [13]. Let X be a smooth complex curve and G a simple, simply connected algebraic group over the complex numbers. Let $\mathcal{P}_0$ denote the trivial principal G-bundle on X. For $S \subset X$ a finite subset, we let $Y_S = (\mathcal{P}, \phi)$ consist of pairs of $\mathcal{P} \to X$ a principal G bundle over X and $\phi : j^*\mathcal{P} \simeq j^*\mathcal{P}_0$ a trivialization away from S, here $j : X \setminus S \hookrightarrow X$ is the open complement to S.

The spaces Y_S satisfy (1.3). Indeed a point in $Y_S \times Y_T$ consists of two principal G-bundles $\mathcal{P}$ and $\mathcal{Q}$ over X, trivialized away from S and T respectively. We can glue these two bundles using the common trivialization away from $S \amalg T$, obtaining thus a point in $Y_{S \amalg T}$. The fact that this map $Y_S \times Y_T \to Y_{S\amalg T}$ is an isomorphism, is a theorem of Beauville and Lazslo [3].

1.3. Let us postpone the discussion of how to do algebraic geometry over these infinite dimensional spaces for the latter sections of this introduction. Given a factorization space Y over X as described in the previous sections, we can consider line bundles, vector bundles or more generally sheaves over them, compatible with the factorization structure. These will amount to a line bundle (respectively a vector bundle, sheaf) $\mathcal{L}_S \to Y_S$ over Y_S for each S. They are required to come with the following compatibility structure. For each disjoint pair $S \amalg T \subset X$ we have $\mathcal{L}_S$ over Y_S, $\mathcal{L}_T$ over Y_T and $\mathcal{L}_{S\amalg T}$ over $Y_{S\amalg T}$. We require that under the isomorphism (1.4) we have an identification

$$\mathcal{L}_S \boxtimes \mathcal{L}_T \simeq \mathcal{L}_{S\amalg T}.$$

Again by simple dimension counting we see that there are no solutions to these equations with vector bundles of rank r unless $r = 1$ or $r = \infty$.

1.4. Even though the spaces Y_S constructed above are infinite dimensional. One can still do algebraic geometry over them. The reason is that they are inductive limits of finite dimensional varieties. Now, looking at the example of the Grassmannian, here is a description of the space Y_x for a given closed point $x \in X$. Let $(\mathcal{P}, \phi)$ be a point in Y_x. Let $\mathscr{O}_x$ be the complete local ring of functions at x and let $D_x = \operatorname{Spec} \mathscr{O}_x$ be its spectrum. Let $\mathcal{K}_x$ be the ring of fractions of $\mathscr{O}_x$ and let $D_x^\times$ be its spectrum. The schemes D_x and $D_x^\times$ are usually called the *formal disk* (respectively *formal punctured disk*) near $x \in X$. The restriction of $(\mathcal{P}, \phi)$ to D_x gives rise to a G-bundle on the disk, which is trivialized on the punctured disk. As any bundle on D_x is trivial, the trivialization ϕ can be viewed as a section of $\mathcal{P}_0$ over $D_x^\times$, that is, an element of $G(\mathcal{K}_x)$. Conversely given any element $\phi \in G(K_x)$, we can glue the trivial bundle on D_x with the trivial G-bundle on $X \setminus x$ over their intersection, $D_x^\times$, obtaining thus a principal G bundle on X, which is trivialized away from x, thus a point in Y_x. We have described a map $G(K_x) \to Y_x$. The fact that this map is surjective is the main theorem of [3].

On the other hand, if $\phi \in G(\mathscr{O}_x) \subset G(\mathcal{K}_x)$, it corresponds to a different trivialization of the trivial bundle on D_x, and thus the corresponding point on Y_x is the same. We see that we have a map

$$Gr_x := G(K_x)/G(\mathscr{O}_x) \xrightarrow{\varphi} Y_x. \tag{1.5}$$

The quotient on the left is called the *affine Grassmanian* of the group G at x. And the map φ turns out to be an isomorphism. A local coordinate t at x is defined as a topological generator of $\mathscr{O}_x$, or equivalently, as an isomorphism $\mathscr{O}_x \simeq \mathbb{C}[[t]]$. This induces an isomorphism $\mathcal{K}_x \simeq \mathbb{C}((t))$. Notice that the space Y_x is therefore isomorphic to $Gr := G((t))/G[[t]]$, which does not depend on x. This isomorphism however is not canonical, as it depends on the chosen coordinate t near x. This space Gr is called the *affine Grassmanian.*

For each $N \in \mathbb{N}$, let Gr_N be the quotient $t^{-N}G[[t]]/G[[t]]$, these are finite dimensional schemes over $\mathbb{C}$. They are badly singular, but we have well defined notions of quasi-coherent sheaves and can do algebraic geometry over them. We have $Gr := \varinjlim_N Gr_N$. So while not really a *scheme*, Gr is just expressed as an *Ind-Scheme*, that is, an inductive limit of schemes.

1.5. The Beilinson-Drinfeld Grassmanian Y of 1.2 come equipped with natural factorization line bundles as described in 1.3. Here is a sketch of their construction. As we have described, the fiber Y_s over a point $x \in X$ is isomorphic to the affine Grassmaniann Gr_x. The Picard group of the affine Grassmanian is easily seen to be isomorphic to $\mathbb{Z}$. Its generator $\mathcal{L}_x$ satisfies the conditions of §1.3. This line bundle is easily described from the point of view of representation theory: $G(\mathcal{K}_x)$ is a Lie group it is the *loop group* of G. Its Lie algebra $\mathfrak{g}(\mathcal{K}_x) \simeq \mathfrak{g}((t))$ is called the *loop algebra* of the finite dimensional Lie algebra $\mathfrak{g}$. This Lie algebra admits a one parameter family of central extensions

$$0 \to \mathbb{C} \cdot K \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0, \tag{1.6}$$

known as the *affine Kac-Moody* lie algebra. The line bundle $\mathcal{L}_x$ is obtained by exponentiating this sequence.

1.6. Defining properly a category of *Ind-Schemes* is out of the scope of this Introduction and we refer the reader to the Appendix B for a short summary of results. For now it suffices to think of an *ind-scheme* as an ind-object in the category of schemes, that is, an inductive limit of a system of schemes. In the above case, the spaces Gr, Gr_x and Y_S are all ind-schemes. Let us heuristically describe the notion of a quasi-coherent sheaf on an ind-scheme is. Let $Y = \varinjlim_n Y_n$ be an ind-scheme, and let X be an arbitrary scheme. If Y were representable as a scheme, a map $f : Y \to X$ would be equivalent to a collection of maps $f_n : Y_n \to X$ making the following diagram commute for every $m > n$.

$$\begin{tikzcd} Y_n \arrow[r, "\iota"] \arrow[dr, "f_n"'] & Y_m \arrow[d, "f_m"] \\ & X \end{tikzcd}$$

Suppose $\mathscr{F}$ is a quasi-coherent sheaf in X. Denote by $\mathscr{F}_n := f_n^* \mathscr{F}$ its restriction to Y_n. We have the natural isomorphisms

$$\mathscr{F}_n \xrightarrow{\sim} \iota^* \mathscr{F}_m. \tag{1.7}$$

In particular, this applies to the identity map $X = Y$. Thus, one can use this as a *definition* of a quasi-coherent sheaf on Y, it is a collection of quasi-coherent sheaves $\mathscr{F}_n \in QCoh(Y_n)$ together with natural isomorphisms (1.7) for each $n < m$ compatible with compositions for $n < m < o$.

In the infinite dimensional setting described above, of configuration spaces of points on a scheme X, we can easily convince ourselves that there aren't many interesting quasi-coherent sheaves. The following variant turns out to be much richer: we will consider the *right adjoint functor* $f^!$ to f_*. These functors are only defined in the derived category of coherent sheaves on schemes. The same argument as above says that if $\mathscr{F} \in DCoh(X)$, we obtain restrictions $\mathscr{F}_n := f_n^! \mathscr{F} \in DCoh(Y_n)$ on each Y_n together with compatibilities $\mathscr{F}_n \simeq \iota^! \mathscr{F}_m$. By adjunction we have natural maps

$$\iota_* \mathscr{F}_n \to \mathscr{F}_m. \tag{1.8}$$

This leads [16] to the notion of an *ind-coherent sheaf* on Y as a collection of sheaves (or complexes of) $\mathscr{F}_n \in DCoh(Y_n)$ together with compatibility homomorphisms (1.8) such that by adjunction they induce isomorphisms $\mathscr{F}_n \simeq \iota^! \mathscr{F}_m$ for $m > n$.

1.7. In the context of factorization, we will be interested in the case where $Y = Ran(X)$ is the collection of subsets $S \subset X$ of a smooth scheme. Although not strictly speaking an ind-scheme, the discussion in the previous section applies verbatim. For each $m > n$ we have many diagonal embeddings $X^n \hookrightarrow X^m$ (one for each surjection

$\{1, \ldots, m\} \twoheadrightarrow \{1, \ldots, n\}$). We let Y be the *direct limit* of this diagram. This space is not algebraic, in fact the configuration space of three points on the complex line $\mathbb{C}$ is not algebraic in any way, as there is no non-trivial formal power series in three variables satisfying $f(x, y, y) = f(x, x, y)$. However, we have the notion of quasi-coherent sheaves and ind-coherent sheaves on $Ran(X)$ as in the previous section. Explicitly we have:

1.8 Definition. An Ind-coherent sheaf on $Ran(X)$ is a collection of complexes of sheaves $\mathscr{F}_n \in DCoh(X^n)$ together with morphisms (1.8) for each $m > n$ and each diagonal embedding $\iota : X^n \hookrightarrow X^m$ inducing isomorphisms $\mathscr{F}_n \simeq \iota^! \mathscr{F}_m$. Similar notions can be defined replacing coherent sheaves by quasi-coherent ones or different categories of sheaves, like sheaves of vector spaces or sheaves of $\mathscr{D}$-modules.

1.9. The above definition is enough to describe the main object of study of this thesis. Let $\mathscr{F}$ be an ind-coherent sheaf on $Y = Ran(X)$ where X is an algebraic scheme. Consider a pair of points $\{x, y\} \in X^2$. We have $\mathscr{F}_2 \in DCoh(X^2)$ and we also have $\mathscr{F}_1 \in DCoh(X)$. Taking fibers in the isomorphism of 1.8 by the diagonal embedding $X \hookrightarrow X^2$ we see that $(\mathscr{F}_2)_{x,x} \simeq (\mathscr{F}_1)_x$, that is, $\mathscr{F}$ satisfies the first isomorphism in (1.1). More generally, taking fibers under arbitrary diagonals we find isomorphisms like the first ones in ((1.2)). So we see that these isomorphisms, obtained as compatibilities between the different sheaves $\mathscr{F}_n$ on X^n by restriction to the diagonals, amount to the collection $\{\mathscr{F}_n\}$ defining an ind-coherent (or quasi-coherent or $\mathscr{D}$-module) on $Ran(X)$.

We proceed now to describe the remaining isomorphisms in (1.1) and (1.2). For this let $\mathscr{F}$ be an ind-coherent sheaf on $Ran(X)$ (or quasi-coherent or $\mathscr{D}$-module). For a surjection $\pi : \{1, \ldots, m\} \twoheadrightarrow \{1, \ldots, n\}$ we obtain a diagonal embedding $\pi : X^n \hookrightarrow X^m$. Explicitly, the image of π consists of those $\{x_i\}_{1 \leq i \leq m}$ such that $x_i = x_j$ if $\pi(i) = \pi(j)$. We define the open subset

$$U^{(\pi)} := \left\{ (x_i)_{1 \leq i \leq m} | x_i \neq x_j \text{ if } \pi(i) \neq \pi(j) \right\} \subset X^m, \tag{1.9}$$

and let $j^{(\pi)} : U^{(\pi)} \hookrightarrow X^m$ be the corresponding open embedding. For each $1 \leq i \leq n$ we let $m_i = |\pi^{-1}(i)|$. Since π is surjective we have $\sum m_i = m$ and the obvious natural isomorphism $\prod X^{m_i} \simeq X^m$. Let $\mathscr{F}_{m_i}$ be the corresponding sheaf on X^{m_i}. Let $\boxtimes_{i=1}^n \mathscr{F}_{m_i} \in DCoh(X^m)$ be the tensor product of the pullbacks of $\mathscr{F}_{m_i}$ under the projections $X^m \to X^{m_i}$ by the above isomorphism. A *factorization structure* on $\mathscr{F}$ is the data of isomorphisms

$$j^{(\pi)*}(\boxtimes \mathscr{F}_{m_i}) \simeq j^{(\pi)^*} \mathscr{F}_m, \tag{1.10}$$

compatible with compositions in the obvious manner.

As examples, consider the surjection $\pi : \{1, 2\} \twoheadrightarrow \{1\}$. We have the corresponding diagonal $\Delta : X \to X^2$ and $U^{(\pi)} = X^2 \setminus \Delta$ is its complement. The isomorphism (1.10) reads

$$(\mathscr{F}_1 \boxtimes \mathscr{F}_1)|_{X^2 \setminus \Delta} \simeq (\mathscr{F}_2)|_{X^2 \setminus \Delta}.$$

Taking fibers at the point $x \neq y$ in $U^{(\pi)} \subset X^2$ we obtain the second isomorphism in (1.1), that is $(\mathscr{F}_2)_{x,y} \simeq (\mathscr{F}_1)_x \otimes (\mathscr{F}_1)_y$.

Similarly, consider the surjection $\pi_{12} : \{1,2,3\} \twoheadrightarrow \{1,2\}$ given by $\pi(1) = \pi(2) = 1$, $\pi(3) = 2$. The corresponding diagonal $\Delta_{12} : X^2 \hookrightarrow X^3$ consists of points (x_1, x_2, x_3) with $x_1 = x_2$. In this case $U^{(\pi)} \subset X^3$ is the complement of this diagonal and taking fibers of the corresponding isomorphism at the point $(x, y, y) \in X^3$ with $x \neq y$ we obtain the second isomorphism in (1.2).

Note however that $U^{(\pi)}$ is not in general the complement of a diagonal embedding. Consider for example the case $\pi : \{1,2,3\} \twoheadrightarrow \{1\}$. In this case the corresponding diagonal $\Delta : X \hookrightarrow X^3$ is the *small diagonal*, and the corresponding open $U^{(\pi)}$ is properly contained in its complement: it consists of points (x, y, z) which are pairwise distinct. Taking fibers on such a point we obtain the third isomorphism in (1.2).

1.10. Given a factorization space Y as described in 1.1 and a functor from the category of spaces to that of vector spaces compatible with the product (that is, sending the product of spaces to the tensor product of vector spaces) we obtain a sheaf on $Ran(X)$ with a factorization structure. This is where the connection with representation theory of infinite dimensional algebras arises: for example, if the factorization space Y is the Beilinson-Drinfeld Grassmaniann described in §1.2 and the functor is the functor of taking global sections, or sheaf cohomology $Y \mapsto H^*(Y, \mathscr{O}_Y)$, one obtains a factorization sheaf $\mathscr{F}$ whose fiber at a point $x \in X$ is naturally identified with the *integrable representation of the affine Kac-Moody Lie algebra* $\hat{\mathfrak{g}}$ at level 0. More generally, if we let $\mathcal{L}$ be the factorization line bundle described in §1.5, we obtain the integrable representations of $\hat{\mathfrak{g}}$ at positive integer level k by considering the cohomology with coefficients in $\mathcal{L}^{\otimes k}$ instead.

1.11. Let $\mathscr{F}$ be a sheaf in $Ran(X)$ (of any of the flavours we have seen above). For each $n \in \mathbb{N}$ we have the pullbacks $\mathscr{F}_n$ to X^n and for each diagonal map $X^n \hookrightarrow X^m$ given by a surjection $\pi : \{1, \dots, m\} \to \{1, \dots, n\}$ we have the corresponding map $\iota_* \mathscr{F}_n \to \mathscr{F}_m$ inducing an isomorphism $\mathscr{F}_n \simeq \iota^! \mathscr{F}_m$. In particular, for a *bijection* π, we obtain an action of the group S_n on each $\mathscr{F}_n$. This means the sheaves $\mathscr{F}_n$ over X^n are S_n-equivariant. That is, there exists a sheaf $\bar{\mathscr{F}}_n$ over $\operatorname{Sym}^n X$ such that $\mathscr{F}_n \simeq p^* \bar{\mathscr{F}}_n$, where $p : X^n \to \operatorname{Sym}^n X$ is the quotient map. The spaces $\operatorname{Sym}^n X$ are the moduli spaces parametrizing degree n divisors on the curve X. We refer the reader to appendix C for a brief explanation of this point. In particular, giving a map $f : Z \to \operatorname{Sym}^n X$ for a test scheme Z is equivalent to giving a $D \subset X \times Z$ which is a relative divisor over X of degree n, that is D is a divisor on $X \times Z$ of degree n, faithfully flat and proper over Z and finite over X (and therefore of degree n) over X. On the other hand we have the sheaf $f^* \bar{\mathscr{F}}_n$ over Z. Summarizing the data of a sheaf $\mathscr{F}$ in $Ran(X)$ has produced the following assignment:

- ◇ For each test scheme Z and each relative divisor $D \subset X \times Z$ we have a sheaf $\mathscr{F}_D$ over Z.

These sheaves are compatible with base change. Namely given a morphism $\pi : Z' \to Z$ and a Cartesian diagram

$$\begin{tikzcd} D' \arrow[d] & \subset & X \times Z' \arrow[d] \\ D & \subset & Z \end{tikzcd} , \tag{1.11}$$

we have a natural isomorphism $\pi^* \mathscr{F}_D \simeq \mathscr{F}_{D'}$, compatible with compositions.

1.12 Definition. A *factorization algebra structure* in $\mathscr{F}$ amounts to the extra data that if $D, D' \subset X \times Z$ are *disjoint*, then we have natural isomorphisms:

$$\mathscr{F}_D \otimes \mathscr{F}_{D'} \simeq \mathscr{F}_{D+D'}.$$

This definition would be seemingly stronger that the one given in 1.9. Indeed for $n \in \mathbb{N}$ we can take for $\mathscr{F}_n := \mathscr{F}_D$ the sheaf associated to the divisor $D \subset X \times X^n$ given as the union of the diagonal divisors $\{x = x_i\}$, where $x \in X$ is a point in the first factor and x_i is a point in the i-th factor of X^n.

1.13. Still in the context of the Beilinson-Drinfeld Grassmanian, the case when the group G is an algebraic torus is of particular interest. In this case let Γ be the character lattice of G. Let X be a smooth algebraic curve. A G bundle on X is equivalent to the datum of a line bundle $\mathcal{L}_\gamma$ for each $\gamma \in \Gamma$, together with isomorphisms

$$\mathcal{L}_\gamma \otimes \mathcal{L}_{\gamma'} \simeq \mathcal{L}_{\gamma+\gamma'}. \tag{1.12a}$$

In this case the notion of a *factorizing line bundle* as described in §1.3 is very nicely described in terms of the construction of the previous section 1.11: it consists of the datum of, for each test scheme Z, relative divisor $D \subset X \times Z$ and element $\gamma \in \Gamma$, of a line bundle $\mathcal{L}_{\gamma,D}$ on Z. These line bundles are equipped with isomorphisms

$$\mathcal{L}_{\gamma,D} \otimes \mathcal{L}_{\gamma',D} \simeq \mathcal{L}_{\gamma+\gamma',D}, \qquad \mathcal{L}_{\gamma,D} \otimes \mathcal{L}_{\gamma',D'} \simeq \mathcal{L}_{\gamma+\gamma',D+D'}, \tag{1.12b}$$

for all $\gamma, \gamma' \in \Gamma$ and all disjoint D, D'.

As an example, we can consider the case $Z = X$, $\Gamma = \mathbb{Z}$ and $\gamma = 1$. We have a natural divisor $\Delta \subset X \times X$ and associated to this divisor we have the line bundle $\mathcal{L}$ on X. For arbitrary integer γ we would obtain $\mathcal{L}^\gamma$.

Consider now the case $Z' = X^2$ and the map $\pi_1 : Z' = X^2 \to X = Z$ being the first projection. We have a Cartesian diagram as in 1.11 where $D' = \Delta_1 = \{x = x_1\} \subset X \times X^2$ and $D = \Delta \subset X \times X$. Thus, it follows that $\mathcal{L}_{\gamma,\Delta_1} = \pi_1^* \mathcal{L}^\gamma$. Similarly using the second projection we have $\mathcal{L}_{\gamma',\Delta_2} \simeq \pi_2^* \mathcal{L}^{\gamma'}$ where $\Delta_2 = \{x = x_2\} \subset X \times X^2$ is the other diagonal divisor.

Notice however that Δ_1 and Δ_2 are not disjoint, their intersection being the small diagonal $\Delta \subset X^3$. We can however restrict our line bundles $\mathcal{L}_{\gamma,\Delta_1}$ and $\mathcal{L}_{\gamma',\Delta_2}$ to $U =$

$X^2 \setminus \Delta$. By compatibility with étale morphisms, these restrictions still correspond to the divisor Δ_i restricted to $\subset X \times U$. We have thus two line bundles $\mathcal{L}_{\gamma,\Delta_1}$ and $\mathcal{L}_{\gamma',\Delta_2}$ on U corresponding to two disjoint divisors Δ_1 and Δ_2 on $X \times U$. On the other hand, we have the line bundle $\mathcal{L}_{\gamma+\gamma',\Delta_1+\Delta_2}$ on X^2 which we can restrict to U. We have isomorphisms

$$\pi_1^* \mathcal{L}^\gamma \otimes \pi_2^* \mathcal{L}^{\gamma'}|_U \simeq \mathcal{L}_{\gamma+\gamma',\Delta_1+\Delta_2}|_U.$$

The crucial observation is that two line bundles defined on X^2 that are isomorphic away from the diagonal, they must differ by a sheaf that is supported on the diagonal. Pulling back this isomorphism along the diagonal divisor $\Delta \subset X^2$ we find

$$\mathcal{L}^\gamma \otimes \mathcal{L}^{\gamma'} \simeq \mathcal{L}^{\gamma+\gamma'} \otimes \omega^{\kappa(\gamma,\gamma')}, \tag{1.13}$$

where $\omega = \omega_X$ is the dualizing sheaf and $\kappa : \Gamma \times \Gamma \to \mathbb{Z}$ is a symmetric bilinear pairing. The datum of a line bundle $\mathcal{L}^\gamma$ for each $\gamma \in \Gamma$ and isomorphisms (1.13), compatible with associativity in the obvious manner, is called a *θ-datum.* In section 3 we cover this construction in detail, in particular in section 3.13 we show the equivalence between factorizing line bundles and theta-datums. Notice that a G-bundle gives rise in particular to a θ-datum using $\kappa = 0$.

1.14. A higher categorical analog of equation (1.13) with $\kappa = 0$ has appeared in a striking different context in the work of Deligne [11]. Let X be a Riemman surface, for a pair of holomorphic functions $f, g : X \to \mathbb{C}P^1 \setminus \{0, \infty\}$, Deligne constructs a line bundle $\mathcal{L}_{f,g}$ with a connection over X such that for every three functions $f, g, h : X \to \mathbb{C}P^1 \setminus \{0, \infty\}$ we have:

$$\mathcal{L}_{f,h} \otimes \mathcal{L}_{g,h} \simeq \mathcal{L}_{fg,h} \qquad \mathcal{L}_{h,f} \otimes \mathcal{L}_{h,g} \simeq \mathcal{L}_{h,fg}. \tag{1.14}$$

We see that for a fixed function h, equations (1.14) are equivalent to (1.13) with $\kappa = 0$, written in a multiplicative form. As an example we can consider the curve X being $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$. In this case there is a global coordinate function z. Both z and $1 - z$ are well defined on X as functions to $\mathbb{C}P^1 \setminus \{0, \infty\}$ and thus we have the corresponding line bundle $\mathcal{L}_{z,1-z}$. This line bundle is trivialized by the *dilogarithm* function $Li_2(z)$. More generally, given any function f we have the line bundle $\mathcal{L}_{f,1-f}$ which has the dilogarithm as trivializing section. The bimultiplicative structure (1.14) is compatible with the *pentagonal identity* of the dilogarithm. We explain these subjects in detail in section 5.

One can extend the above constructions to the more general case of Γ-valued functions for a lattice Γ, that is, one considers pairs $\gamma \otimes f$ where $\gamma \in \Gamma$ and $f : X \to \mathbb{C}P^1 \setminus \{0, \infty\}$. Alternatively, such a pair can be thought of as the function $\bar{f} : X \to \mathbb{C}P^1 \setminus \{0, \infty\} \times \Gamma$, where the projection to the first factor is f and to the second factor is the constant function γ. Equation (1.14) holds without modifications.

1.15 Gerbes. Equation (1.14) satisfied by the collection of line bundles $\mathcal{L}_{f,g}$ constructed by Deligne can be considered as a cocycle equation. If one replaces line bundles $\mathcal{L}_{f,g}$ by invertible functions $f_{i,j} \in \Gamma(U_i \cap U_j, \mathscr{O}_X)$ indexed by some open covering $\{U_i\}$ of X, and one replaces isomorphisms in (1.14) by equalities, the collection of functions $f_{i,j}$ defines the gluing data to construct a *line bundle* over X.

Similarly, the collection of line bundles $\mathcal{L}_{f,g}$ defines the gluing data to construct a *gerbe with connection and lien* $\mathbb{G}_m$ on X. More generally, for a lattice Γ the line bundles $\mathcal{L}_{f,g}$ for Γ-valued functions f, g define the gluing data for a gerbe with lien the torus with character group Γ. We describe gerbes and their connections with Picard groupoids in section 2.

1.16 Factorizing Gerbes. The main objective of this thesis is to construct a higher categorical version of that of a *factorizing line bundle* of §1.13. That is we will construct a *factorizing gerbe*. Local sections of line bundles over a space X can be identified with functions on X. Some special line bundles like those constructed by Deligne and described in the previous section, have natural trivializing sections given by special functions on X. In the example of §1.15, where $X = \mathbb{C}P^1 \backslash \{0, 1, \infty\}$, these functions are the classical *dilogarithm functions*. Similarly, local sections of gerbes (with lien $\mathbb{G}_m$) can be identified with line bundles on X. We will construct a collection of gerbes satisfying analogous factorization properties as those in (1.12a)–(1.12b) having Deligne's line bundles as local sections. This example can be found in §5.

There is a deep connection between factorizing line bundles and representation theory of infinite dimensional Lie algebras or more generally vertex algebras as we will briefly describe in the following subsections. Under this connection, local sections of factorizing line bundles arise as *n-point functions* of vertex algebras. In [2] the authors construct the dilogarithm functions as *n-point functions* of certain physical system, it is natural to ask if the corresponding line bundles have a natural factorization structure. Our main example shows that while these line bundles to not form a factorization line bundle in the sense of §1.13, they are local sections of a factorizing gerbe.

1.17 Vertex Algebras. Factorization structures as informally described in the previous sections are geometric objects by nature, that "live" over a space X. When this space is taken to be the affine line $\mathbb{A}^1$, and the factorization structure is compatible with the group of translations $\mathbb{G}_a$, these can be described in terms of linear algebra. The corresponding structure is called a *vertex algebra* and has been studied in the mathematics literature since the 80's work by Borcherds. In section 4 we recall the definition and basic properties of vertex algebras. In this section we simply mention two aspects that bridge the connection to factorizing structures. One of the principal ways of constructing vertex algebras, and the reason they were invented, is that many representations of infinite dimensional Lie algebras have this structure. In fact all integrable representations of the affine Kac-Moody Lie algebra are examples of vertex algebras. We see thus that vertex algebras arise as *fibers of a sheaf on* $Ran(X)$ as described in section 1.10, equivalently, they appear as cohomologies of a factorization space: the Beilinson Drinfeld Grasmannian.

Formally, a vertex algebra is a vector space V with a “multiplication” that takes values in Laurent series:

$$V \otimes V \to V\left((z)\right).$$

This multiplication satisfies axioms analogous to skew-symmetry and the Jacobi identity of Lie algebras. Vertex algebras are “unital” in the sense that there is a unit vector $|0\rangle \in V$ which acts as a left identity for the above multiplication.

As we have seen vertex algebras arise as fibers of natural factorizing sheaves, not surprisingly, vectors in a vertex algebra should give rise to sections of said sheaves. And the “multiplication” in the vertex algebra is translated to the factorization of these sections. Indeed, given n points on the line $z_1, \dots, z_n$ and n vectors in V, $a_1, \dots, a_n$, we can construct the following V-valued function of n-variables:

$$a_1(z_1)\dots a_n(z_n)|0\rangle,$$

where we have taken n products. These functions are called n-point functions and we cover their properties in section 4.12. Typically, vertex algebras are graded and satisfy some natural finiteness dimensions

$$V = \oplus_{n\geq 0} V_n, \qquad V_0 = \mathbb{C}|0\rangle, \quad \dim V_n < \infty.$$

In this case we have a natural functional $\varphi \in V^*$ which consists of the projection to $V_0 \simeq \mathbb{C}$ (the algebra is naturally *augmented*). We obtain thus $\mathbb{C}$-valued n-point functions

$$\varphi\Big(a_1(z_1)\dots a_n(z_n)|0\rangle\Big). \tag{1.15}$$

These functions satisfy

- **a**. For fixed $a_1, \dots, a_n$ they are rational, meromorphic functions of $z_1, \dots, z_n$.
- **b**. They are possibly singular, having poles at $z_i = z_j$, $z_i = 0$.
- **c**. They are symmetric with respect to $a_i \leftrightarrow a_j$, $z_i \leftrightarrow z_j$.
- **d**. The collection of all these functions determine completely the vertex algebra V.

1.18. Lattices Γ together with an even, symmetric, bilinear mapping $\kappa : \Gamma \times \Gamma \to \mathbb{Z}$ provide examples of vertex algebras that are more indirectly related to Lie theory. As a vector space, the vertex algebra V_Γ associated to the pair (Γ, κ) is constructed as follows. One considers the torus $T = \Gamma \otimes \mathbb{R}/\Gamma$. The space of functions $L^2(T)$ is naturally a completion of $\mathbb{C}[\Gamma]$, having the exponential functions e^γ, $\gamma \in \Gamma$ as basis. For the sake of simplicity we abuse notation and identify $L^2(T)$ with $\mathbb{C}[\Gamma]$. The torus T acts on itself by translations, therefore it acts on $L^2(T)$ and so does its complexified Lie algebra $\mathfrak{h} := \operatorname{Lie} T \otimes_{\mathbb{R}} \mathbb{C}$. We consider the Kac-Moody affinization $\hat{\mathfrak{h}}$ of $\mathfrak{h}$. This is an infinite dimensional Heisenberg Lie algebra which is a central extension of the Lie algebra of loops into T, that is, maps $S^1 \to T$. It has a subalgebra $\hat{\mathfrak{h}}_+$, the Lie algebra of the subgroup consisting on regular, or contractible, loops, that is those that can be extended to the whole disk $D \to T$. We identify $\mathfrak{h} \subset \hat{\mathfrak{h}}_+$ as the Lie algebra of the constant loops. It follows that $L^2(T)$ is $\hat{\mathfrak{h}}_+$-representation, and therefore it induces a representation

$$V_\Gamma := \operatorname{Ind}_{\hat{\mathfrak{h}}_+}^{\hat{\mathfrak{h}}} L^2(T).$$

This vector space is naturally Γ-graded:

$$V_\Gamma = \oplus_{\gamma \in \Gamma} V_\gamma,$$

And as a $\hat{\mathfrak{h}}$-module, V_γ is generated by the exponential function e^γ. The relation between the abelian group structure of Γ, the bilinear pairing κ and the vertex algebra multiplication of V_Γ is manifest in the vertex algebra multiplication

$$e^\gamma(z) e^{\gamma'} = z^{\kappa(\gamma,\gamma')} e^{\gamma+\gamma'} + o\left(z^{\kappa(\gamma,\gamma')}\right). \tag{1.16}$$

Compare this equation with the geometric counterpart of factorization line bundles (1.13).

1.19. The connection between the factorizing line bundles of Beilinson and Drinfeld, satisfying (1.13) and those of Deligne satisfying (1.14) comes from the work of Aldi and Heluani [2]. In op. cit. the authors attempt to carry out Beilinson and Drinfeld's construction in the case where Γ is not commutative. In particular, they look at the case where Γ is a non-commutative self extension of the rank three lattice $\Lambda = \mathbb{Z}^3$. That is an exact sequence of groups

$$0 \to \Lambda \to \Gamma \to \Lambda \to 0,$$

where Γ is not commutative (it is two step unipotent) and the Λ factor in the left is central. The authors follow verbatim the description of the previous section 1.18, this time instead of a torus T, the manifold $X = \Gamma \otimes \mathbb{R}/\Gamma$ is not a group, but it is rather a 6-dimensional nilmanifold. It is a non-trivial $T^3 := \Lambda \otimes \mathbb{R}/\Lambda$ over T^3. The nilpotent group $\Gamma \otimes \mathbb{R}$ acts on $L^2(X)$, and so does its complexified Lie algebra $\mathfrak{h}$, which is a two-step nilpotent Lie algebra. The space

$$V_\Gamma := \operatorname{Ind}_{\hat{\mathfrak{h}}_+}^{\hat{\mathfrak{h}}} L^2(X)$$

arises. Describing a good basis for $L^2(X)$ is not as simple as in the torus case. In general it involves harmonic analysis and a good basis is constructed in terms of theta functions and different choices of isomorphisms $T^3 \simeq S^1 \times E$ between the three-dimensional real torus, and the circle times an elliptic curve. However, the functions on the fibers of the fibration $X \to T^3$ admit a natural basis by exponential functions e^λ, $\lambda \in \Lambda \simeq \mathbb{Z}^3$. The striking fact obtained in [2] is that the corresponding 3-point function

$$e^{\lambda_1}(z)e^{\lambda_2}(w)e^{\lambda_3}(t)|0\rangle = \exp\left(\det(\lambda_1, \lambda_2, \lambda_3) Li_2\left(\frac{z-t}{z-w}\right)\right) e^{\lambda_1+\lambda_2+\lambda_3} + \dots$$

Thus, these are not 3-point functions of a vertex algebra, since they fail a) in the list in section 1.17, as the dilogarithm functions are not rational functions but are analytic.

1.20. It follows that for non-commutative Γ, the space V_Γ is not necessarily a vertex algebra. The n-point functions (1.15) are not sections of $\mathscr{O}_X$ but rather sections of a line bundle with a connection $\mathcal{L}$. In the case of [2], these line bundles are precisely Deligne's line bundles $\mathcal{L}_{f,1-f}$ over $\mathbb{C}P^1$ as described in 1.14. As in general these line bundles provide the gluing data to construct gerbes on curves, this begs the following questions:

- **a**. Does there exist a natural *factorization gerbe* whose sections are described by Deligne's line bundles $\mathcal{L}_{f,1-f}$?
- **b**. Does Aldi-Heluani's structure arise as taking global sections/cohomology of a *factorization gerbe* over the Ran space of the affine line?

In this thesis we answer positively the first question, providing an effective definition of a factorization gerbe and giving the example in section 5, where the dilogarithm line bundles are taken as structure constants of a factorization gerbe.

1.21. We end this introduction with the following disclaimer, many of the objects we define and work on in this thesis make sense in many different categorical contexts, the notion of a principal G-bundle makes sense over a topological space, a topological manifold, a differentiable manifold, a holomorphic manifold, an algebraic variety, an algebraic scheme or even an ind-scheme. At times we do exploit techniques from topos theory to treat all the possible categorical applications (see for example 2.21), however, we refrain from introducing the language and the machinery of algebraic topoi and we simply say *bundle* when the context should dictate if we are talking about a principal bundle for a Lie group over a differentiable manifold, or an algebraic group over a scheme. Unless otherwise noted, we are mostly interested in the algebraic situation so that our base spaces will be schemes, an our sheaves will be sheaves in the étale or fppf topology.

2 Gerbes

In this section we recall the basic definitions and properties of *gerbes* and their connections. For a detailed discussion on the topic the reader is referred to [7].

Let G be a group. A *G-torsor* is a set S endowed with a simply transitive action of G. That is an action map[1]

$$G \times S \to S, \qquad g, s \mapsto g \cdot s,$$

satisfying

$$(g \cdot h) \cdot s = g \cdot (h \cdot s), \qquad \forall s \in S,\ g, h \in G,$$

and such that for each $s, t \in S$ there exists a unique $g \in G$ such that $g \cdot s = t$.

Any such set S is in bijection with G, but non-canonically so, that is, for each choice of an element $s \in S$, the map

$$G \to S, \qquad g \mapsto g \cdot s$$

is an isomorphism. The group G itself, with the multiplication, is a G-torsor called the *trivial torsor*. This notion admits a relative version as follows. Let G be a group, a G-torsor, or a G-principal bundle over a topological space, consists of sheaf $\mathscr{F}$ over X, with a simply transitive fiberwise action of the group G, that is locally trivial in the sense that for each $x \in X$, there exists an open neighborhood $x \in U \subset X$ such that $\mathscr{F}|_U$ is the sheaf of sections of $U \times G$ with the obvious fiberwise action of G. G-torsors over X form a category, a morphism $\mathscr{F} \to \mathscr{G}$ between G-torsors is a morphism of sheaves that commutes with the G action. Any such morphism is an isomorphism, so that the category of G-torsors is a *groupoid*.

One example of such groupoid is the groupoid $Pic(X)$ of line bundles over X, which is equivalent to the groupoid of $\mathbb{G}_m$-torsors over X. In this case, the underlying group $G = \mathbb{G}_m$ is commutative, this endows the groupoid of G-torsors with an extra structure. Let $\mathscr{F}$ and $\mathscr{G}$ be two G-torsors over a commutative group G. Then the fiber product $\mathscr{F} \otimes \mathscr{G} := \mathscr{F} \times_G \mathscr{G}$ over the diagonal action of G carries an action of G (either on the first or second factor) making it into a G-torsor. In the case of $Pic(X)$ this structure is identified with the usual tensor product of line bundles. These groupoids, endowed with this extra tensor structure are called *Picard groupoids*.

Roughly speaking, a *gerbe* is a higher categorical analog of the above construction. One starts with a groupoid $\mathcal{P}$, to make things simpler we start with a Picard groupoid $(\mathcal{P}, \otimes)$ like $Pic(X)$. A $\mathcal{P}$-torsor consists of a category $\mathcal{C}$ with a simply transitive action of $\mathcal{P}$, that is for each object $C \in \mathcal{C}$ and $P \in \mathcal{P}$ we have an object $P \otimes C$ of $\mathcal{C}$ and *natural isomorphisms*

$$P \otimes (P' \otimes C) \simeq (P \otimes P') \otimes C, \qquad P, P' \in \mathcal{P},\ C \in \mathcal{C}.$$

These isomorphisms satisfy commuting diagrams for each three objects of $\mathcal{P}$. The category $\mathcal{P}$ itself, with its tensor product, is an example of a $\mathcal{P}$-torsor, called the *trivial torsor*.

[1] We consider *left actions*, similar definitions apply for actions on the right.

A *gerbe* over X consists of a sheaf of categories, locally isomorphic with the trivial sheaf of sections of $\mathcal{P} \times X$, with an action of $\mathcal{P}$. In the case when $\mathcal{P} = Pic(X)$, this notion has appeared with several incarnations, in the study of the Brauer group of X as Azumaya algebras, twisted sheaves, and in the context of differential geometry in the study of Poisson structures on Lie groupoids and related structures.

We start this section describing the classical theory of principal bundles in sections 2.1–2.11, we describe their classification by Čech cohomology classes in 2.12–2.15. We recall their connections in 2.16 and define gerbes in 2.22. We study Picard groupoids in 2.36 and connections on gerbes in 2.49. We end this section with the example of the determinantal gerbe of a certain infinite dimensional bundle in section 2.56.

2.1 Principal bundles

In this subsection we will relate different approximations to the notion of G-torsors or principal bundles.

2.2 Definition. Let X be a complex manifold and G a commutative complex Lie group. A *principal G-bundle* on X is a pair (P, π) where P is a complex manifold and π is a map $\pi\colon P \to X$ of complex manifolds together with an *action map*

$$G \times P \to P, \qquad (g, p) \mapsto g \cdot p,$$

such that

i. the group G acts freely and transitively on the fibers, i.e. $\pi(g.p) = \pi(p)$ and if $\pi(p) = \pi(q)$ there is a unique $g \in G$ such that $p = g.q$,
ii. for each $x \in X$ there is an open neighborhood $x \in U$ and an equivariant isomorphism $\phi_U\colon \pi^{-1}(U) \to G \times U$ commuting with the projections, i.e. $\phi_U(p) = (\psi(p), \pi(p))$ and $\psi(g \cdot p) = g \cdot \psi(p)$.

We denote it (P, π).

2.3 Example. The projection $G \times X \to X$ with the action by left multiplication on the first factor is a principal G-bundle called the *trivial* G-bundle.

2.4. A *morphism of principal G-bundles* (P, π), (P', π') is a holomorphic map ϕ : $P \to P'$ such that:
a. $\pi' \circ \phi = \pi$ and,
b. $g \cdot \phi(p) = \phi(g \cdot p)$ for every $g \in G$ and $p \in P$.

2.5. A *G-set* is a set endowed with a freely transitive action of the group G. Denote

$$Hom_G(A, B) := \{f : A \to B | f(g \cdot a) = g \cdot f(a) \text{ for all } a \in A, g \in G\}.$$

Since G is commutative,

$$End_G(A) \simeq Aut_G(A) \simeq G,$$

$$Hom_G(A,B) \simeq Iso_G(A,B)$$

where Iso, End, Aut denote invertible morphisms, endomorphisms and invertible endomorphisms, as usual.

2.6 Example. The group G is a G-set with the left multiplication action. We call this the *trivial* G-set.

2.7. In the category of sheaves of sets over a topological space X, there is the analogous notion. Namely, given a sheaf of commutative groups $\mathcal{G}$, a *$\mathcal{G}$-sheaf* is a sheaf of sets $\mathcal{F}$, such that $\mathcal{F}(U)$ is a $\mathcal{G}(U)$-set for all open sets $U \subseteq X$ such that if $V \subseteq U$, $g \in \mathcal{G}(U)$, $a \in \mathcal{F}(U)$, then $(g \cdot u)\big|_V = g\big|_V \cdot u\big|_V$. A *morphism of $\mathcal{G}$-sheaves* is an equivariant morphism of sheaves.

2.8 Example. Given a commutative group G, the sheaf defined by $G_X(U) = \{f : U \to G\}$ is a sheaf of groups. As in the case of sets, G_X is a G_X-sheaf and we call it *trivial*.

2.9. Let (P, π) be a principal G-bundle and a covering $\{U_i\}_{i\in I}$ with isomorphisms $\phi_i \colon \pi^{-1}(U_i) \to G \times U_i$. Let $e \in G$ be the unit. Composing the section $x \mapsto (e, x) \in G \times X$ with ϕ_U^{-1}, we obtain a *trivializing* section $s_U : U \to \pi^{-1}(U)$. Trivializing in the sense that given a section $t : V \to \pi^{-1}(U)$, $V \subseteq U$ there exists a unique $f : V \to G$ such that $t(v) = f(v) \cdot s_U(v)$, for each $v \in V$, i.e. it gives a local isomorphism $P \to G \times X$ between P and the trivial G-bundle.

2.10 Definition. Let $\mathcal{G}$ be a sheaf of groups, a *$\mathcal{G}$-torsor* on X is a locally trivial $\mathcal{G}$-sheaf, i.e. a $\mathcal{G}$-sheaf $\mathcal{F}$ verifying that for every $U \subset X$ there is a covering $\{U_\alpha\}$ of U such that $\mathcal{F}(U_\alpha) \neq \emptyset$.

2.11. It follows from 2.9, that given a G-bundle, its sheaf of sections is a G_X-torsor and this map is clearly functorial.

2.12. Let $\mathcal{G}$ be a sheaf on X and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering. Recall the Čech complex defined as follows. We set

$$\check{C}^p(\mathcal{U}, \mathcal{G}) = \prod_{i_0, i_1, \ldots, i_p} \mathcal{G}(U_{i_0} \cap \ldots \cap U_{i_p})$$

and define the differential by letting for $\alpha \in \check{C}^p(\mathcal{U}, \mathcal{G})$,

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}\Big|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}},$$

where $\widehat{i_j}$ denotes that the index i_j is missing. We denote by $\check{H}(\mathcal{U}, \mathcal{G})$ its cohomology.
The *Čech cohomology* is defined as the colimit $\check{H}(X, \mathcal{G}) := \varinjlim \check{H}(\mathcal{U}, \mathcal{G})$ taken over the set of open coverings which are partially ordered by refinement.

2.13. Consider a G-torsor $\mathcal{F}$, so there exist isomorphisms $\phi_i \colon \mathcal{F}(U_i) \to G_X(U_i)$ for an open covering $\mathcal{U} := \{U_i\}$. Denote $U_{ij} := U_i \cap U_j$. The morphisms $\phi_{ji} := \phi_j \mid_{U_{ij}} \circ \phi_i^{-1} \mid_{U_{ij}} \colon G_X(U_{ij}) \to G_X(U_{ij})$ are determined by objects $g_{ij} \in G_X(U_{ij})$ i.e. an element of $\check{C}^1(\mathcal{U}, G_X)$. Since $\phi_{ji} \circ \phi_{ik} = \phi_{jk}$, $d(g_{ij}) = 0$, i.e. the collection $\{g_{ij}\}$ is a cocycle. If we choose another trivialization, say $\psi_i \colon \mathcal{F} \mid_{U_i} \to G_X(U_i)$ (we can assume it on the same covering by refining if necessary) given by a cocycle $f_{ij} \in G(U_{ij})$, we have that $\phi_i \circ \psi_i^{-1} : G(U_i) \to G(U_i)$ are determined by $h_i \in G(U_i)$ and $d(h_i)_{ij} = g_{ij} \cdot f_{ij}^{-1}$. We obtain therefore a map from the set of G-torsors to $\check{H}^1(X, G_X)$, that sends $\mathcal{F}$ to $\{g_{ij}\}$.

2.14. Conversely, consider a cocycle $\{g_{ij}\}$ representing the cohomology class $\bar{g}_{ij} \in \check{H}^1(\mathcal{U}, G_X)$. Define P as the quotient of $\amalg_i G \times U_i$ by the relation $(f, x, i) \sim (h, y, j)$ if an only if $x = y \in U_i \cap U_j$ and $f \mid_{U_{ij}} = g_{ij}(x) \cdot h \mid_{U_{ij}}$. P, endowed with its projection map $(g, x, i) \mapsto x$ is a principal G-bundle, the action being by multiplication on the first factor. It is easily seen that the isomorphism class of P does not depend on $\{g_{ij}\}$ but rather on the Čech cohomology class it represents. Summarizing, we have proved the following lemma.

2.15 Lemma. The map that assigns to a principal G bundle P its sheaf of sections, is an equivalence of categories between principal G-bundles and G-torsors on X. The isomorphism classes of either category are classified by the first Čech cohomology group $\check{H}^1(X, G)$.

2.16 Connections

There are several ways of defining a connection on a principal bundle on X. We here give a *crystalline* definition due to Grothendieck that generalizes vastly to other contexts, see for example the appendix for the situation where X is an ind-scheme. Denote by $\Delta \subset X \times X$ the diagonal. If $U \subset X$ is an open subscheme such that Δ is defined by a sheaf of ideals $\mathcal{I}$, recall that the first infinitesimal neighbourhood $\Delta^{(2)}$ is defined locally by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

2.17 Definition. Let $p_1, p_2 : \Delta^{(2)} \to X$ the projections of the first order infinitesimal neighbourhood of the diagonal in $X \times X$ and $\Delta : X \to X \times X$ the diagonal morphism. A *connection* on a G-torsor F is a group isomorphism

$$\alpha : p_1^*(F) \to p_2^*(F)$$

such that $\Delta^*(\alpha) = Id$. An *integrable connection* is a connection such that $p_{13}^*(\alpha) = p_{23}^*(\alpha) \circ p_{12}^*(\alpha)$ where $p_{ij} : X \times X \times X \to X \times X$ are the projections to the i, j-factors.

2.18. Consider a line bundle L on X with connection $\alpha : p_1^*(L) \to p_2^*(L)$. Consider a covering U_i with sections $s_i \in L(U_i)$, $s_i \neq 0$, then there exists $A_i : \Delta^{(2)}|_{U_i} \to \mathbb{C}$ such that

$$\alpha(p_1^*(s_i)) = A_i + p_2^*(s_i),$$

since α restricts to the identity on Δ $A_i \in \omega_X$ and if g_{ij} are such that $s_i = g_{ji}.s_j$ then

$$A_j - A_i = g_{ij}^{-1} dg_{ij}.$$

In this terms, the *curvature* of the connection is the 2-form K such that $dA_i = K$. If the connection is integrable, $dA_i = 0$. We have the following result (see [7], [10]).

2.19. Proposition.

a. The isomorphism classes of $\mathbb{G}_m$-torsors with connection are identified with $\check{H}^1(X, \mathbb{G}_m \overset{dlog}{\longrightarrow} \Omega^1_X)$

b. The isomorphism classes of $\mathbb{G}_m$-torsors with integrable connection are identified with $\check{H}^1(X, \underline{\mathbb{C}}_X)$

2.20. Observe that the definition 2.17 can be made for any sheaf or space over X, and also if $X \to Y$ we can define a connection relative to Y considering $X \times_Y X$ in the previous definition.

2.21 Sites. Here is a different approach to thinking about sheaves and principal G-bundles. Let X be a *space*. It could be in any category like topological spaces, differentiable or holomorphic manifolds, etc. Here we think for simplicity on an algebraic scheme. Let $\mathscr{F}$ be a sheaf of sets (groups, vector spaces, etc) on X. For each $f : Y \to X$ we have a sheaf $\mathscr{F}_Y := f^*\mathscr{F}$ on X. Taking global sections we obtain a set (resp. group, vector space, etc) $F_Y := \Gamma(Y, f^*\mathscr{F})$. Here we are abusing notation and should be denoting this set $F_{Y,f}$ to specify its dependence on f. Suppose $Y \subset X$ is an open subspace, then $F_Y = \mathscr{F}(Y)$ are the sections of $\mathscr{F}$ on Y.

Now let $g : Z \to Y$ be another map. We have isomorphisms

$$g^* \circ f^* \simeq (g \circ f)^*, \tag{2.1}$$

and thus

$$F_Z = \Gamma(Z, g^* f^* \mathscr{F}) \simeq \Gamma(Z, (g \circ f)^* \mathscr{F}).$$

Note however that f^* and g^* are not necessarily exact, and thus the isomorphism (2.1) needs to be derived if we were to work with complexes of sheaves in the derived category of X instead of just sheaves. On the other hand, when $Y \subset X$ is open and f is the corresponding embedding, f^* is exact.

There are other situations where f^* is exact, this happens for example for étale morphisms. This allows one to consider topologies where the "open embeddings" $Y \subset X$ are replaced by étale morphisms $f : Y \to X$. The reader may want to stick to the Zarisky topology of schemes, or the usual topology of differentiable manifolds. But still the following *functor of points* description of the sheaf $\mathscr{F}$ on X is useful. From now on, we restrict our maps $f : Y \to X$ to be *open* in some unspecified sense, it could be étale for the étale topology, open embedding for the Zariski topology, finitely presented and quasi-compact for the fpqc-topology, etc.

Let $f' : Y' \to X$ be another such map. We have the Cartesian diagram

$$\begin{tikzcd} Z := Y' \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Y' \arrow[r] & X. \end{tikzcd}$$

We see that F_Z is well defined, either pulling back from the upper side of the diagram or the lower side. Similarly, under the isomorphism

$$(Y'' \times_X Y') \times_X Y =: Z' \times_X \simeq Y'' \times_X Z := Y'' \times_X (Y' \times_X Y),$$

we find that $F_{Z' \times_X Y} \simeq F_{Y'' \times_X Z}$. We notice that we have maps $F_Y \to F_Z$, $F_{Y'} \to F_Z$ and thus two projections

$$F_Y \times F_{Y'} \xrightarrow{\pi_1, \pi_2} F_Z.$$

Notice that the map $F_X \to F_Y \times F'_Y$ factors through the equalizer of π_1 and π_2.

Conversely, the collection of F_Y for each $f : Y \to X$ and isomorphisms as above gives rise to a *pre-sheaf* on X, whose sections on Y is F_Y. The sheaf condition is requiring that for a covering $\coprod Y_i \to X$, the corresponding sequence

$$F_X \longrightarrow \prod_i F_{Y_i} \rightrightarrows \prod_{i,j} F_{Y_i \times_X Y_j} \tag{2.2}$$

is exact, in the sense that the first arrow is the equalizer of the right two arrows.

This machinery can be applied to other settings. If the F_Y are vector spaces and the maps in (2.2) are linear, we obtain a sheaf of vector spaces. If for a covering $Y_i = \operatorname{Spec} R_i$ by affine schemes, the F_{Y_i} are free R_i-modules of finite rank n, the corresponding sheaf $\mathscr{F}$ on X is a *vector bundle* of rank n. If the F_Y are G-sets and there exists an étale-surjection $Y \twoheadrightarrow X$ with $F_Y \neq \emptyset$, then the corresponding $\mathscr{F}$ is a *G-local system*, etc.

2.22 Introducing Gerbes

2.23. Throughout this section E will denote a Grothendieck site with fibered products (see Appendix A). We will think of the étale site of a scheme over $\mathbb{C}$ or a complex manifold and its usual topology.

2.24 Definition. A *pseudofunctor* $\mathcal{F} : E^{op} \dashrightarrow Cat$ is an assignment:

a. For an object $U \in E$ a category $\mathcal{F}(U)$.

b. For an arrow $U \xrightarrow{f} V$ a functor $\mathcal{F}(V) \xrightarrow{f^*} \mathcal{F}(U)$.

c. For a pair of composable arrows f, g of E an isomorphism of functors $c_{f,g} : g^* f^* \to (fg)^*$.

Verifying:

a. $c_{f,id_U} = id_{f*} = c_{f,id_V}$

b. The following diagram commutes

$$\begin{tikzcd} h^*g^*f^* \arrow[r, "c_{g,h}f^*"] \arrow[d, "h^*c_{f,g}"] & (gh)^*f^* \arrow[d, "c_{f,gh}"] \\ h^*(fg)^* \arrow[r, "c_{fg,h}"] & (fgh)^*. \end{tikzcd}$$

A *morphism* of pseudofunctors $\alpha : \mathcal{F} \to \mathcal{F}'$ is

a. a functor $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for each $U \in E$

b. a natural isomorphism $\eta_f : \alpha_U \circ f^* \to f^* \circ \alpha_V$ for each $U \xrightarrow{f} V$

such that $\eta_{gf} \circ c_{gf} = c'_{gf} \circ \eta_g \circ \eta_f$ for composable arrows f, g of E.

If $\mathcal{C}$ is a category, $a \in \mathcal{C}$ will denote a is an object of $\mathcal{C}$.

2.25 Definition. A *prestack* is a pseudofunctor $\mathcal{F} : E^{op} \dashrightarrow Cat$ such that for all $a, b \in \mathcal{F}(V)$ the presheaf of sets over $E|_V$ defined by

$$f : U \to V \longmapsto Hom_{\mathcal{F}(U)}(f^*a, f^*b)$$

is a sheaf (i.e. morphisms can be glued).

2.26 Remark. On notation. If U_α, U_β are objects of $E|_U$, then $U_{\alpha\beta}$ denotes the pullback:

$$\begin{tikzcd} U_{\alpha\beta} \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"] & U_\alpha \arrow[d] \\ U_\beta \arrow[r] & U. \end{tikzcd}$$

And if we have $V \xrightarrow{i} U$ in E, then the image of $a \xrightarrow{f} b$ through i^* is denoted $a|_V \xrightarrow{f|_V} b|_V$.

2.27 Definition. Given a covering $\{U_\alpha\}$ of $U \in E$, we define the *category of descent data* $\mathrm{Des}(\{U_\alpha)\}, \mathcal{F})$:

- ◇ The objects are pairs of collections $(a, \theta) = (\{a_\alpha\}, \{\theta_{\alpha\beta}\})$ where $a_\alpha \in \mathcal{F}(U_\alpha)$ and

  $$\theta_{\alpha\beta} : a_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} a_\alpha|_{U_{\alpha\beta}}$$

  are isomorphisms, such that $\theta_{\alpha\alpha} = id$ and $\theta_{\alpha\beta} \circ \theta_{\beta\gamma}|_{U_{\alpha\beta\gamma}} = \theta_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$
- ◇ An arrow $(a, \theta) \xrightarrow{f} (b, \rho)$ is a collection $\{f_\alpha : a_\alpha \to b_\alpha\}$ such that

  $$\begin{tikzcd} a_\beta|_{U_{\alpha\beta}} \arrow[r, "\theta_{\alpha\beta}"] \arrow[d, "f_\beta|_{U_{\alpha\beta}}"] & a_\alpha|_{U_{\alpha\beta}} \arrow[d, "f_\alpha|_{U_{\alpha\beta}}"] \\ b_\beta|_{U_{\alpha\beta}} \arrow[r, "\rho_{\alpha\beta}"] & b_\alpha|_{U_{\alpha\beta}} \end{tikzcd}$$

  commutes.

2.28 Definition. A prestack $\mathcal{F} : E^{op} \dashrightarrow Cat$ is a *stack* if for all $U \in E$ the natural functor:

$$\mathcal{F} \xrightarrow{D} \operatorname{Des}(U_\alpha, \mathcal{F})$$

is an equivalence of categories for any covering $\{U_\alpha\}$. Roughly, $\mathcal{F}(U)$ is defined locally and we can glue up to isomorphism.

2.29. Let's define the functor D precisely. Consider the pullback diagram

$$\begin{tikzcd} U_{\alpha\beta} \arrow[r, "i_\alpha"] \arrow[d, "i_\beta"] & U_\alpha \arrow[d, "j_\alpha"] \\ U_\beta \arrow[r, "j_\beta"] & U \end{tikzcd}$$

and let

$$\begin{tikzcd} a|_{U_\beta}|_{U_{\alpha\beta}} \arrow[r, "\sim"] \arrow[dr, "\theta_{\alpha\beta}"] & a|_{U_{\alpha\beta}} \arrow[d, "\wr"] \\ & a|_{U_\alpha}|_{U_{\alpha\beta}} \end{tikzcd}$$

where the isomorphisms are given by the pseudofunctor condition. Finally, define

$$\mathcal{F} \xrightarrow{D} \operatorname{Des}(U_\alpha, \mathcal{F})$$

$$a \longmapsto (a|_{U_\alpha}, \theta_{\alpha\beta}).$$

2.30 Example. Consider $(X,\ E)$ a scheme with a Grothendieck topology and let G be an group scheme. Define $\mathcal{L} : E^{op} \dashrightarrow Cat$:

- ⋄ For $U \in E$, $\mathcal{L}(U)$ is the category of G-torsors over U.
- ⋄ For a morphism $i : U \to U'$, $i^* : \mathcal{L}(U') \to \mathcal{L}(U)$ is given by the pullback.

It is a pseudofunctor. It is a prestack because morphisms can be glued. Given G-torsors L_α on U_α and $\theta_{\alpha\beta} : L_\beta|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} L_\alpha|_{U_\alpha \cap U_\beta}$ verifying the cocycle condition, then the sheaf given by $L(V) := \varprojlim \Gamma(L_\alpha, V \cap U_\alpha)$ is a G-torsor over $U = \bigcup U_\alpha$. This defines a quasi inverse of the functor D in the definition of stack. This is true for the Zariski, fppf, étale topologies.

2.31. A *2-category* is a category $\mathcal{C}$ such that for every $a, b \in \mathcal{C}$, the set of morphims $Hom(a, b)$ is a category and the composition $Hom(a, b) \times Hom(b, c) \to Hom(a, c)$ is functorial. The canonical example is the category of categories $\mathcal{C}at$. Observe that in the definition of pseudofunctor the target can be any 2-category. So, there are stacks on categories, on groupoids, on any 2-category. Recall that a *groupoid* is a category such that every morphism is an isomorphism.

2.32 Definition. A *gerbe* is a stack of groupoids $\mathcal{F} : E^{op} \dashrightarrow Grpds$ such that for any $U \in E$ there exist a covering $\{U_\alpha\}$ such that $\mathcal{F}(U_\alpha)$ is nonempty and for a, $b \in \mathcal{F}(U)$

$$a|_{U_\alpha} \xrightarrow{\sim} b|_{U_\alpha}.$$

If $\mathcal{G}$ is a sheaf of abelian groups, a gerbe with *band* or *lien* $\mathcal{G}$ (or a $\mathcal{G}$-gerbe) is such that

$$\mathcal{G}|_U \simeq Aut(a) \text{ for } a \in \mathcal{F}(U), U \in E.$$

2.33 Remark. The lien of a gerbe is not necessarily abelian. We refer to [17], for the study of gerbes in the general setting.

2.34 Example. If G is commutative. The stack of (2.30) is a gerbe.

a. It is on groupoids since locally every morphism is the multiplication by an element of the group G.

b. It is a gerbe because there are trivializing coverings.

c. Let $\mathcal{L}$ be a G-torsor. For $V \subset U$, an automorphism $\phi \in \underline{\mathrm{Aut}}(\mathcal{L})(V)$ is an automorphisms of G-torsors. Then, its lien is the sheaf of G-valued functions.

We call it the *trivial G-gerbe* on X and denote it $\mathcal{T}_X$.

Analogously, if $\mathcal{G}$ is a sheaf of commutative groups, the trivial $\mathcal{G}$-gerbe is the stack of $\mathcal{G}$-torsors.

2.35. If $\mathcal{F}$ is a $\mathcal{G}$-gerbe, it is locally trivial. Let $\{U_\alpha\}$ be a covering of X, such that $U_\alpha \neq \emptyset$, we will sketch how $\mathcal{F}|_{U_\alpha} \simeq \{\mathcal{G}|_{U_i} - torsors\}$. Consider $V \subset U_\alpha$ and take $a \in \mathcal{F}(U_\alpha)$. Since $\mathcal{F}$ is a gerbe, for $b \in \mathcal{F}(V)$ there exist a covering $\{V_i\}$ of V and isomorphisms $\phi_i : a|_{V_i} \to b|_{V_i}$. On intersections we have

$$b|_{V_j \cap V_i} \simeq b|_{V_j}|_{V_j \cap V_i} \simeq a_\alpha|_{V_j}|_{V_j \cap V_i} \simeq a_\alpha|V_j \cap V_i \simeq a_\alpha|_{V_i}|_{V_j \cap V_i} \simeq b|_{V_i}|_{V_j \cap V_i} \simeq b|_{V_j \cap V_i}. \tag{2.3}$$

Namely, we have automorphisms $\psi_{ij} \in \underline{\mathrm{Aut}}(b|_{V_j \cap V_i})$ and corresponding $g_{ij} \in \mathcal{G}(V_j \cap V_i)$ that verify the cocycle condition on the triple intersections. We have associated to an element $b \in \mathcal{F}(V)$ a $\mathcal{G}|_V$-torsor. The existence of the inverse of this morphism is guaranteed by the descent property of the stack, take $a \in \mathcal{F}(V)$, $\{g_{ij}\} \in \check{H}(V, \mathcal{G})$ and consider the descent data $a_i := a|_{U_i}$ and θ_{ij} given by $\{g_{ij}\}$. In particular, a G-gerbe is trivial (isomorphic to the trivial gerbe) if and only if it has a global object.

2.36 Picard Groupoids

In this subsection, we present another approach to gerbes. The idea is to define a 1-categorical version of a $\mathcal{G}$-torsor. The first step is to define a categorical version of group.

2.37 Definition. A *Picard groupoid* $\mathcal{P}$ is a groupoid endowed with a bifunctor

$$\otimes : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}$$

with isomorphims

$$a : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z \tag{2.4}$$

$$c : X \otimes Y \xrightarrow{\sim} Y \otimes X \tag{2.5}$$

for all objects $X, Y, Z \in \mathcal{P}$, verifying:

a. $c \circ c = id$

b. the pentagon diagram

$$\begin{tikzcd}[column sep=small]
& X \otimes (Y \otimes (Z \otimes T)) \arrow[dl] \arrow[dr] & \\
X \otimes ((Y \otimes Z) \otimes T)) \arrow[d] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd} \tag{2.6}$$

commutes,

c. the hexagon diagram

$$\begin{tikzcd}[column sep=small]
& X \otimes (Y \otimes Z) \arrow[dl] \arrow[dr] & \\
X \otimes (Z \otimes Y) \arrow[d] & & (X \otimes Y) \otimes Z \arrow[d] \\
(X \otimes Z)) \otimes Y \arrow[dr] & & Z \otimes (X \otimes Y) \arrow[dl] \\
& (Z \otimes X) \otimes Y &
\end{tikzcd} \tag{2.7}$$

commutes,

d. the functor that assigns $Y \mapsto X \otimes Y$ is an isomorphism for all objects $X \in \mathcal{P}$.

A *morphism of Picard groupoids* is a functor $F : \mathcal{P} \to \mathcal{P}'$ with isomorphisms

$$F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$$

compatible with the associativity and commutativity constraints, i.e. such that

$$\begin{array}{ccc} F(X \otimes (Y \otimes Z)) & \longrightarrow & F(X) \otimes (F(Y) \otimes F(Z)) \\ \downarrow & & \downarrow \\ F((X \otimes Y) \otimes Z) & \longrightarrow & (F(X) \otimes F(Y)) \otimes F(Z) \end{array}$$

and

$$\begin{array}{ccc} F(X \otimes Y) & \xrightarrow{\sim} & F(X) \otimes F(Y) \\ \downarrow & & \downarrow \\ F(Y \otimes X) & \longrightarrow & F(Y) \otimes F(X) \end{array}$$

commutes. Denote the 2-category of Picard Groupoids by $\mathcal{P}Grpds$.

2.38. Examples.

a. A group G can be made into a Picard groupoid.

b. Given a commutative algebraic group G and a scheme X, the category of G_X-torsors over X is a Picard groupoid with the usual tensor product. Moreover, this is a stack on Picard groupoids. Namely, the rule that assigns to $U \to X$ the category of G_X-torsor on U is a pseudofuntor with target $\mathcal{P}Grpds$ with the descent property. Denote this stack G_X-*tors*.

c. We denote by $Pic(X)$ the stack of $\mathbb{G}_m$-torsors i.e. of line bundles on a scheme X. Let $Pic^{\mathbb{Z}}$ be the category of graded lines with morphisms

$$Hom_{Pic^{\mathbb{Z}}}((l,n),(l',n')) = \begin{cases} Hom_k(l,l') \setminus 0 & \text{if } n = n' \\ \emptyset & \text{if } n \neq n'. \end{cases}$$

It is a Picard groupoid with the product

$$(l,n) \otimes (l',n') := (l \otimes l', n+n'),$$

and the commutativity constraint

$$c(v \otimes v') = (-1)^{n+n'} v' \otimes v.$$

We denote by $Pic^{\mathbb{Z}}(X)$, the stack of graded line bundles over X.

2.39 Proposition. Every Picard groupoid has a unit.

Proof. Let ϕ be the inverse of $Y \mapsto X \otimes Y$, then

$$\phi(X) \otimes X \simeq X,$$

and

$$\phi(X) \otimes \phi(X) \otimes X \simeq \phi(X) \otimes X,$$

Thefore

$$\phi(X) \otimes \phi(X) \simeq \phi(X).$$

$\square$

2.40 Example. A unit of G-*tors* is the trivial G-torsor i.e. the sheaf of G-valued functions.

2.41 Definition. A $\mathcal{P}$-*torsor* is a category $\mathcal{C}$ endowed with a bifunctor

$$\otimes : \mathcal{P} \times \mathcal{C} \to \mathcal{C}$$

and isomorphisms

$$a : X \otimes (Y \otimes A) \xrightarrow{\sim} (X \otimes Y) \otimes A \tag{2.8}$$

for all objects $X, Y \in \mathcal{P}$, $A \in \mathcal{C}$, verifying:

a. the pentagon diagram

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes A)) \arrow[dl] \arrow[dr] & \\
X \otimes ((Y \otimes Z) \otimes A)) \arrow[d] & & (X \otimes Y) \otimes (Z \otimes A) \arrow[d] \\
(X \otimes (Y \otimes Z)) \otimes A \arrow[rr] & & ((X \otimes Y) \otimes Z) \otimes A
\end{tikzcd} \tag{2.9}$$

commutes, for $X, Y, Z \in \mathcal{P}$, $A \in \mathcal{C}$,

b. the functor $A \mapsto I \otimes A$ is an equivalence, where $I \in \mathcal{P}$ is a unit,

c. for all $A \in \mathcal{C}$, the functor $X \mapsto X \otimes A$ is an equivalence of categories.

A *morphism of* $\mathcal{P}$-*torsors* is a functor $F : \mathcal{C} \to \mathcal{C}'$ provided with isomorphisms

$$F(X \otimes A) \xrightarrow{\sim} X \otimes F(A)$$

such that

$$\begin{tikzcd}
F((X \otimes Y) \otimes A) \arrow[r] \arrow[d] & (X \otimes Y) \otimes F(A) \arrow[d] \\
F(X \otimes (Y \otimes A)) \arrow[r] & X \otimes (Y \otimes F(A))
\end{tikzcd}$$

for all $X, Y \in \mathcal{P}$, $A \in \mathcal{C}$. Denote by $\mathcal{BP}$ the category of $\mathcal{P}$-torsors.

2.42 Proposition. The category $\mathcal{BP}$ is a Picard groupoid.

Proof. (Sketch) The product of two $\mathcal{P}$-torsors $\mathcal{C}$, $\mathcal{C}'$ is defined by

$$Ob(\mathcal{C} \otimes_{\mathcal{P}} \mathcal{C}') := Ob(\mathcal{C}) \times Ob(\mathcal{C}')$$

$$Hom(A \times A', B \times B') := (\phi, \psi, g)/\sim$$

where $g \in \mathcal{P}$, $\phi \in Hom(A, g \otimes B)$, $\psi \in Hom(g \otimes A', B')$ and $(\phi, \psi, g) \sim (\phi', \psi', g')$ if and only if there exists $f : g \to g'$ such that $f \otimes B \circ \phi = \phi'$ and $\psi' \circ f \otimes A' = \psi$. The definition of the associativity and the commutativity constraints are straightforward. $\square$

Observe that $(g \otimes A, A') \simeq (A, g \otimes A')$ in $\mathcal{C} \otimes_{\mathcal{P}} \mathcal{C}'$. As usual we will omit the subindex whenever its clear by the context what product we are considering.

2.43 Definition. The category $\mathcal{BP}$ is a 2-category. Given $F, F' : \mathcal{C} \to \mathcal{C}'$ morphisms of $\mathcal{P}$-torsors, a morphism $F \to F'$ is a natural transformation η such that the following diagram commutes

$$\begin{tikzcd}
F(X \otimes A) \arrow[r] \arrow[d, "\eta_{X \otimes A}"] & X \otimes F(A) \arrow[d, "i \otimes \eta_A"] \\
F'(X \otimes A) \arrow[r] & X \otimes F(A).
\end{tikzcd}$$

2.44. Let $\mathcal{A}$ be a sheaf of commutative groups and $\mathcal{P}$ be the stack of $\mathcal{A}$-torsors on a space X. A stack $\mathcal{G}$ on $\mathcal{BP}$ is for each $U \to X$ a $\mathcal{P}(U)$-torsor $\mathcal{G}(U)$. If $\mathcal{G}$ is locally nonempty this is the same as a $\mathcal{A}$-gerbe. Notice that for $a \in \mathcal{G}(U)$ we have $Aut(a) \simeq \mathcal{A}|_U$. This is a direct consequence of item **(b)** and **(c)** in the definition of $\mathcal{P}$-torsor. A *morphism* of $\mathcal{A}$-gerbes is a morphism between its underlying pseudofunctors that is a morphism of $\mathcal{P}$-torsors. Observe that every morphism of gerbes is an equivalence.

2.45 Proposition. Given a sheaf of commutative groups $\mathcal{A}$ on a a space X. There is a group isomorphism between $\check{H}^2(X, \mathcal{A})$ and the equivalence classes of $\mathcal{A}$-gerbes. Moreover,

$$\check{H}^2(X, \mathcal{A}) \simeq \check{H}^1(X, \mathcal{A}\text{-tors}).$$

Therefore, for a good covering U_i, an $\mathcal{A}$-gerbe is specified by a collection of $\mathcal{A}$-torsors L_{ij} defined on the intersections $U_i \cap U_j$ such that in the triple intersections $L_{ij} \otimes L_{jk} \simeq L_{ik}$.

Proof. We will define isomorphisms:

$$\begin{tikzcd}
\check{H}^1(X, \mathcal{A}\text{-tors}) \arrow[r, "\alpha"] & \check{H}^2(X, \mathcal{A}) \arrow[dl, "\gamma"] \\
\{\mathcal{A} - gerbes\}/\sim \arrow[u, "\beta"] &
\end{tikzcd}$$

Let L_{ij} be $\mathcal{A}$-torsors on $U_i \cap U_j$, such that in the triple intersections $U_i \cap U_j \cap U_k := U_{ijk}$

$$L_{ij} \otimes L_{jk}|_{U_{ijk}} \simeq L_{ik}|_{U_{ijk}}.$$

These isomorphisms of $\mathcal{A}$-bundles are given by $h_{ijk} \in \mathcal{A}_X(U_{ijk})$ and $\{h_{ijk}\}$ is a cocycle. The assignment $\{L_{ij}\} \mapsto \{h_{ijk}\}$ is clearly a morphism of groups.
Now, for a $\{h_{ijk}\} \in \check{H}^2(X, \mathcal{A})$ we define: the objects of $\mathcal{G}(V)$ are families $\{L_i\}$ of $\mathcal{A}$-torsors over U_i with isomorphims $L_i|_{U_i \cap U_j} \simeq L_j|_{U_i \cap U_j}$ given by $g_{ij} \in \mathcal{A}_X(U_i \cap U_j)$ such that $g_{ij}.g_{ik}^{-1}.g_{jk} = h_{ijk}$ and the morphisms are families of line bundle morphisms compatible with the isomorphisms on the intersections.
Finally, given a gerbe $\mathcal{G}$ we have seen that is locally isomorphic to the trivial gerbe. Consider a covering and isomorphisms $\psi_i : \mathcal{G}(U_i) \to \mathcal{A}^*_{U_i}$, then $\psi_j^{-1} \circ \psi_i : \mathcal{A}^*_{U_{ij}} \to \mathcal{A}^*_{U_{ij}}$ are given by $\mathcal{A}$-torsors $\{L_{ij}\}$ on $U_i \cap U_j$ that verifies the cocycle condition. □

2.46 Proposition. Given a morphism of Picard Groupoids $F : \mathcal{P} \to \mathcal{P}'$ and $\mathcal{C} \in \mathcal{BP}$ the product of $\mathcal{P}$-torsors $\mathcal{P}' \otimes \mathcal{C}$ inherits a stucture of $\mathcal{P}'$-torsor. Moreover, this gives a morphism $\mathcal{BP} \to \mathcal{BP}'$.

Proof. On objects:

$$\mathcal{P}' \times \mathcal{P}' \times \mathcal{C} \to \mathcal{P}' \times \mathcal{C}$$

$$(x, (y, A)) \mapsto (x \otimes y, A)$$

The properties (i) an (ii) of the definition of torsor are deduced from the fact that $\mathcal{P}'$ is a Picard groupoid. And for (iii), the inverse of the functor

$$\mathcal{P}' \to \mathcal{P}' \otimes \mathcal{C}$$

$$y \mapsto (y \otimes x, A)$$

is

$$\mathcal{P}' \otimes \mathcal{C} \to \mathcal{P}'$$

$$(y, B) \mapsto F(\phi(B)) \otimes y \otimes x$$

where $\phi : \mathcal{C} \to \mathcal{P}$ is the inverse of $X \mapsto X \otimes A$. □

2.47. Pullback of gerbes

Now, let X be a scheme, we denote by $\mathcal{B}Pic(X)$ the category of $\mathbb{G}_m$-gerbes over X. Let $f : X \to Y$ be a morphism of schemes. Then the *pullback* functor

$$f^* : \mathcal{B}Pic(Y) \to \mathcal{B}Pic(X)$$

is defined by considering in 2.46 $F = f^* : Pic(Y) \to Pic(X)$. And this can also be defined for $\mathcal{A}$-gerbes with $\mathcal{A}$ a constant sheaf.

2.48. A pushforward of $\mathbb{G}_m$-gerbes Let $f : X \to Y$ be a faithfully flat of finite type morphism of schemes. We want to define a morphism

$$Pic(X) \to f^{-1}Pic(Y)$$

to apply the previous proposition.

For $\mathcal{L} \in Pic(X)$, the norm ([18] 6.4-6.5) $N_{X/Y}(\mathcal{L}) = det(f_*\mathcal{L})$ gives a multiplicative morphism $\phi : Pic(X) \to f^{-1}Pic(Y)$. To prove that ϕ is multiplicative, consider a covering $\mathcal{U}$ of X such that $U_i = f^{-1}(V_i) \simeq \mathcal{O}_{V_i}^{\oplus r}$. If the class of $\mathcal{L}$ in $\check{H}^1(\mathcal{U}, \mathbb{G}_m)$ is represented by $g_{ij} \in \mathcal{O}_{U_{ij}}$, through the previous isomorphism can be thought as a matrix $g_{ij} \in M_{r\times r}(\mathcal{O}_{V_{ij}})$. Then $N_{X/Y}(\mathcal{L})$ is represented by $det(g_{ij}) \in \mathcal{O}_{V_{ij}}$ and this is clearly multiplicative.

Given $\mathcal{G} \in \mathcal{B}Pic(X)$, $V \to Y$ we define a pushforward

$$N_{X/Y}(\mathcal{G})(V) := \phi(\mathcal{G})(f^{-1}(V)).$$

More over,

$$N_{X/Y} : \mathcal{B}Pic(X) \to \mathcal{B}Pic(Y)$$

is a morphism of Picard groupoids.

2.49 Connections on gerbes

2.50 Definition. Let $\mathcal{G}$ be an $\mathcal{A}$-gerbe on X. A *connection* on $\mathcal{G}$ is an equivalence of gerbes

$$\alpha : p_1^*\mathcal{G} \to p_2^*\mathcal{G}$$

and a natural isomorphism $\Delta^*(\alpha) \Rightarrow Id$. The connection is integrable if and only if there is a natural isomorphism $p_{13}(\alpha) \Rightarrow p_{23}(\alpha) \circ p_{12}(\alpha)$ where $p_{ij} : X \times X \times X \to X \times X$ are the projections.

2.51. It can be proved (e.g. [8, 5.3.11]) that the isomorphism classes of gerbes with connection is

$$H^2(X, \mathcal{O}_X^* \xrightarrow{dlog} \Omega_X^1 \longrightarrow \Omega_X^2)$$

i.e. if the gerbe is given by a 3-cocycle g_{ijk} the connection are given by forms

$$A_{ij} + A_{jk} + A_{ki} = g_{ijk}^{-1} dg_{ijk}$$

$$F_i - F_j = dA_{ij}$$

And a connection is flat if and only if $dF_i = 0$

2.52. Family of examples [7] Given a line bundle $\mathcal{L}$ and $q \in \mathcal{O}_X^*$, the gerbe $\mathcal{L}^{log(q)}$ is the image of $\mathcal{L}$ through the morphism

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X^*)$$

where the first map is the induced by the exponential sequence and the second is induced by

$$\mathbb{Z} \to \mathcal{O}_X^*$$

$$n \mapsto q^n.$$

More explicitly, for Čech cocycles representatives,

$$f_{ij} \mapsto n_{ijk} = log(f_{ij}) + log(f_{jk}) + log(f_{ki}) \mapsto q^{n_{ijk}}.$$

2.53 Proposition. The gerbe $\mathcal{L}^{log(q)}$ is endowed with a connection.

Proof. The connection is given by

$$A_{ij} = log(f_{ij})q^{-1}dq$$

and

$$F_i = dlog(s_i) \wedge dlog(q)$$

where $s_i \in \mathcal{L}(U_i)$ is a section.

2.54. This can be defined the same way for $q \in \mathcal{A}(X)$, $\mathcal{A}$ any sheaf of commutative groups. And if $f : Y \to X$ is an holomorphic function we have

$$f^*(\mathcal{L}^{log(q)}) \simeq f^*(\mathcal{L})^{log(q \circ f)}.$$

Let $D \subset X$ a divisor, $Y = supp(D)$. And consider the short exact sequence:

$$0 \to \mathcal{O}_X^* \to \varinjlim \mathcal{O}(nY) \to \nu_{Y*}\mathbb{Z}_{\tilde{Y}} \to 0$$

where $\nu_Y : \tilde{Y} \to Y$ is the normalization. Then, the boundary map

$$\delta : H^1(\tilde{Y}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X^*)$$

gives as for each divisor D and $\mathbb{Z}$-torsor on $\tilde{Y}$ a gerbe on X. For $q \in \mathbb{C}^*$ consider the $\mathbb{Z}$-torsor of logarithms of q and, if $D = \sum n_i D_i$ is the decomposition in irreducible divisors, this morphisms gives us the class of the gerbe $\bigotimes \mathcal{O}(D_i)^{\log q}$. In [7], they prove that this gerbes has a connective structure. In [27], the author proves that for $\mathbb{C}^*$-gerbe these gerbes classifies gerbes with trivilizations away the divisor. More generally,

2.55 Proposition. Let A be a commutative group, $\mathcal{G}$ be an A-gerbe over a scheme X, $D \subset X$ a divisor. Consider $Z = supp(D)$ and $U = X \setminus Z$. Suppose $\mathcal{G}|_U \simeq \mathcal{T}_U$ then there exists $q \in A$ such that $\mathcal{G} \simeq \mathcal{O}(D)^{log(q)}$.

2.56 The determinantal gerbe of a Tate bundle

2.57. We know how to compute the determinant of a finite dimensional vector space. We have the functor

$$det : Vect_0 \to Pic^{\mathbb{Z}}$$

$$det(V) = (\bigwedge^n V, n)$$

where $n = dim(V)$ and $Vect_0$ denotes the category of vector spaces with isomorphisms. For every short exact sequence $0 \to V' \to V \to V''$

$$det(V) \otimes det(V'') \xrightarrow{\sim} det(V') \tag{2.10}$$

verifying:

a. For each commutative diagram of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "\wr"] & V' \arrow[r] \arrow[d, "\wr"] & V'' \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & U \arrow[r] & U' \arrow[r] & U'' \arrow[r] & 0 \end{tikzcd}$$

the corresponding diagram

$$\begin{tikzcd} det(V) \otimes det(V'') \arrow[r] \arrow[d] & det(V') \arrow[d] \\ det(U) \otimes det(U'') \arrow[r] & det(U') \end{tikzcd}$$

is commutative.

2.58. The category of vectors spaces endowes $Vect_0$ with the direct sum is symmetric monoidal category and $det : Vect_0 \to Pic^{\mathbb{Z}}$ is symmetric monoidal (sends the sum to the product respecting the symmetry). That's the reason of including the grading on $\mathcal{P}ic$.

2.59. This can be done locally. That is to say, if S be a scheme, the determinant of a finitely generated $\mathcal{O}_S$-module is a line bundle that verifies the same conditions.

2.60. We want to define the determinant of a infinite dimensional vector bundle. We dont́ know how to do so. Kapranov [22] realized that given two lattices $L_1 \subset L_2$ in a locally compact vector space (or a Tate vector sapce), we have "$det(L_1) \otimes det(L_2)^{-1} := det(L_1/\, L_2)$" and this rule verifies some properties. For a certain kind of $\mathcal{O}_S$-modules, this will give us another source of examples of $\mathcal{O}_S^*$-gerbes.

2.61. First, a baby example. Consider $V = \mathbb{C}((t))$, and define the subspaces $V_n := t^{-n}\mathbb{C}[[t]]$. Then, $V_n \subset V_m$ for $n \leq n$ and $V = \varinjlim V_n$. Observe that $V_{n,m} := V_n/V_m$ are finite dimensional. We want to give a $\mathcal{P}ic^{\mathbb{Z}}(k)$- torsor $\mathcal{C}$. The objects of the category are the *determinantal theories* on $\mathbb{C}((t))$, i.e. a compatible way to define a "determinant" of each V_n. A *determinantal theory* on $\mathbb{C}((t))$ is for each $m \in \mathbb{Z}$ a line $\Delta(V_m)$, and for $m \geq n$ an isomorphism

$$\Delta(V_m) \otimes det(V_{n,m}) \xrightarrow{\sim} \Delta(V_n) \tag{2.11}$$

such that for $m \geq n \geq l$ the diagram

$$\begin{array}{ccc} \Delta(V_m) \otimes det(V_{n,m}) \otimes det(V_{l,n}) & \longrightarrow & \Delta(V_n) \otimes det(l,n) \\ \downarrow & & \downarrow \\ \Delta(V_m) \otimes det(V_{l,m}) & \longrightarrow & \Delta(V_l) \end{array}$$

commutes.

For example, define

$$\Delta_0(m) = \begin{cases} det(V_{0,m}) & if m \geq 0 \\ det(V_{m,0}) & if 0 \geq m \end{cases}$$

2.62 Definition. Let R be a commutative ring and Mod_R the category of finitely generated R-modules,

$$Tate(R) := \lim_{\leftrightarrow} Mod_R$$

is the category of *Tate* R-modules.

We refer to the appendix for the definition of the double limit, Ind and Pro objects in a Category.

2.63 Example. For k a field, these are Tate vector spaces ([12]) or locally compact vector spaces (Lefschetz). And the main example is the Laurent series $k((t)) = \varinjlim \varprojlim t^{-n}k[t]/(t^{m+1})$. More generally, $R((t))^d$. More generally, $R((t))^n$ is a Tate R-module. Then every finitely generated free $R((t))$-module and, more generally, every projective finitely genrated $R((t))$-module is a Tate R-module.

2.64 Definition. Let $\mathcal{V} \in Tate(R)$, a *lattice* L is a submodule of $\mathcal{V}$ such that $L \in Pro(Mod_R$ and $\mathcal{V}/L \in Ind(Mod_R)$. Analogously, a *colattice* L' is a submodule of $\mathcal{V}$ such that $L' \in Ind(Mod_R)$ and $\mathcal{V}/L' \in Pro(Mod_R)$.

2.65 Definition. A *determinantal theory* on $\mathcal{V} \in Tate(R)$ is for each lattice $L \subset \mathcal{V}$ a $\mathbb{Z}$-graded invertible R-module, $\Delta(L)$, and for every two lattices $L_1 \subset L_2$ an isomorphism

$$\Delta(L_1) \otimes det(L_2/L_1) \xrightarrow{\sim} \Delta(L_2) \tag{2.12}$$

such that the diagram

$$\begin{array}{ccc} \Delta(L_1) \otimes det(L_2/L_1) \otimes det(L_3/L_2) & \longrightarrow & \Delta(L_2) \otimes det(L_3/L_2) \\ \downarrow & & \downarrow \\ \Delta(L_1) \otimes det(L_3/L_1) & \longrightarrow & \Delta(L_3) \end{array}$$

commutes, for $L_1 \subset L_2 \subset L_3$.

The groupoid of determinantal theories on $\mathcal{V}$ is defined as you imagine.

2.66 Example. Given two lattices L and L', $L \cap (V/L')$ is both in $Ind(Mod_R)$ and $Pro(Mod_R)$. Then, $L/L \cap L' \in Mod_R$.
Denote $Pic^{\mathbb{Z}}(R)$ the category of $\mathbb{Z}$-graded invertible R-modules. Choosing $\mathcal{L}_0 \in Pic^{\mathbb{Z}}(R)$ and L_0 a lattice, we can define a determinantal theory given by

$$\Delta(L) := \mathcal{L}_0 \otimes det\left({}^{L}\!/_{L \cap L_0} \right) \otimes det\left({}^{L_0}\!/_{L \cap L_0} \right).$$

Therefore, if we have a canonical way to choose a lattice we have an object of the category, then a trivialization.

2.67 Theorem. Consider the functor

$$\mathcal{D}et \colon Tate(R) \to \mathcal{B}Pic^{\mathbb{Z}}(R)$$

that assigns to a $\mathcal{V} \in Tate(R)$ to the category of determinantal theories on $\mathcal{V}$. Then, for each $\mathcal{V} \hookrightarrow \mathcal{V}'$, there is an isomorphism

$$\mathcal{D}et(\mathcal{V}) \otimes \mathcal{D}et(\mathcal{V}'/\mathcal{V}) \xrightarrow{\sim} \mathcal{D}et(\mathcal{V}') \tag{2.13}$$

verifying:

a. For each diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V} \arrow[r] \arrow[d, "\wr"] & \mathcal{V}' \arrow[r] \arrow[d, "\wr"] & \mathcal{V}'/\mathcal{V} \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{U}' \arrow[r] & \mathcal{U}'/\mathcal{U} \arrow[r] & 0 \end{tikzcd}$$

the corresponding diagram

$$\begin{tikzcd} \mathcal{D}et(\mathcal{V}) \otimes \mathcal{D}et(\mathcal{V}'/\mathcal{V}) \arrow[r] \arrow[d] & \mathcal{D}et(\mathcal{V}') \arrow[d] \\ \mathcal{D}et(\mathcal{U}) \otimes \mathcal{D}et(\mathcal{U}'/\mathcal{U}) \arrow[r] & \mathcal{D}et(\mathcal{U}') \end{tikzcd}$$

is commutative.

b. For

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{V}' & \longrightarrow & \mathcal{V}'/\mathcal{V} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{U}' & \longrightarrow & \mathcal{U}'/\mathcal{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{W}' & \longrightarrow & \mathcal{W}'/\mathcal{W} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

there is an equivalence (2-isomorphism) between the functors given by

$$\begin{array}{ccc} \mathcal{D}et(\mathcal{V}) \otimes \mathcal{D}et(\mathcal{V}'/\mathcal{V}) \otimes \mathcal{D}et(\mathcal{W}) \otimes \mathcal{D}et(\mathcal{W}'/\mathcal{W}) & \longrightarrow & \mathcal{D}et(\mathcal{V}') \otimes \mathcal{D}et(\mathcal{W}') \\ \downarrow & & \\ \mathcal{D}et(\mathcal{V}) \otimes \mathcal{D}et(\mathcal{W}) \otimes \mathcal{D}et(\mathcal{V}'/\mathcal{V}) \otimes \mathcal{D}et(\mathcal{W}'/\mathcal{W}) & & \downarrow \\ \downarrow & & \\ \mathcal{D}et(\mathcal{U}) \otimes \mathcal{D}et(\mathcal{U}'/\mathcal{U}) & \longrightarrow & \mathcal{D}et(\mathcal{U}') \end{array} .$$

c. If $\mathcal{V} = 0$ (or $\mathcal{V} = \mathcal{V}'$), then the isomorphism 2.13 is the canonical isomorphism

$$Pic^{\mathbb{Z}}(A) \otimes \mathcal{D}et(\mathcal{V}') \to \mathcal{D}et(\mathcal{V}')$$

(resp.$\mathcal{D}et(\mathcal{V}') \otimes Pic^{\mathbb{Z}}(A) \to \mathcal{D}et(\mathcal{V}')$)

Proof. Idea. It is enough to consider the case $S = Spec(A)$, A a noetherian ring. Given a short exact sequence of Tate modules

$$0 \longrightarrow \mathcal{V} \xrightarrow{i} \mathcal{V}' \xrightarrow{\pi} \mathcal{V}'/\mathcal{V} \longrightarrow 0$$

and a lattice $L \in \mathcal{V}'$, the submodules $i^{-1}(L)$ and $\pi(L)$ are lattices then we define

$$\Delta(L) := \Delta'(i^{-1}(L)) \oplus \Delta''(\pi(L))$$

where $(\Delta', \Delta'') \in \mathcal{D}et(\mathcal{V}) \otimes \mathcal{D}et(\mathcal{V}'/\mathcal{V})$.

□

2.68 Definition. Let X be a k-scheme. A *Tate*-sheaf is for each k-algebra A and each $Spec(A) \to X$ a Tate A-module $\mathcal{V}_A$ such that for $\phi : A \to B$, $\mathcal{V}_A \otimes B \simeq \mathcal{V}_B$.

Observe that this definition is thought from the point of view of the functor of points $h_X(A) = Hom(Spec(A), X)$ and it can be made for any $F : \mathcal{A}\!f\!f^{op} \to \mathcal{S}ets$ i.e. for more general spaces. And the above construction gives a gerbe on this space.

2.69 Example. **a**. Let $X = Spec(R)$ be affine of finite type and consider the loop space $\tilde{\mathcal{L}}(X)$ that Ind-represents the functor from $\mathbb{C}$-algebras defined by:

$$\lambda_X(A) = Hom(R, A((t))).$$

Observe that $\tilde{\mathcal{L}}(X) \simeq \varprojlim_n \varinjlim_m \tilde{\mathcal{L}}^n_m(X)$, where $\tilde{\mathcal{L}}^n_m(X)$ is the affine scheme of finite type representing:

$$\lambda_X^{n,m}(A) := Hom(R, t^{-n}A[t]/(t^{m+1})).$$

And $\tilde{\mathcal{L}}(X) = \varinjlim \tilde{\mathcal{L}}^n$ is the inductive limit of schemes of infinite type. To give an example of Tate sheaf we want to "linearize" this space. If we consider $\Omega^1_{\tilde{\mathcal{L}}^n}$ is not true that $i^*\Omega^1_{\tilde{\mathcal{L}}^n} \simeq \Omega^1_{\tilde{\mathcal{L}}^{n'}}$ then, the cotangent bundle of $\tilde{\mathcal{L}}(X)$ (or any Ind-scheme), $\Omega^1_{\tilde{\mathcal{L}}(X)}|_{\tilde{\mathcal{L}}^n}$ is defined as the projective system $i^*\Omega^1_{\tilde{\mathcal{L}}^{n'}}$, $n' \geq n$. Each $\Omega^1_{\tilde{\mathcal{L}}(X)}|_{\tilde{\mathcal{L}}^n}$ is a Tate sheaf on $\tilde{\mathcal{L}}^n$ and we obtain a gerbe $\mathcal{G}^n = \mathcal{D}et(\Omega^1_{\tilde{\mathcal{L}}(X)}|_{\tilde{\mathcal{L}}^n})$. Moreover, this gerbes verify $i^*\mathcal{G}^{n'} \simeq \mathcal{G}^n$ for $n' \geq n$ defining a gerbe on $\tilde{\mathcal{L}}(X)$.

b. If Y is an Ind-scheme over X with certain conditions of regularity (reasonable $\aleph_0$-Ind- scheme in [12] or locally locally compact Ind-schemes in [23]) the cotanget sheaf Ω_Y is a Tate sheaf (op. cit.) and the determinantal gerbe $\mathcal{D}et(\Omega_Y)$ we have defined coincides with the one defined on [23].

3 Factorizing Structures

In this section we define and recall basic properties of different factorization structures as defined in [4] and informally described in the introduction. We start with *factorization spaces* in sections 3.1–3.6, we introduce the affine Grassmanian in section 3.7 and describe a factorization space associated to loop spaces in 3.8. Finally in sections 3.9–3.16, after a short digression on central extensions, we define factorizing line bundles over the *Picard factorization space.* We construct an example by means of Deligne's *push forward of divisors* [19, Exp XVIII] and recall Beilinson and Drinfeld's notion of a *theta-datum.*

By *space* in this section we mean a formally smooth ind-scheme. For an overview of this notion we refer the reader to Appendix B.

3.1 Factorizing spaces

In this section we define *factorization spaces* following [4]. As mentioned in the introduction, by *space* we mean a formally smooth ind-scheme, but at the present moment these can be taken to be objects of any category with fibered products. In particular, a space Y over X can be thought as a fibered space, i.e. as a collection of spaces Y_x for each point $x \in X$. Indeed for $Y \to X$ and a closed point $x : \operatorname{Spec} k \hookrightarrow X$, we let $Y_x := Y \times_X x$. The notion of Y being *formally-smooth* says that the spaces Y_x ought to be reasonably nice and vary smoothly when x varies within X. Intuitively, a factorizing space over X is a collection of spaces Y_x, $Y_{\{x_1,x_2\}}$ such that $Y_{\{x,x\}} \simeq Y_x$ and, if $x_1 \neq x_2$, $Y_{\{x_1,x_2\}} \simeq Y_{x_1} \times Y_{x_2}$ (parametrized by points of X and pairs of points of X, respectively). More precisely, spaces $Y_{\{x_1,...x_n\}}$ for each set of n points $\{x_1, x_2..., \}$ such that

a. $Y_{x_1,...,x_i,...,x_n} \simeq Y_{x_1,...,\hat{x_i},...,x_n}$ if $x_i = x_j$ for some $j \leq n$,

b. $Y_{\{x_1,...x_n\}} \simeq Y_{x_1} \times Y_{x_2} \times ... \times Y_{x_n}$ if $x_i \neq x_j$ for all $i, j \leq n$.

The first condition says that the spaces depend on the subset $\{x_1, \ldots, x_n\} \subset X$ as a subset, not counting multiplicities. The second is the factorization property. It follows from these two conditions that the space $Y_{x_1,...,x_n}$ depends on $\{x_1, \ldots, x_n\} \subset X$ as an *unordered* set. There are several ways of formally defining factorization, we will first need some notation in order to give two equivalent definitions.

Let **Fset** be the category of nonempty finite sets and surjections as morphisms. For $p : J \to I$, $q : K \to J$, $p' : J' \to I'$ in **Fset**, define:

$$X^p = \{(x_j) \in X^J \mid x_j \neq x_{j'} \text{ if } p(x_j) \neq p(x_{j'})\}$$

$$\Delta_{p,q} : X^p \to X^{p \circ q}, (x_j) \mapsto (x_{q(k)})$$

$$j_{p,q} : X^q \to X^{p \circ q}, (x_k) \mapsto (x_k)$$

$$i_{p,p'} : X^{p \amalg p'} \to X^p \times X^{p'}, (x_j, x_{j'}) \mapsto (x_j, x_{j'})$$

3.2 Definition. A *factorizing space over* X is, for each $p : J \to I$ in **Fset**, a formally smooth Ind-space $Y_p \to X^p$ with integrable connection along X^p, and isomorphisms:

$$\xi_{p,q} : \Delta^*_{p,q}(Y_{pq}) \to Y_p$$

$$\kappa_{p,q} : j^*_{p.q}(Y_{pq}) \to Y_q$$

$$\phi_{p,p'} : i^*_{p,p'}(Y_p \times Y_{p'}) \to Y_{p \amalg p'}$$

with the obvious compatibilities with triple products and compositions. A factorization space over X is *unital* if there are sections $s_p : X^p \to Y_p$ compatible with all the structure above. Namely, for example, $\Delta^*_{p,q}(s_p) \circ \xi_{p,q} = s_p$.

3.3 Remarks. Let $\{Y_p\}$ be a factorization space over X. If p_n is the surjection $p_n : \{1, 2, ..., n\} \to \{1\}$ then $X^{p_n} = X^n$. We put $Y_n := Y_{p_n}$.

a. For $\sigma \in S_n$ a permutation, then $\Delta_{p_n,\sigma} : X^n \to X^n$ is the induced action on X^n and we have symmetries $\Delta^*_{p_n,\sigma}(Y_n) \simeq Y_n$. Then the group of permutations S_n acts on Y_n.

b. Consider $p_2 : \{1,2\} \to \{1\}$ and $p_1 : \{1\} \to \{1\}$, then $\Delta_{p_1,p_2} = \Delta$ is the diagonal $\Delta : X \to X^2$, $\Delta(x) = (x,x)$. And we have

$$\Delta^*(Y_2) \simeq Y_1.$$

c. Consider $id_2 : \{1,2\} \to \{1,2\}$ then $X^{id_2} = X^2 \setminus \Delta$ and the isomorphism $j^*_{p_2,id_2}(Y_2) \simeq Y_{id_2}$ means that $Y_2|_{X^2\setminus\Delta} \simeq Y_{id_2}$. Now, observe that $p_1 \amalg p_1 = id_2$ and $i_{p_1,p_1} : X^{p_1 \amalg p_1} \to X \times X$ is the inclusion. Then it is verified that

$$Y_2|_{X^2\setminus\Delta} \simeq Y_{id_2} \simeq Y_1 \times Y_1|_{X^2\setminus\Delta}.$$

d. In general, consider the following situation. Denote by

$$\Delta_{ij} = \{(x_k) \in X^k \mid x_i = x_j\}$$

$$\Delta^{nm} = \sum_{i\leq n, j>n} \Delta_{ij} \subset X^{n+m}.$$

Observe that $X^{p_n \amalg p_m} = X^{n+m} \setminus \Delta^{nm}$. Let p be the surjection $p : \{1\} \cup \{1\} \to \{1\}$. Then, $j_{p,p_n\amalg p_m} : X^{n+m} \setminus \Delta^{nm} \to X^{n+m}$ is the inclusion and we have the isomorphism

$$\kappa_{p,p_n\amalg p_m} : Y_{n+m}|_{X^{n+m}\setminus\Delta^{nm}} \to Y_{p_n\amalg p_m}.$$

Also, $i_{p_n,p_m} : X^{p_n\amalg p_m} \to X^{p_n} \times X^{p_m}$ is again the inclusion and we have:

$$\phi_{p_n,p_m} : Y_n \times Y_m|_{X^{n+m}\setminus\Delta^{nm}} \to Y_{p\amalg p'}.$$

Let

$$\psi_{nm} : Y_n \times Y_m|_{X^{n+m}\setminus\Delta^{nm}} \to Y_{n+m}|_{X^{n+m}\setminus\Delta^{nm}}$$

be the resulting isomorphism.

The second definition follows the functor of points approach along the lines described in 2.21. Given an affine scheme Z, consider the set of divisors $S \subset Z \times X$ finite, flat and proper over Z and denote it $\mathcal{D}iv(X)_Z$. It has a equivalence relation $S \sim S'$ if $S_{red} = S'_{red}$. For each such Z we let $\mathcal{C}(X)_Z$ be the quotient $\mathcal{D}iv(X)_Z/\sim$.

3.4 Proposition. The following data is equivalent to a *unital factorization space over* X. For each $S \in \mathcal{C}(X)_Z$, a formally smooth ind-space $Y_S \to Z$ with closed embeddings $i_{S,S'} : Y_S \hookrightarrow Y_{S'}$ when $S_{red} \subset S'_{red}$ and, for S_1, S_2 disjoint, isomorphisms

$$c_{S_1,S_2} : Y_{S_1} \times Y_{S_2} \to Y_{S_1+S_2} \tag{3.1}$$

commutative, associative and compatible with base change.

Proof. First, we define a factorizing space from the data of the proposition. For each I in **Fset**, we define $Y_I := Y_\Delta$ where $\Delta \subset X^I \times X$ is the union of the diagonals $x_i = x$ (where x denotes the last coordinate). And in general,

$$Y_p := j^*_{p_I,p}(Y_I)$$

where $p_I : I \to \{1\}$. One readily verifies that it is indeed a factorizing space.
Conversely, given a factorizing space as in definition 3.2, define the following space over $Sym^n X$:

$$Y_{Sym^n X} := Y_n \big/ S_n.$$

A divisor $S \subset Z \times X$ of degree n, induces a morphism $Z \to Sym^n X$ by the universal property of $Sym^n X$ (here we need the fact that X is a smooth curve, see Appendix C). We define

$$Y_S := Z \times_{Sym^n X} (Y_{Sym^n X}).$$

Observe that the morphism $\psi_{nm} : Y_n \times Y_m|_{X^{n+m} \setminus \Delta^{nm}} \to Y_{n+m}|_{X^{n+m} \setminus \Delta^{nm}}$ is $S_n \times S_m$-invariant and we obtain an isomorphism over $Sym_n X \times Sym_m X \setminus \Delta^{mn}$. If $S_1, S_2 \subset Z \times X$ are disjoint divisors of degree n and m respectively, then the morphism of the sum $S_1 + S_2$, $Z \to Sym^{n+m} X$ factors through $Sym_n X \times Sym_m X \setminus \Delta^{mn}$ and we obtain the isomorphims c_{S_1,S_2}. Now, let $S \subset S'$ of degree n and $n+m$. To obtain the morphisms $i_{S,S'}$, compose the morphisms ψ_{nm} with

$$id \times s_m : Y_n \times X^m \to Y_n \times Y_m.$$

That these maps are mutually inverse reduces to $p_n^* Y_{Sym^n X} \simeq Y_n$.

□

3.5 Remarks.

a. By replacing the category of *spaces* with their Cartesian product, with vector spaces with their tensor product one arrives to the definition of factorizing vector spaces. These are called *locally constant* factorization algebras and are studied by Lurie and his students [26].

b. By considering finitely generated quasi-coherent $\mathcal{O}$-modules instead, one is led to the notion of *factorization algebras* originally defined by Beilinson and Drinfeld [4]. They are particularly interesting from a representation theory perspective because their category (or rather unital factorization algebras) is equivalent to that of *chiral algebras*, a global notion, whose local counterpart is that of a *vertex algebra*.

c. Observe that an ind-space over X^I is the same that an S_n-invariant ind-space over X^n ($n = |I|$) i.e. $Y_n \to X^n$ such that $\sigma^* Y_n = Y_n$ for all permutations $\sigma \in S_n$. And it follows from the equivalence that to define a factorizing space it is sufficient to define a S_n-invariant Y_Δ for each diagonal $\Delta \subset X^n \times X$.

3.6 Definition. A factorizing space is *commutative* if the isomorphisms 3.1 are defined for every pair of divisors (not just disjoint).

3.7. The affine Grassmannian.
Let $x \in X$ and consider $\hat{\mathcal{O}}_x$ the completed local ring of x and $\mathcal{K}_x$ its field of fractions, the *disc* is

$$D_x := Spec(\hat{\mathcal{O}}_x) \simeq Spec\ \mathbb{C}[[t]]$$

and the puntured disc

$$D_x^{\times} := Spec(\mathcal{K}_x) \simeq Spec\ \mathbb{C}((t)).$$

Let G be a complex algebraic group, the *affine Grassmannian* is the quotient

$$Gr_G = G((t))/G[[t]].$$

Given a point $x \in X$, the fiber over x of the *Beilinson and Drinfeld Grassmannian* is the quotient

$$Gr_{G,x} = G(D_x^{\times})/G(D_x).$$

Observe that the affine Grassmannian may be viewed as the space of G–bundles on the disc D_x together with a trivialization on $D_x^{\times}$. By a lemma given by Beauville-Laszlo [3], we obtain that

$$Gr_{G,x} \simeq \{(\mathcal{L}, \phi) | \mathcal{L} \text{ is a } G - \text{torsor on } X \text{ and } \phi \text{ is a trivialization of } \mathcal{L}|_{X \setminus x}\}.$$

If $S \subset Z \times X$ a divisor, define the functor:

$$Sch_Z \xrightarrow{Gr_{G,S}} Sets$$

$$Gr_{G,S}(Y) = \{(\mathcal{L}, \phi) | \mathcal{L} \text{ is a } G\text{–torsor on } Y \times X,\ \phi \text{ is a trivialization on } Y \times X \setminus y^*(S)\}.$$

When G is reductive, the Grassmannians are ind-representable (ie, they are representable as a union of a directed system $Z_1 \subset Z_2 \subset ...$ of projective, finite dimensional schemes Z_i, see for example [28]). The tensor product of torsors gives the factorizing structure.
We will be interested in the case of G being a torus T. Observe that in this case we have

$$Gr_{G,S}(Y) = Div(X, \Gamma)_S(Y)$$

where $T := Spec(\mathbb{C}[\Gamma]) \simeq \mathbb{G}_m \otimes \Gamma^{\vee}$ and $Div(X, \Gamma)_S(Y)$ is the set of divisors $D \subset Y \times X$ contained in the pullback of S tensorized (over $\mathbb{Z}$) by Γ. The affine Grassmannian is a *commutative* factorizing space.

3.8. Factorizing spaces associated to loop spaces.
Let M be affine of finite type, consider the loop space $\tilde{\mathcal{L}}(M)$ defined in example 2.69. By definition, there is an action of $Aut\ \mathbb{C}[[t]]$ on $\tilde{\mathcal{L}}(M)$. Explicitly, we want for each $\mathbb{C}$-algebra A, an action

$$Aut\ \mathbb{C}[[t]] \to Aut(\lambda_X(A), \lambda_X(A)).$$

An automorphism of $\mathbb{C}[[t]]$ is determined by its image on t, i.e. an element $\phi(t) \in \mathbb{C}[[t]]$ and, since it is invertible, if $\phi(t) = \sum a_i t^i$, then $a_0 = 0$ and $a_i \neq 0$. Then, the action is defined composing with:

$$\begin{aligned} A((t)) &\longrightarrow A((t)) \\ \sum a_i t^i &\mapsto \sum a_i(\phi(t))^i \end{aligned}.$$

Now, let X be a smooth curve over $\mathbb{C}$ and $\hat{X} \to X$ be the scheme of pairs (x, t_x) where t_x is a coordinate arround x i.e. $t_x : \mathscr{O}_x \xrightarrow{\sim} \mathbb{C}[[t]]$. Then, the group $Aut\ \mathbb{C}[[t]]$ also acts on $\hat{X}$, an element $\rho \in Aut\ \mathbb{C}[[t]]$ sends (x, t_x) to $(x, \rho \circ t_x)$. Define:

$$\tilde{\mathcal{L}}(M)_X := \hat{X} \times_{Aut\ \mathbb{C}[[t]]} \tilde{\mathcal{L}}(Y).$$

The Ind-scheme $\tilde{\mathcal{L}}(M)_X$ represents the following functor. Let $f : S \to X$, consider $\hat{\mathcal{O}}_f$ the sheaf of functions on the formal neighborhood of $\Gamma(f)$, the graph of f and $\mathcal{K}_f$ the sheaf of functions on the punctured formal neighborhood of $\Gamma(f)$. Then, $\tilde{\mathcal{L}}(M)_X$ represents the functor (see [24] for the details):

$$\tilde{\lambda}_{M,X}(S) = \{(f, \rho)/f : S \to X, \rho : (\Gamma(f), \mathcal{K}_f) \to M \text{ of locally ringed spaces}\}.$$

Now, we will define a factorizing space analogously. Let $f_I : S \to X^I$, consider $\Gamma(f_I) \subset S \times X$ the union of the graphs of the coordinates of f_I. Let $\hat{\mathcal{O}}_{f_I}$ the sheaf of functions on the formal neighborhood of $\Gamma(f)$, and $\mathcal{K}_{f_I}$ the sheaf of functions on the punctured formal neighborhood of $\Gamma(f)$. Then, the functors

$$\tilde{\lambda}_{M,X^I}(S) = \{(f_I, \rho)/f_I : S \to X^I, \rho : (\Gamma(f_I), \mathcal{K}_{f_I}) \to M \text{ of locally ringed spaces}\}$$

are represented by Ind-schemes over X^I, $\tilde{\mathcal{L}}(M)_{X^I}$ and, by construction of the functors $\tilde{\lambda}_{M,X^I}$, this collection form a factorizating space over X.

3.9 Factorizing line bundles

In this section we define and expand on the notion of *factorizing line bundles* as described informally in 1.3. We start with a brief recall of Grothendieck's interpretation of central extensions as principal bundles in section 3.10. We then describe factorizing line bundles in the special case where the base factorization space is $\mathcal{D}iv(X, \Gamma)$ in section 3.11

3.10 Central extensions of groups. Let G be a group and A be an Abelian group. Consider a central extension of the form

$$0 \to A \overset{\iota}{\hookrightarrow} \hat{G} \overset{\pi}{\to} G \to 1.$$

G may not be commutative and we use multiplicative notation for it, both 0 and 1 in the above sequence denote the trivial group. For each $g \in G$, the set $L_g := \pi^{-1}(g)$ is an A-torsor. The multiplication map in $\hat{G}$ restricts to a map

$$L_g \times L_h \to L_{gh}.$$

By the associativity in $\hat{G}$ and the fact that $\pi\iota(A) = 1 \in G$, this map factors through a map

$$L_g \times_A L_h \to L_{gh}.$$

Recall that since A is commutative, the category of A-torsors is a Picard groupoid, in particular, this map is a map of A-torsors, and therefore an isomorphism

$$\pi_{g,h} : L_g \otimes L_h \xrightarrow{\sim} L_{gh}. \tag{3.2}$$

Associativity of $\hat{G}$ implies the associativity of the isomorphisms (3.2). That is the following diagram commutes:

$$\begin{tikzcd} L_f \otimes L_g \otimes L_h \arrow[r, "\pi_{f,g}\otimes 1"] \arrow[d, "1\otimes\pi_{g,h}"'] & L_{fg} \otimes L_h \arrow[d, "\pi_{fg,h}"] \\ L_f \otimes L_{gh} \arrow[r, "\pi_{f,gh}"'] & L_{fgh} \end{tikzcd} \tag{3.3}$$

If in addition $\hat{G}$ (and therefore G) is commutative, then the isomorphisms (3.2) are also symmetric, namely the following diagram commutes

$$\begin{tikzcd} L_f \otimes L_g \arrow[rr, "\sigma"] \arrow[dr, "\pi_{f,g}"'] & & L_g \otimes L_f \arrow[dl, "\pi_{g,f}"] \\ & L_{fg} = L_{gf} & \end{tikzcd} \tag{3.4}$$

Conversely, given the A-torsors L_g and the isomorphisms (3.2) satisfying (3.3) we obtain a central extension $\hat{G}$ of G by A. Indeed we set $\hat{G} = \amalg_G L_g$. For $a \in L_g$ and $b \in L_h$ we set

$$a \cdot b := \pi_{f,g}(a \otimes b).$$

By (3.3) this multiplication is associative. The map $\pi : \hat{G} \to G$ sending $L_g \to g \in G$ is a group homomorphism and $A = L_1$ is its kernel. If in addition G is multiplicative and $\pi_{f,g}$ satisfies (3.4), then $\hat{G}$ is also commutative.

3.11 Definition of factorizing line bundle. Consider the functor that assigns to each quasi-compact scheme Z the group of Cartier divisors of $Z \times X$ flat, finite and proper over Z. As above, this functor is ind-representable (for each degree d is represented by the symmetric power $Sym^d(X)$, see appendix B) and is a sheaf with respect to the flat topology, denote it $\mathcal{D}iv(X)$. Let Γ be a lattice, denote by $\mathcal{D}iv(X, \Gamma)$ the sheaf given by $\mathcal{D}iv(X,\Gamma)(Z) := \mathcal{D}iv(X)(Z) \otimes \Gamma$.

Definition. A *factorizing line bundle* over $\mathcal{D}iv(X,\Gamma)$ is the datum of:
- ⋄ for each Z and $D \in \mathcal{D}iv(X,\Gamma)(Z)$ a line bundle λ_D over Z
- ⋄ given $D, D' \in \mathcal{D}iv(X)(Z)$ disjoint divisors an isomorphism

$$\lambda_D \otimes \lambda_{D'} \xrightarrow{\sim} \lambda_{D+D'} \tag{3.5}$$

such that, given disjoint $D, D', D'' \in \mathcal{D}iv(X)(Z)$, the diagrams:

a.

$$\begin{tikzcd} \lambda_D \otimes \lambda_{D'} \otimes \lambda_{D''} \arrow[r] \arrow[d] & \lambda_{D+D'} \otimes \lambda_{D''} \arrow[d] \\ \lambda_D \otimes \lambda_{D'+D''} \arrow[r] & \lambda_{D+D'+D''} \end{tikzcd} \tag{3.6}$$

b.

$$\begin{tikzcd} \lambda_D \otimes \lambda_{D'} \arrow[rr, "\sigma"] \arrow[dr] & & \lambda_{D'} \otimes \lambda_D \arrow[dl] \\ & \lambda_{D+D} & \end{tikzcd} \tag{3.7}$$

are commutative. All compatible with base change.
Denote $\mathcal{P}ic^f(X,\Gamma)$ the category of factorizing line bundles.

3.12. Observe that if 3.5 is defined for all pairs of divisors, not only disjoint, the collection of all λ_D gives rise to a group central extension:

$$1 \to \mathbb{G}_m \to \lambda \to \mathcal{D}iv(X,\Gamma) \to 0$$

as described in 3.10. We call it a *commutative* factorizing line bundle. There is an equivalence of categories:

$$\{\text{commutative factorizing line bundles over } \mathcal{D}iv(X,\Gamma)\} \to \{T\text{-torsors over } X\}$$

where $T = \mathbb{G}_m \otimes_{\mathbb{Z}} \Gamma^\vee$. Let $\{\lambda\}$ be a commutative facctorizing line bundle. The equivalence is constructed in the following way. An isomorphism $\mathbb{G}_m^{\otimes n} \to T$ given by $\gamma_i^\vee \in \Gamma^\vee$ induces an isomorphism $\phi : \prod_i H^1(X,\mathbb{G}_m) \to H^1(X,T)$. Then, we assign to $\{\lambda\}$ the T-torsor associated to $\phi(\lambda_{\Delta\otimes\gamma_i})$, where γ_i is the dual of $\gamma_i^\vee$, $\Delta \subset X^2$ is the diagonal. Now, consider a T-torsor $\mathcal{L}$ and $\gamma \in \Gamma$. Denote $\mathcal{L}^\gamma$ the image of $\mathcal{L}$ through the morphism $H^1(X,T) \to H^1(X,\mathbb{G}_m)$ induced by γ. The inverse functor is given by:

$$\lambda_{D\otimes\gamma} := det(p^D_{Z*}(p_X^*(\mathcal{L}^\gamma)|_D))$$

where p^D_Z, p_X denote the following projections:

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "p^D_Z"'] & Z \times X \arrow[d] \arrow[dr, "p_X"] & \\ & Z & X. \end{tikzcd}$$

Since D flat, finite and proper over Z, the line bundle $det(p^D_{Z*}(p^*_X(\mathcal{L}^\gamma)|_D))$ is well defined. Note $\mathcal{L}^\gamma_Z := p^*_X(\mathcal{L}^\gamma)$, then,

$$\begin{aligned} det(p^{D+D'}_{Z*}(\mathcal{L}^\gamma_Z|_{D+D'})) &\simeq det(p^D_{Z*}(\mathcal{L}^\gamma_Z|_D)) \oplus p^{D'}_{Z*}(\mathcal{L}^\gamma_Z|_{D'})) \\ &\simeq det(p^D_{Z*}(\mathcal{L}^\gamma_Z|_D)) \otimes det(p^{D'}_{Z*}(\mathcal{L}^\gamma_Z|_{D'})). \end{aligned}$$

In [4], the authors prove that the category of factorizing line bundles over $\mathcal{D}iv(X,\Gamma)$ is equivalent to the category of θ-data, defined below.

3.13 Definition. A *θ-datum* on X is a triple (λ, κ, c), where $\kappa : \Gamma \times \Gamma \to \mathbb{Z}$ is a symmetric bilinear form, λ is a rule that assigns to each $\gamma \in \Gamma$ a line bundle λ^γ on X, and for each pair γ_1, $\gamma_2 \in \Gamma$, an isomorphism

$$c^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \to \lambda^{\gamma_1+\gamma_2} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)}$$

such that the following diagrams commute:

$$\begin{tikzcd} \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \lambda^{\gamma_3} \arrow[r, "c^{\gamma_1,\gamma_2}\otimes Id"] \arrow[d, "Id\otimes c^{\gamma_2,\gamma_3}"] & \lambda^{\gamma_1+\gamma_2} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)} \otimes \lambda^{\gamma_3} \arrow[d, "c^{\gamma_1+\gamma_2,\gamma_3}"] \\ \lambda^{\gamma_1} \otimes \lambda^{\gamma_2+\gamma_3} \otimes \omega_X^{\kappa(\gamma_2,\gamma_3)} \arrow[r] & \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \lambda^{\gamma_3} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)+\kappa(\gamma_1,\gamma_3)+\kappa(\gamma_2,\gamma_3)} \end{tikzcd} \tag{3.8}$$

$$\begin{tikzcd} \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \arrow[r, "c^{\gamma_1,\gamma_2}"] \arrow[d] & \lambda^{\gamma_1+\gamma_2} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)} \\ \lambda^{\gamma_2} \otimes \lambda^{\gamma_1} \arrow[ur, "c^{\gamma_2,\gamma_1}"'] & \end{tikzcd} \tag{3.9}$$

A *morphism* $\phi : (\lambda, \kappa, c) \to (\lambda', \kappa, c')$ is for each $\gamma \in \Gamma$, a morphism $\phi^\gamma : \lambda^\gamma \to \lambda'^\gamma$ such that $\phi^{\gamma_1+\gamma_2} \circ c^{\gamma_1,\gamma_2} = c'^{\gamma_1,\gamma_2} \circ (\phi^{\gamma_1} \otimes \phi^{\gamma_2})$, and $Hom((\lambda, \kappa, c), (\lambda', \kappa', c') = \emptyset$ if $\kappa \neq \kappa'$. Denote this category $\mathcal{P}^\theta(X,\Gamma)$. It is a Picard groupoid with $(\lambda, \kappa, c) \otimes (\lambda', \kappa, c') = (\lambda \otimes \lambda', \kappa + \kappa', c \otimes c')$.

3.14. Denote by $\mathcal{P}^\theta(X,\Gamma)^\kappa$ the category of θ-datum with κ fixed. The idea of the equivalence is the following. Consider the diagram:

$$\begin{tikzcd} \{\text{commutative f. line bundles over } \mathcal{D}iv(X,\Gamma)\} \simeq T-\text{torsors} \arrow[r, "\sim", "①"'] \arrow[d, hook] & \mathcal{P}^\theta(X,\Gamma)^0 \arrow[d, hook] \\ \mathcal{P}ic^f(X,\Gamma) \arrow[r, "\sim", "②"'] & \mathcal{P}^\theta(X,\Gamma) \end{tikzcd} \tag{3.10}$$

We will show a fully faithful essentially surjective morphism ② that restricts to ①. The equivalence ② is given by:

$$\lambda \mapsto \lambda^\gamma := \lambda_{\Delta\otimes\gamma}.$$

First, we see that it restricts to an equivalence ①. As before, its inverse is given by:

$$\lambda \mapsto \phi(\lambda^{\gamma_i})$$

where $\phi : \prod_i H^1(X, \mathbb{G}_m) \to H^1(X, T)$ is induced by an isomorphism $\mathbb{G}_m^{\otimes n} \to T$ given by $\gamma_i^\vee \in \Gamma^\vee$.

3.15. Now, let λ be a factorizing line bundle over $\mathcal{D}iv(X, \Gamma)$. Denote by $\Delta_{i3} \subset X^2 \times X$ the diagonal $x_i = x_3$. By 3.5, there is an isomorphism of line bundles:

$$\lambda_{\Delta_{13}\otimes\gamma_1} \otimes \lambda_{\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta} \xrightarrow{\sim} \lambda_{\Delta_{13}\otimes\gamma_1+\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta}. \tag{3.11}$$

Then, there exists $n = \kappa(\gamma_1, \gamma_2)$ such that

$$\lambda_{\Delta_{13}\otimes\gamma_1} \otimes \lambda_{\Delta_{23}\otimes\gamma_2} \xrightarrow{\sim} \lambda_{\Delta_{13}\otimes\gamma_1+\Delta_{23}\otimes\gamma_2} \otimes \mathcal{O}_{X^2}(-n\Delta) \tag{3.12}$$

Restricting to the diagonal, we obtain the isomorphism:

$$\lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \to \lambda^{\gamma_1+\gamma_2} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)}.$$

Its clear that it is symmetric. Let's see that it is bilinear. First, some notation. For $1 \leq i \leq n-1$, $\Delta_{in} \subset X^{n-1} \times X$ are the diagonals $x_i = x_n$, $p_i^{n-1} : X^{n-1} \to X$ the i-th projections and $\Delta^i : X \to X^i$ the diagonal map. Because of the compatibility with base change we have:

$$\lambda_{\Delta_{14}\otimes\gamma_1} \otimes \lambda_{\Delta_{24}\otimes\gamma_2} \otimes \lambda_{\Delta_{34}\otimes\gamma_3} \xrightarrow{\sim} \lambda_{\Delta_{14}\otimes\gamma_1+\Delta_{24}\otimes\gamma_2+\Delta_{34}\otimes\gamma_3} \otimes \mathcal{O}_{X^3}(-\textstyle\sum_{i<j} \kappa(\gamma_i, \gamma_j)\Delta_{ij}). \tag{3.13}$$

Pulling back through Δ^3, we obtain

$$\lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \lambda^{\gamma_3} \simeq \lambda^{\gamma_1+\gamma_2+\gamma_3} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)+\kappa(\gamma_2,\gamma_3)+\kappa(\gamma_1,\gamma_3)}.$$

But, on the other hand,

$$\lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \otimes \lambda^{\gamma_3} \simeq \lambda^{\gamma_1+\gamma_2}\lambda^{\gamma_3} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)} \simeq \lambda^{\gamma_1+\gamma_2+\gamma_3} \otimes \omega_X^{\kappa(\gamma_1,\gamma_2)+\kappa(\gamma_1+\gamma_2,\gamma_3)}.$$

Then, we have ② well defined.

3.16. Since each $\mathcal{P}^\theta(X, \Gamma)^\kappa$ is a $\mathcal{P}^\theta(X, \Gamma)^0$-torsor, to see that ② is essentially surjective is sufficient to give for each κ a factorizing line bundle associated to κ.
First, consider the case $\Gamma = \mathbb{Z}$. For $D \in \mathcal{D}iv(X)(Z)$, let $D' \in \mathcal{D}iv(X)(Z)$ effective such that $D + D' \geq 0$ and define

$$\lambda_D := det(p_{Z*}\mathcal{O}_{X\times Z}(D)/\, O_{X\times Z}(-D')) \otimes det(p_{Z*}\mathcal{O}_{D'})^{-1}.$$

The bilinear form associated is the product in $\mathbb{Z}$. In general, any $\kappa(\gamma_1, \gamma_2) = \sum m_i\alpha_i^\vee(\gamma_1).\alpha_i^\vee(\gamma_2)$ for some $\alpha_i^\vee : \Gamma \to \mathbb{Z}$ (this follows from the computation $a_i.b_j + a_jb_i = (a_i + a_j)(b_i + b_j) - a_ib_i - a_jb_j$). Then, $\lambda_{D\otimes\gamma} := \bigotimes \lambda_{\alpha_i^\vee(\gamma)D}^{\otimes m_i}$ is the factorizing line bundle over $\mathcal{D}iv(X, \Gamma)$ we were looking for.

4 Vertex Algebras

In this section we recall the basic notions about vertex algebras and their connection with factorization structures. We follow [9] and [20]. We start by defining quantum fields, locality and vertex algebras in sections 4.1–4.3. We give the examples of affine Kac-Moody and Virasoro vertex algebras in 4.4–4.6. We then give an equivalent definition of vertex algebras introducing *state-field correspondence* in 4.8. We define morphisms of vertex algebras in 4.9 and *n-point functions* in section 4.12. We then define modules over a vertex algebra in 4.13 .

4.1. Fields Let V be a vector space. A *formal distribution* with values in $End(V)$ is a formal power series $a(z) \in End(V)[[z, z^{-1}]]$. We write

$$a(z) = \sum_{n \in \mathbb{Z}} z^{-1-n} a_{(n)}, \qquad a_{(n)} \in End(V).$$

The coefficients $a_{(n)}$ are called the *Fourier modes* of $a(z)$. Similarly we define *two-variable* $End(V)$-valued formal distributions as formal series

$$a(z, w) \in End(V)[[z, z^{-1}, w, w^{-1}]].$$

Notice that unlike Laurent series with coefficients in $End(V)$, the space of formal distributions with values in $End(V)$ is not a $\mathbb{C}((z))$-module. However, we can still multiply a formal distribution with values in $End(V)$ by a Laurent polynomial, that is, an element of $\mathbb{C}[z, z^{-1}]$, and in particular, by a polynomial. Similarly, the space of two-variable $End(V)$-valued formal distributions is not a module over $\mathbb{C}((z, w))$, but we can multiply any such two-variable distribution by a Laurent polynomial in two variables, that is, an element of $\mathbb{C}[z, z^{-1}, w, w^{-1}]$.

A formal distribution $a(z)$ is called a *quantum-field* on V if we have $a(z) \in Hom(V, V((z)))$. That is, for each $v \in V$, $a_{(n)}(v) = 0$ for $n \gg 0$. Formal distributions cannot be composed, however given a pair of fields $a(z), b(z)$, their composition makes sense as a two-variable $End(V)$-valued formal distribution, namely

$$a(z)b(w) \in End(V)[[z, z^{-1}, w, w^{-1}]],$$

is well defined. Similarly $b(w)a(z)$ is well defined and therefore their commutator $[a(z), b(w)]$ is a well defined formal distribution in two variables with values in $End(V)$. By the discussion above, it makes sense to multiply this commutator by any Laurent polynomial in two variables, in particular, by a polynomial in z and w.

A pair of fields $a(z), b(z)$ is said to be a *local pair* if there exists $n \in \mathbb{N}$ such that

$$(z - w)^n [a(z), b(w)] := (z - w)^n \big(a(z)b(w) - b(w)a(z)\big) = 0.$$

Here the LHS is viewed as a two-variable formal formal distribution with values in $End(V)$.

4.2. More generally, a two-variable $End(V)$-valued formal distribution $a(z,w)$ is called *local* if there exists $n \in \mathbb{N}$ such that

$$(z-w)^n a(z,w) = 0.$$

The space of *local* two variable formal distributions is invariant under taking formal partial derivatives. Indeed we have

$$(z-w)^{n+1}\partial_z a(z,w) = \partial_z (z-w)^{n+1} a(z,w) - (n+1)(z-w)^n a(z,w) = 0,$$

and similarly for $\partial_w a(z,w)$.

The typical example of a local, two-variable formal distribution (in this case one can take it with coefficients in $End(V)$ for any vector space V by simply multiplying by the identity operator) is given by the *formal delta function* defined as

$$\delta(z,w) = \sum_{n\in\mathbb{Z}} z^n w^{-1-n}.$$

One verifies immediately that

$$(z-w)\delta(z,w) = 0.$$

We can now give the first definition of vertex algebras following [9]:

4.3 Definition. A *vertex algebra* is the data of a vector space V (the space of states), an element $|0\rangle \in V$ (the vacuum), a morphism $T \in End(V)$ (the translation morphism) and a collection of quantum fields $\mathcal{F}$. Verifying:

a. the vacuum axiom: $T(|0\rangle) = 0$

b. translation covariance: $[T, a(z)] = \partial_z a(z)$ for all $a \in \mathcal{F}$

c. locality: the fields of $\mathcal{F}$ are pairwise local

d. completeness: $V = \langle a^1_{(n_1)} a^2_{(n_2)} \ldots a^j_{(n_j)} |0\rangle\rangle_{a^i\in\mathcal{F}}$, $n_i \in \mathbb{Z}_{<0}$

4.4 Example. Let $\mathfrak{g}$ be a finite dimensional Lie algebra endowed with an invariant, symmetric bilinear form $\kappa(.,.)$. The *affine Kac-Moody* algebra associated to this pair $\mathfrak{g}, \kappa$ is the central extension:

$$0 \to \mathbb{C}\cdot K \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

with the bracket defined by:

$$[a\otimes f, b\otimes g] := [a,b]\otimes fg - K\kappa(a,b)\int f dg \quad \text{for } a,b \in \mathfrak{g};\ f,g \in \mathbb{C}((t)),$$

and K is central. If we define $a_n := a \otimes t^n$ then the bracket is given by

$$[a_n, b_m] = [a,b]_{m+n} + m\kappa(a,b)\delta_{m,-n}K. \tag{4.1}$$

Let $\hat{\mathfrak{g}}_+ := \mathfrak{g}[[t]] \oplus \mathbb{C} \cdot K$ and consider $\mathbb{C} \cdot |0\rangle$, the one-dimensional representation of $\hat{\mathfrak{g}}_+$ on which K acts as the identity and $\mathfrak{g}[[t]]$ by 0. The *Fock representation*

$$V^\kappa(\mathfrak{g}) = Ind_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}|0\rangle = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+)} \mathbb{C}|0\rangle$$

is a vertex algebra.

Indeed for each $a \in \mathfrak{g}$ we have a quantum field with values in V given by

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-1-n}.$$

The fact that these are indeed fields and the completeness axiom d) in Definition 4.3 are trivially verified. To check locality we compute directly using (4.1) the commutator

$$[a(z), b(w)] = [a, b](w)\delta(z, w) + \kappa(a, b)K\partial_w \delta(z, w), \qquad a, b \in \mathfrak{g},$$

where $\delta(z, w)$ is the formal delta function defined in 4.2. It follows from the results in 4.2 that the pair $a(z), b(z)$ is local, since

$$(z - w)^2 [a(z), b(w)] = 0. \tag{4.2}$$

One introduces the translation operator T by defining $T|0\rangle = 0$ (thus the vacuum axiom is trivially satisfied) and imposing the relation

$$[T, a_{(n)}] = -na_{(n-1)}, \qquad a \in \mathfrak{g},\ n \in \mathbb{Z}. \tag{4.3}$$

By the PBW theorem, the $\hat{\mathfrak{g}}$-representation $V^\kappa(\mathfrak{g})$ comes with an increasing filtration, hence by induction we see that $T \in End(V)$ is well defined by (4.3). Translation covariance is immediate to check.

4.5 Example. A particular example of the above construction is the *Heisenberg* vertex algebra. In this case we consider $\mathfrak{g} = \langle \alpha \rangle_{\mathbb{C}}$ a one dimensional Lie algebra. We take as κ the bilinear form defined by $\kappa(\alpha, \alpha) = 1$, and denote this form by 1. By the Poincaré-Birkhoff-Witt theorem, as a vector space:

$$\pi := V^1(\mathfrak{g}) \simeq \mathbb{C}[\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, ...].$$

Through this identification, α_n acts by:

$$\begin{cases} \alpha_n f = n \frac{\partial}{\partial \alpha_{-n}} f & \text{if } n > 0 \\ \alpha_0 f = 0 & \\ \alpha_n f = \alpha_n f & \text{if } n < 0 \end{cases}$$

The operators α_n for $n < 0$ are called the "creation operators" since acting on the vacuum, they generate the full space. Define $\alpha(z) = \sum_{n \in \mathbb{C}} \alpha_n z^{-n-1}$. The structure of vertex algebra is given by:

a. the vacuum is $|0\rangle := 1$

b. the translation operator is defined by $T(|0\rangle) := 0$ and

$$T = \sum_{n>0} -n\alpha_{-n-1}\frac{\partial}{\partial\alpha_{-n}}.$$

Notice that this sum is finite acting on any polynomial.

c. The space of generating fields is defined by $\mathcal{F} := \{\alpha(z), K\}$

Translation covariance is evident. We compute explicitly (4.2):

$$\begin{aligned}
[\alpha(z), \alpha(w)] &= \sum_{n,m\in\mathbb{Z}} [\alpha_m, \alpha_n] z^{-m-1} w^{-n-1} \\
&= \sum_{n,m\in\mathbb{Z}} ([\alpha, \alpha]_{m+n} + m\delta_{m,-n}(\alpha, \alpha)K) z^{-m-1} w^{-n-1} \\
&= K \sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m-1} \\
&= K\partial_w \sum_{m\in\mathbb{Z}} z^{-m-1} w^m = K\partial_w \delta(z, w).
\end{aligned}$$

4.6 Example. The *Virasoro Lie algebra* is the central extension

$$0 \to \mathbb{C}.C \to \mathcal{V}ir \to \mathcal{D}er\mathbb{C}((t)) \to 0$$

where $\mathcal{D}er\ \mathbb{C}((t))$ is the Lie algebra of derivations of $\mathbb{C}\ ((t))$. As a vector space $\mathcal{D}er\mathbb{C}((t)) = span\{t^n\partial_t\}_{n\in\mathbb{Z}}$, we define $L_n := -t^{n+1}\partial_t$. The bracket is given by:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C$$

and C is central. Let

$$\mathcal{V}ir_+ := \mathbb{C}[[t]]\partial_t \oplus \mathbb{C}.C$$

and consider $\mathbb{C}|0\rangle$ the one dimensional representation of $\mathcal{V}ir_+$ where $\mathbb{C}[[t]]\partial_t$ acts trivially and C acts by multiplication by $c \in \mathbb{C}$. The *Virasoro Vertex Algebra with central charge c* is the induced representation:

$$\mathcal{V}ir_c = Ind_{\mathcal{V}ir_+}^{\mathcal{V}ir}\mathbb{C}|0\rangle$$

with

a. $T = L_{-1}$

b. $\mathcal{F} = \{L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, C\}$

A computation shows:

$$[L(z), L(w)] = 2L(w)\partial_w \sum_{m\in\mathbb{Z}} z^{-m-1}w^m + (\partial_w L(w)) \sum_{m\in\mathbb{Z}} z^{-m-1}w^m + \frac{C}{12}\partial_w^3 \sum_{m\in\mathbb{Z}} z^{-m-1}w^m$$

then

$$(z-w)^4[L(z), L(w)] = 0.$$

4.7. Now, we will give an alternative definition of vertex algebras. For an outline of the equivalence between both definitions we refer the reader to [9]. Let $(V, \mathcal{F}, T, |0\rangle)$ be a vertex algebra, it is possible to enlarge $\mathcal{F}$ so that the map:

$$\begin{aligned} \mathcal{F} &\to V \\ a(z) &\mapsto a_{-1}|0\rangle \end{aligned}$$

becomes bijective. We obtain a state-field correspondence:

$$\begin{aligned} V &\to \mathcal{F} \\ a &\mapsto Y(a,z). \end{aligned}$$

The following definition is based on this correspondence.

4.8 Theorem. A vertex algebra as defined in 4.3 is equivalent to the datum of a tuple $(V, T, Y, |0\rangle)$:

a. a pointed vector space $|0\rangle \in V$
b. T an endomorphism of V
c. a linear morphism $Y : V \to End(V)[[z, z^{-1}]]$

verifying:

Vacuum axioms **a**. $Y(|0\rangle, z) = Id_V$
b. $Y(a,z)|0\rangle \in V[[z]]$
c. $Y(a,z)|0\rangle|_{z=0} = a$

Translation axioms **a**. $[T, Y(a,z)] = \partial_z Y(a,z)$
b. $T(|0\rangle) = 0$

Locality For $a, b \in V$, there exists $n \in \mathbb{N}$ such that

$$(z-w)^n [Y(a,z), Y(b,w)] = 0.$$

We denote

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$$

With this definition we can define the *category of vertex algebras* by definining

4.9 Definition. A *morphism of vertex algebras* $\phi : (V, T, Y, |0\rangle) \to (V', T', Y', |0\rangle')$ is a morphism of vector spaces $\phi : V \to V'$ such that:

a. $\phi(|0\rangle) = |0\rangle'$
b. $\phi \circ T = T' \circ \phi$
c. $\phi(Y(a,z)b) = Y(\phi(a), z)(\phi(b))$

4.10 Example. The simplest example with this definition is a *commutative* vertex algebra. Let V be a unital commutative algebra with a derivation T. Put $|0\rangle := 1$ and

$$Y(a,z)b := e^{zT}(a).b = \sum_{n\geq 0} \frac{z^n}{n!} T^n(a).b$$

If $T = 0$, is just the structure of a unital commutative algebra $Y(a,z)b = a.b$. Conversely, if V is a vertex algebra such that

$$[Y(a,z), Y(b,w)] = 0$$

for all $a, b \in V$ then $a.b := a_{(-1)}b$ endows V with a structure of commutative algebra, T is a derivation and these constructions are mutually inverse.

4.11. To give the state-field correspondence of the previous examples, we need to define the *normally ordered product* of two fields $a(z), b(w) \in End[[z, z^{-1}]]$:

$$: a(z)b(w) : := a_+(z)b(w) + b(w)a_-(z)$$

where $a_+(w) = \sum_{n<0} a_{(n)} z^{-n-1}$, $a_-(w) = \sum_{n\geq 0} a_{(n)} z^{-n-1}$.
Observe that, in general, $a(w)b(w)c$ is not well defined while $: a(w)b(w) : c$ is.
Consider our first example, the vertex algebra $V^k(\mathfrak{g})$, then

$$\begin{aligned} &Y(\alpha_{j_1}\alpha_{j_2}...\alpha_{j_k}, z) := \\ &\qquad : \partial^{(-j_1-1)}\alpha(z) : \partial^{(-j_2-1)}\alpha(z) ... : \partial^{(-j_{k-1}-1)}\alpha(z)\partial^{(-j_k-1)}\alpha(z) : ... :: \end{aligned}$$

Observe that the normally ordered product is not associative, then the order above matters. It is not a simple computation to check that this definition verifies the axioms of a vertex algebra, it is a consequence of general results on fields such as Dong's Lemma and formulas known as OPE (operator product expansion). For the details, see [13] or [20].

4.12. n-point functions Let $(V, T, Y, |0\rangle)$ be a vertex algebra. Intuitively, a vertex algebra carries a factorizing structure in the following way (we refer the reader to [13], [4] for a rigorous exposition of this relation). As a consequence of locality and the translation covariance axiom we have the following result known as *associativity.* The fields $Y(a,z)Y(b,w)c$, $Y(b,w)Y(a,z)c$ and $Y(Y(a, z-w)b, w))c$ are images of the same object through the morphisms:

$$\begin{tikzcd} & V[[z]][[w]][z^{-1}, w^{-1}, (z-w)^{-1}] \arrow[dl, "i_{z,w}"'] \arrow[d, "i_{w,z}"] \arrow[dr, "i_{w,z-w}"] & \\ V((z))((w)) & V((w))((z)) & V((w))((z-w)). \end{tikzcd}$$

More generally, for each $v_1, ... v_n \in V$, $\phi \in V^*$ there exists a polynomial
$f \in \mathbb{C}[z_1, ..., z_n]$ such that

$$\langle \phi | Y(v_1, z_1) .... Y(v_n, z_n) |0\rangle \text{ and } \langle \phi | Y(v_{\sigma(1)}, z_{\sigma(1)}) .... Y(v_{\sigma(n)}, z_{\sigma(n)} |0\rangle$$

are expansions on the respective domains of

$$f_{v_1,...,v_n}(z_1, ..., z_n) = \frac{f(z_1, ..., z_n)}{\prod (z_i - z_j)^{n_{ij}}}$$

where the n_{ij} does not depend on ϕ . The n-point functions verify that the expansion of $f_{v_1,...,v_n}(z_1, ..., z_n)$ near the diagonal $z_i = z_j$ is

$$f_{Y(v_j,z_j-z_i)v_i,v_1,...,\hat{v_i},...,\hat{v_j}..,v_n}(z_j, z_1, ..., \hat{z_i}, ..., \hat{z_j}...., z_n)$$
$$:= \sum_{m\in\mathbb{Z}} f_{v_j(m)v_i,v_1,...,\hat{v_i},...,\hat{v_j}..,v_n}(z_j, z_1, ..., \hat{z_i}, ..., \hat{z_j}...., z_n)(z_j - z_i)^m.$$

That is, the n-point functions can be constructed from the $(n-1)$-point functions near the diagonals. Now, given points in a curve $x_1, ...x_n \in X$, choosing a formal coordinate z_i around x_i, can be associated an infinite dimensional bundle $\mathcal{V}_n$ on X^n such that the n-point functions are the matrix elements of its sections, with isomorphisms

$$\Delta^*_{x_i=x_j}\mathcal{V}_n \simeq \mathcal{V}_{n-1}$$
$$\mathcal{V}_2|_{X^2\setminus\Delta} \simeq \mathcal{V}_1 \boxtimes \mathcal{V}_1.$$

This bundle is defined in 4.16.

4.13 Definition. A *module over a vertex algebra* V is a vector space M together with a map $Y^M : V \to End(M)[[z, z^{-1}]]$ such that

a. $Y^M(a, z)m \in M((z))$

b. $Y^M(|0\rangle, z)m = m$

c. $Y^M(Y(a, z-w)b, w)m$, $Y^M(a, z)Y^M(b, w)m$ and $Y^M(b, w)Y^M(a, z)m$ are images of the same object through the morphisms:

$$\begin{tikzcd}
 & M[[z]][[w]][z^{-1}, w^{-1}, (z-w)^{-1}] \arrow[dl, "i_{z,w}"'] \arrow[d, "i_{w,z}"] \arrow[dr, "i_{w,z-w}"] & \\
M((z))((w)) & M((w))((z)) & M((w))((z-w)).
\end{tikzcd}$$

4.14 Example. Let Γ be an even lattice i.e. a free finite rank $\mathbb{Z}$-module with $(.,.) : \Gamma\times\Gamma \to \mathbb{Z}$ such that (α, α) is even and let $\mathfrak{h} = \Gamma\otimes_{\mathbb{Z}}\mathbb{C}$ be the commutative Lie algebra associated to it. Let $\mathbb{C}_\epsilon[\Gamma] \simeq \mathbb{C}[\Gamma]$ as a vector space, we denote its generators by e^α, $\alpha \in \Gamma$ and define a product:

$$e^\alpha.e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}$$

where

$$\epsilon : \Gamma \times \Gamma \to \{1, -1\}$$

is a cocycle satisfying $\epsilon(\alpha, \beta) = (-1)^{(\alpha,\beta)}\epsilon(\beta, \alpha)$ and $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 0$. Define

$$V_\Gamma := V^1(\mathfrak{h}) \otimes \mathbb{C}_\epsilon[\Gamma] \simeq \bigoplus_{\gamma\in\Gamma}\mathfrak{h}_\gamma = \bigoplus_{\gamma\in\Gamma} Ind^{\hat{\mathfrak{h}}}_{\hat{\mathfrak{h}}_+}\mathbb{C}e^\gamma$$

where $\mathfrak{h}_\gamma$ is the highest weight module generated by the highest weight vector e^γ, the action is given by:

$$\begin{cases} h_m e^\gamma = 0 & h \in \mathfrak{h},\ m > 0 \\ h_0 e^\gamma = (h,\gamma) e^\gamma & h \in \mathfrak{h} \\ C e^\gamma = 0 \end{cases}$$

Observe that $\mathfrak{h}_0$ is just $V^1(\mathfrak{h})$, again we have the generating fields $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-1-n}$ for $a \in \mathfrak{h}$ and V_Γ is a module for $V^1(\mathfrak{h})$. We can extend the map

$$Y : V^1(\mathfrak{h}) \to End(V_\Gamma)[[z, z^{-1}]]$$

to V_Γ. For each $\gamma \in \Gamma$, define

$$Y(|0\rangle \otimes e^\gamma) := \Gamma_\alpha(z) = \Gamma_\alpha(z) := e^\alpha z^{\alpha_0} exp(-\sum_{n<0} \frac{\alpha_n}{n} z^{-n}) exp(-\sum_{n>0} \frac{\alpha_n}{n} z^{-n}).$$

Then the vertex algebra structure is:

a. $T(a \otimes |0\rangle) = T_{V^1(\mathfrak{h})}(a) \otimes |0\rangle$, $T(|0\rangle \otimes e^\alpha) = t^{-1}\alpha \otimes e^\alpha$

b. $\mathcal{F} = \{a(z), \Gamma_\alpha(z)\}_{a\in\mathfrak{h},\alpha\in\Gamma}$

Some computations give the following formulas that will be useful in our exposition. Denote

$$\Gamma_{\alpha,\beta}(z,w) = e^{\alpha+\beta} z^{\alpha_0} w^{\beta_0} exp(-\sum_{n<0} \frac{\alpha_n}{n} z^{-n} + \frac{\beta_n}{n} w^{-n}) exp(-\sum_{n>0} \frac{\alpha_n}{n} z^{-n} + \frac{\beta_n}{n} w^{-n}) \tag{4.4}$$

then

$$\Gamma_\beta(w)\Gamma_\alpha(z) = \epsilon(\alpha,\beta) i_{z,w}(z-w)^{(\alpha,\beta)} \Gamma_{\alpha,\beta}(z,w) \tag{4.5}$$

and $\Gamma_\alpha, \Gamma_\beta$ is a local pair.

4.15. Summarizing, given a lattice Γ and a cocycle ϵ we have a structure of vertex algebra on the direct sum of the Fock modules $\mathfrak{h}_\gamma$. In [4], the authors prove that there is a correspondence between lattice vertex algebras and factorizing line bundles over $\mathcal{D}iv(X,\Gamma)$. We give a rough idea of how these objects are related. Heuristically, suppose that $\Gamma_\alpha(z)$ is a section of a line bundle λ_α over a curve X, then $\Gamma_\alpha(z)\Gamma_\beta(w)$ is a section of $\lambda_{\alpha,\beta} = \lambda_\alpha \boxtimes \lambda_\beta$ on X^2. Equation (**??**) tells us that, outside the diagonal, $\Gamma_{\alpha,\beta}(z,w)$ is also a section of $\lambda_{\alpha,\beta}$, but restricting this section to the diagonal

$$\Gamma_{\alpha,\beta}(z,z) = \Gamma_{\alpha+\beta}(z).$$

In addition, the order of the pole of $\Gamma_\alpha(z)\Gamma_\beta(w)\Gamma_{\alpha,\beta}(z,w)^{-1}$ on the diagonal is (α,β).

4.16 Vertex algebra bundle

Let X be a smooth curve. In the following we will associate to a vertex algebra V a infinite dimensional vector bundle over X. As in example 3.8, the vertex algebra bundle is the product $\mathcal{V} = \hat{X} \times_{Aut\ \mathbb{C}[[t]]} V$. We first describe sufficient conditions for the vertex algebra to be endowed with a natural action of $Aut\ \mathbb{C}[[t]]$, then we show how the structure of the vertex algebra is reflected.

4.17 Definition. A $\mathbb{Z}$-graded vertex algebra V is called *conformal*, of central charge $c \in \mathbb{C}$ if there is a vector $\omega \in V$ such that the Fourier coefficients $L_n := \omega_{(n+1)}$ satisfy the relations of the Virasoro algebra with central charge c, and $L_{-1} = T$, $L_0|_{V_n} = n.Id$

4.18 Examples. **a**. The Virasoro vertex algebra $\mathcal{V}ir_c$ is conformal, with central charge c and conformal vector

$$\omega = L_{-2}|0\rangle.$$

b. The Heisenberg vertex algebra has a family of conformal vectors. Given $\lambda \in \mathbb{C}$, the vector

$$\omega_\lambda = \frac{1}{2}\alpha_{-1}^2 + \lambda\alpha_{-2}$$

is conformal with central charge $c_\lambda = 1 - 12\lambda^2$.

c. Let $\mathfrak{g}$ be a simple Lie algebra. The vertex algebra $V^\kappa(\mathfrak{g})$ is conformal. Let κ_0 be the invariant bilinear form such that $\kappa_0(\alpha, \alpha) = 2$ for α the highest root of $\mathfrak{g}$, then $\kappa = k.\kappa_0$ for a $k \in \mathbb{C}$. Consider $\{a_1, ..., a_r\}$ a basis of $\mathfrak{g}$ and $\{a^1, ..., a^r\}$ its dual. Then $a_i(z) = \sum_{n\in\mathbb{Z}} a_{i,n} z^{-n-1}$, $a^i(z) = \sum_{n\in\mathbb{Z}} a^i_n z^{-n-1}$ and

$$\omega := \frac{1}{k + h^\vee} \sum_{i=1}^{r} a_{i,-1} a^i_{-1}|0\rangle$$

is conformal, where $h^\vee$ is the dual Coxeter number and $k \neq h^\vee$.

4.19. Let $\mathcal{O} = \mathbb{C}[[t]]$. Consider the subalgebras of the Virasoro Lie algebra

$$\mathcal{D}er_+\mathcal{O} := t^2\mathbb{C}[[t]]\partial_t \qquad \mathcal{D}er_0\mathcal{O} := t\mathbb{C}[[t]]\partial_t \qquad \mathcal{D}er\mathcal{O} := \mathbb{C}[[t]]\partial_t.$$

$\mathcal{D}er_0\ \mathcal{O}$. If V is a conformal vertex algebra, then the Virasoro Lie algebra acts on V. Then, the above Lie subalgebras also act on V. Since the algebra $\mathcal{D}er_0\ \mathcal{O}$ is the Lie algebra of the group $Aut\ \ \mathcal{O}$, to define an action of $Aut\ \mathcal{O}$ on V, we will exponentiate the action of its Lie algebra on V.

Since a continuous automorphism is determined by the image of t, the group of automorphisms can be characterized by

$$Aut\ \mathcal{O} = \{\phi(t) = a_1 t + a_2 t^2 + .../a_1 \neq 0\}.$$

Consider the subgroup

$$Aut_+ \ \mathcal{O} := \{\phi(t) = t + a_2 t^2 + ...\},$$

and observe that

$$Aut \ \mathcal{O} = Aut_+ \ \mathcal{O} \rtimes \mathbb{G}_m.$$

Now,

$$Lie \ \mathbb{G}_m \simeq \mathbb{C}t\partial_t \subset \mathcal{D}er_0\mathcal{O}$$

and an action of this algebra may be exponentiated if the action of $t\partial_t$ is diagonal with integer eigenvalues. On the other side, the exponential map

$$exp : \mathcal{D}er_+\mathcal{O} \to Aut_+ \ \mathcal{O}$$

is an isomorphims and an action of $\mathcal{D}er_+\mathcal{O}$ can be exponentiated if the exponential formula is a finite sum i.e. if for each $v \in V$, $\phi \in \mathcal{D}er_+\mathcal{O}$ there exists $n_0 \in \mathbb{N}$ such that $\phi^n.v = 0$ for $n \geq n_0$. Then, if V is a conformal vertex algebra, there is an action of $Aut \ \mathcal{O}$ on V.

4.20 Definition. Let X be a smooth curve and let V be a conformal vertex algebra. The *vertex algebra bundle* is:

$$\mathcal{V} = \hat{X} \times_{Aut \ \mathcal{O}} V.$$

If the $\mathbb{Z}$-gradation of the vertex algebra V satisfies $V = \bigoplus_{n \geq n_0} V_n$ then there is a filtration $V_{n_0} \subset V_{n_0} \oplus V_{n_0+1} \subset ...V$ by $Aut \ \mathcal{O}$-submodules of finite rank and

$$\mathcal{V}_X = \varinjlim \hat{X} \times_{Aut \ \mathcal{O}} V^{\leq m}$$

is the inductive limit of vector bundles. Define

$$\mathcal{V}_X^* = \varprojlim \hat{X} \times_{Aut \ \mathcal{O}} V^{\leq m,*}.$$

Now, let $j : X^2 \setminus \Delta \to X^2$ be the inclusion of the complement and define

$$\Delta_+\mathcal{V}_X := j_*j^*(\mathcal{O}_X \boxtimes \mathcal{V}_X) / \mathcal{O}_X \boxtimes \mathcal{V}_X.$$

Let $x \in X$ and denote by $\hat{\mathcal{O}}_x$ the completion of the local ring of x and let

$$D_x = Spec(\hat{\mathcal{O}}_x)$$

be the formal disc of x. Choose a formal coordinate at x i.e. an isomorphism $\hat{\mathcal{O}}_x \simeq \mathbb{C}[[z]]$ then,

$$\mathcal{V}_X|_{D_x} \simeq V$$

and there are isomorphisms of $\mathbb{C}[[z, w]]$-modules

$$j_*j^*(\mathcal{V}_X \boxtimes \mathcal{V}_X)|_{D_x^2} \simeq V \otimes V[[z, w]][(z - w)^{-1}]$$

$$\Delta_+(\mathcal{V}_X)|_{D_x^2} \simeq V[[z, w]][(z - w)^{-1}] / V[[z, w]].$$

4.21 Theorem. The map $\mathcal{Y}^2_x : j_*j^*(\mathcal{V}_X \boxtimes \mathcal{V}_X)|_{D^2_x} \to \Delta_+(\mathcal{V}_X)|_{D^2_x}$ defined by

$$\mathcal{Y}^2_x(f(z,w)A \otimes B) = f(z,w)Y(A, z-w)B$$

is independent of the coordinate and gives a morphism of bundles:

$$\mathcal{Y}^2 : j_*j^*(\mathcal{V}_X \boxtimes \mathcal{V}_X) \to \Delta_+(\mathcal{V}_X).$$

We refer to [13] for a proof and the details on how this bundle has a factorization structure (see [4]). Roughly, the sheaf $\mathcal{V}^2$ on X^2 is defined as

$$\mathcal{V}^2 := ker(\mathcal{Y}^2)$$

and the sheaves $\mathcal{V}^n$ on X^n are the intersections of the kernels of different ways of composing $\mathcal{Y}^2$. Observe that $j_*j^*(\mathcal{V}_X \boxtimes \mathcal{V}_X)|_{X^2\setminus\Delta} \simeq \mathcal{V}_X \boxtimes \mathcal{V}_X$ and $\Delta_+(\mathcal{V}_X)$ is supported on the diagonal, then

$$\mathcal{V}^2|_{X^2\setminus\Delta} \simeq \mathcal{V}_X \boxtimes \mathcal{V}_X|_{X^2\setminus\Delta}$$
$$\mathcal{V}^2|_{\Delta} \simeq \mathcal{V}.$$

5 Factorizing gerbes and dilogarithms

In this section we state and prove the main results of this thesis. We start by recalling the classical dilogarithm function in 5.1. In section 5.2 we expand on the description of Deligne's construction described initially in 1.14. In section 5.5–5.9 we recall the main result of [2] obtaining the dilogarithm function as 3-point functions on the sigma model with target certain nilmanifold. In sections 5.10–5.12 we define *factorizing gerbes*. We give an example of the determinantal gerbe as a factorizing gerbe in 5.13. In section 5.14 we describe factorizing gerbes over $\mathcal{D}iv(X,\Gamma)$ from a functor of points approach akin to definition 3.11. We prove that this point of view is equivalent to the more general definition in 5.15. In sections 5.17–5.22 we classify factorizing gerbes over $\mathcal{D}iv(X,\Gamma)$ by combinatorial data, this data is a gerbe version of the theta datum of 3.13. In section 5.23 we construct our main example of a factorizing gerbe whose sections are Deligne's line-bundles associated to the dilogarithm function.

5.1. Dilogarithms The higher logarithm functions are be defined recursively by

$$Li_k(z) = \int_0^z \frac{Li_{k-1}(t)}{t} dt$$

with $Li_0(z) = log(\frac{1}{z-1})$. The integral is taken on a contour from 0 to z, with $|z| < 1$. These functions are singular at $z = 0$ and can be analytically continued to $\mathbb{C}$ minus the negative real axis and the interval $[1, \infty)$. Roger's dilogarithm is defined by

$$L(z) = Li_2(z) + \frac{1}{2} log(z) log(1-z).$$

It satisfies the following functional equations:

$$\begin{aligned} L(z)+L(1-z) &= \frac{\pi^2}{6} \\ L(z)+L(\frac{1}{z}) &= \frac{\pi^2}{3} \\ L(x)+L(y) &= L(xy)+L(\frac{x-xy}{1-xy})+L(\frac{y-xy}{1-xy}). \end{aligned} \tag{5.1}$$

The last of these equations is known as the *pentagonal identity*.

5.2. Deligne's construction

Given two invertible functions $f,g:\Sigma\to\mathbb{C}^*$ from a Riemann surface Σ, Deligne defines in [11], a line bundle (f,g) on Σ. This line bundle can be described in the following way. Consider

$$M=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \mid x,y,z\in\mathbb{C}\right\}$$

This is the Heisenberg 3 group over $\mathbb{C}$. It is a three dimensional nilpotent complex Lie group. It has a discrete subgroup

$$\Lambda=\left\{\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \mid x,y\in 2\pi i\mathbb{Z}, z\in(2\pi i)^2\mathbb{Z}\right\}.$$

Let $N=\Lambda\backslash M$ be the quotient and

$$\begin{aligned} \pi : N &\longrightarrow \mathbb{C}^*\times\mathbb{C}^* \\ \begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} &\longmapsto (e^{-x},e^{y}). \end{aligned}$$

is a line bundle over $\mathbb{C}^*\times\mathbb{C}^*$ Then we let

$$(f,g):=(f,g)^*N,$$

that is, (f,g) is the pullback of N by the map $\Sigma\to\mathbb{C}^*\times\mathbb{C}^*$ given by (f,g).

In the special case $\Sigma=\mathbb{C}^*\setminus\{1\}$ $f(z)=1-z$, $g(z)=z$, Roger's dilogarithm $L(z)$ is a section of $(z-1,z)$. In general, for any morphism $(f,g):Y\to\mathbb{C}^*\times\mathbb{C}^*$ such that $f+g=1$, we will denote the line bundle (f,g) simply as $\mathcal{L}(g)$. It has a distinguished section $L(g)$. Identify $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}$ via the exponential, and using the additive notation, these line bundles verify the following equations:

$$\mathcal{L}(f)\simeq\mathcal{L}(f^{-1})$$

$$\mathcal{L}(f) \simeq \mathcal{L}(1-f)$$

$$\mathcal{L}(f) + \mathcal{L}(g) \simeq \mathcal{L}(fg) + \mathcal{L}(\frac{f-fg}{1-fg}) + \mathcal{L}(\frac{g-fg}{1-fg})$$

Each involved line bundle is trivialized by the corresponding dilogarithm. The trivializations on both sides are identified by the functional equations (5.1).

5.3. Here is a generalization of the above construction. Let X be a complex curve and G a complex commutative group. Let G_X denote the sheaf of functions from X with target G and $\underline{G}_X$ the constant sheaf G. For $A \to B$ a morphism, we denote by $[A \to B]_{-1,0}$ the complex reduced to A and B at degrees -1 and 0.

In [11], Deligne constructs a morphism:

$$H^i(X, \mathcal{O}_X^*) \times H^j(X, G_X) \to \mathbb{H}^{i+j}(X, [G_X \to \Omega^1 \otimes \mathrm{Lie}(G)]_{-1,0}). \tag{5.2}$$

We are interested in the case $G = \mathbb{G}_m$. If $i = j = 0$, we obtain for f, g invertible functions the line bundle (f, g) defined previously and interpreted in terms of Čech cocycles is given by:

$$c_{ij} = g^{log_j(f) - log_i(f)} g^{\frac{1}{2\pi i}}.$$

Here log_i, log_j are branches of the logarithm.

5.4 A gerbe version. In the case $i = j = 1$, the construction 5.2 associates for line bundles $\mathcal{L}$, $\mathcal{L}'$ a line bundle $\langle \mathcal{L}, \mathcal{L}' \rangle$. Explicitly, $\langle \mathcal{L}, \mathcal{L}' \rangle$ is the $\mathcal{O}_X$-module generated by symbols $\langle l, l' \rangle$, l, l' rational sections of $\mathcal{L}$ and $\mathcal{L}'$ respectively with disjoint divisors and the relations

$$\langle l, f.l' \rangle = f(div(l)) \langle l, l' \rangle$$

$$\langle f.l, l' \rangle = f(div(l')) \langle l, l' \rangle,$$

where, $f(\sum n_i p_i) = \prod f(p_i)^{n_i}$.

This definition makes sense in the relative setting. Let $X \to Z$ faithfully flat of codimension one and $\mathcal{L}$, $\mathcal{L}'$ line bundles over X. We want to construct a line bundle $\langle \mathcal{L}, \mathcal{L}' \rangle$ over X. For this let $D \subset X$ be a relative divisor over Z, that is a divisor of X, finite over Z. Let f be a rational function on X, we define $N_{D/Z}(f)$, the *norm*, locally following [18]. Consider a covering $\mathcal{U}$ of X such that $U_i = f^{-1}(V_i) \simeq \mathcal{O}_{V_i}^{\oplus r}$, then we can think of $f|_{U_i} \in M_{r \times r}(\mathcal{O}_{V_i})$ and $N_{D/Z}(f)|_{U_i} = det(f|_{U_i}) \in \mathcal{O}_{V_i}$.

We define $f(D)$ by

$$f(D) := N_{D/Z}(f) \qquad \text{if } D \text{ is effective}$$

$$f(D) := N_{D_1/Z}(f).N_{D_2/Z}(f)^{-1} \qquad \text{if } D = D_1 - D_2.$$

Let l be a rational section of $\mathcal{L}$ relative to Z, that is $div(l) \subset X$ is finite over Z. We then define the line bundle $\langle \mathcal{L}, \mathcal{L}' \rangle$ as the $\mathscr{O}_X$-module generated by symbols $\langle l, l' \rangle$, with l (resp. l') a local rational section of $\mathcal{L}$ (resp. $\mathcal{L}'$) and the same relations above. Observe that if D is effective,

$$\langle \mathcal{O}(D), \mathcal{L} \rangle = N_{D/Z}(\mathcal{L}).$$

Where the norm $N_{D/Z}(\mathcal{L})$ was defined in 2.48. There are isomorphisms:

$$\begin{aligned} \langle \mathcal{L} \otimes \mathcal{L}', \mathcal{L}'' \rangle &\simeq \langle \mathcal{L}, \mathcal{L}'' \rangle \otimes \langle \mathcal{L}', \mathcal{L}'' \rangle \\ \langle \mathcal{L}, \mathcal{L}' \otimes \mathcal{L}'' \rangle &\simeq \langle \mathcal{L}, \mathcal{L}' \rangle \otimes \langle \mathcal{L}, \mathcal{L}'' \rangle \\ \langle \mathcal{L}, \mathcal{L}' \rangle &\simeq \langle \mathcal{L}', \mathcal{L} \rangle. \end{aligned} \tag{5.3}$$

5.5. Sigma model on a nilmanifold Let G be a group over $\mathbb{R}$, $\mathfrak{g}$ its Lie algebra over $\mathbb{C}$ with $(.,.) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ a bilinear non-degenerate invariant form. Let $\Gamma \subset G$ be a lattice, the Lie algebra $\mathfrak{g}$ acts on $L^2(G/\Gamma)$. Let

$$\hat{\mathfrak{g}} := \mathfrak{g}[[t, t^{-1}]] \oplus \mathbb{C}.K$$

be the Kac-Moody affinization of $\mathfrak{g}$. It has a subalgebra

$$\hat{\mathfrak{g}}_+ = \mathfrak{g}[[t]] \oplus \mathbb{C}K,$$

called the annihilation subalgebra of $\hat{\mathfrak{g}}$. We let $\hat{\mathfrak{g}}_+$ act on $L^2(G/\Gamma)$ by extending the action of $\mathfrak{g} \subset \hat{\mathfrak{g}}_+$, that is, we let

$$\begin{aligned} a(t) \cdot f &:= a(0) \cdot f, \qquad a(t) \in \mathfrak{g}[[t]], \quad f \in L^2(G/\Gamma), \\ K \cdot f &:= f. \end{aligned}$$

We obtain $\hat{\mathfrak{g}}$-module M by induction.

$$M := Ind_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} L^2(G/\Gamma). \tag{5.4}$$

Notice that we have

$$V^1(\mathfrak{g}) \subset M$$

and M is a $V^1(\mathfrak{g})$-module. Since $L^2(G/\Gamma)$ is not irreducible as $\mathfrak{g}$-module, M is not irreducible. However, there is a decomposition into irreducible modules [25]:

$$L^2(G/\Gamma) \simeq \bigoplus_{i \in I} \mathcal{H}_i$$

such that

$$M \simeq \bigoplus_{i \in I} Ind_{\hat{\mathfrak{g}}}^{\hat{\mathfrak{g}}} \mathcal{H}_i$$

and $Ind_{\hat{\mathfrak{g}}}^{\hat{\mathfrak{g}}} \mathcal{H}_o \simeq V^1(\mathfrak{g})$. Denote $M_i := Ind_{\hat{\mathfrak{g}}}^{\hat{\mathfrak{g}}} \mathcal{H}_i$.

5.6 Definition. Let V be a vertex algebra and let M_1, M_2, M_3 be V-modules. An *intertwining operator* of type $\begin{pmatrix} M_3 \\ M_1 \; M_2 \end{pmatrix}$ is a morphism

$$\phi : M_1 \otimes M_2 \to z^h M_3((z))$$

for some $h \in \mathbb{Q}$ such that

$$\phi(Y^1(v, z-w)m_1, w)m_2, \; Y^3(v,z)\phi(m_1,w)m_2, \; \phi(m_1,w)Y^2(v,z)m_2$$

are images of the same object through the morphisms:

$$\begin{tikzcd} & M^3[[z]][[w]][z^{-1}, w^{-1}, (z-w)^{-1}] \arrow[dl, "i_{z,w}"'] \arrow[d, "i_{w,z}"] \arrow[dr, "i_{w,z-w}"] & \\ M^3((z))((w)) & M^3((w))((z)) & M^3((w))((z-w)). \end{tikzcd}$$

5.7 Example. Let V be a vertex algebra and $M = V$ viewed as a module over itself. Then the state field correspondence $Y(\cdot, z)$ is an interwining operator of type $\begin{pmatrix} M \\ M \; M \end{pmatrix}$.

We are interested in constructing a vertex operator of type $\begin{pmatrix} M \\ M \; M \end{pmatrix}$ where $V = V^1(\mathfrak{g})$ and M is the induced module 5.4.

5.8 Example. If $G = \mathbb{R}^n$ is abelian and $\Gamma = \mathbb{Z}^n$, then $V^1(\mathfrak{g})$ is the Heisenberg vertex algebra. The decomposition is

$$L^2(G/\Gamma) \simeq \bigoplus_{\alpha \in \Gamma} \mathbb{C}e^\alpha$$

and M is the lattice vertex algebra defined in 4.14 and $\phi(e^\alpha, z) = \Gamma_\alpha(z)$.

5.9. Consider the case where G is nilpotent,

$$0 \to \mathbb{R}^3 \to G \to \mathbb{R}^3 \to 0$$

with the product defined

$$(x^i, x^*_i).(y^i, y^*_i) = (x^i + y^i, x^*_i + y^*_i + 12\epsilon_{ijk}x^j y^k)$$

where ϵ is the totally antisymmetric tensor. The lattice is

$$0 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}^3 \to 0,$$

then,

$$0 \to \mathbb{T}^3 \to G/\Gamma \to \mathbb{T}^3 \to 0.$$

Functions on the fiber $\mathbb{T}^3$ are parametrized by $\alpha \in \mathbb{Z}^3$. We have

$$\mathbb{C}[\mathbb{Z}^3] \hookrightarrow L^2(G/\Gamma),$$

denote the generators e^α. In [2], the authors compute the 3-point function:

$$\langle \alpha+\beta+\gamma | \phi(\alpha, z)\phi(\beta, w)\phi(\gamma, t)|0\rangle = exp(det(\alpha, \beta, \gamma)L(\frac{z-t}{w-t})).$$

And the obtain the following factorization property:

$$\begin{aligned}\langle e^\psi | e^\alpha(z)e^\beta(w)e^\gamma(t)|e^\delta\rangle = \\ &\langle e^{-\alpha}e^\psi | e^\beta(w)e^\gamma(t)|e^\delta\rangle \times e^{-\gamma}e^\psi|e^\alpha(z)e^\beta(w)|e^\delta\rangle \\ &\times \langle e^{-\beta}e^\psi|e^\alpha(z)e^\gamma(t)|e^\delta\rangle \times \langle e^{-\delta}e^\psi|e^\alpha(z-t)e^\beta(w-t)|e^\gamma\rangle. \quad (5.5)\end{aligned}$$

5.10 Factorizing gerbes

Throughout this section X denotes a smooth complex curve and Γ a lattice (i.e. a commutative group isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$). We will define a factorizing gerbe over a factorizing space and describe some equivalences in the special case of factorizing spaces over $\mathcal{D}iv(X, \Gamma)$.

5.11. Recall the definition of the box product. Let $\mathcal{G}_i$ be gerbes on X_i for $1 \le i \le n$, then the box product is the gerbe on $X_1 \times X_2 \times ... \times X_n$:

$$\mathcal{G}_1 \boxtimes \mathcal{G}_2 \boxtimes ... \boxtimes \mathcal{G}_n := p_1^*\mathcal{G}_1 \otimes p_2^*\mathcal{G}_2 \otimes ... \otimes p_n^*\mathcal{G}_n$$

where $p_i : X_1 \times X_2 \times ... \times X_n$ are the projections to the i-th factors.

5.12 Definition. Let $\mathcal{A}$ be a constant commutative group G or $\mathbb{G}_m$. A factorizing $\mathcal{A}$-gerbe on a factorizing space (Y, c) over X is for each $S \in \mathcal{C}(X)_Z$ a gerbe $\mathcal{G}_S$ on Y_S and

a. for S_1, S_2 disjoint, an equivalence of gerbes on $Y_{S_1} \times Y_{S_2}$

$$(S_1, S_2) : \mathcal{G}_{S_1} \boxtimes \mathcal{G}_{S_2} \xrightarrow{\sim} c^*(\mathcal{G}_{S_1+S_2}) \quad (5.6)$$

b. for S_1, S_2, S_3 disjoint a natural isomorphism between the functors

$$\begin{tikzcd} & \mathcal{G}_{S_1} \boxtimes \mathcal{G}_{S_2} \boxtimes \mathcal{G}_{S_3} \arrow[dl] \arrow[dr] & \\ c^*(\mathcal{G}_{S_1+S_2}) \boxtimes \mathcal{G}_{S_3} \arrow[dr] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{G}_{S_1} \boxtimes c^*(\mathcal{G}_{S_2+S_3}) \arrow[dl] \\ & c^*(\mathcal{G}_{S_1+S_2+S_3}) & \end{tikzcd} \quad (5.7)$$

Such that given four disjoint divisors the associated diagram of natural isomorphisms commutes.

A *morphism of factorizing gerbes* on (Y, c) is for each S a functor $\phi_S : \mathcal{G}_S \to \mathcal{G}'_S$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}_{S_1} \boxtimes \mathcal{G}_{S_2} \arrow[r, "\sim"] \arrow[d, "\phi_{S_1} \boxtimes \phi_{S_2}"] & c^*(\mathcal{G}_{S_1+S_2}) \arrow[d, "c^* \phi_{S_1+S_2}"] \\
\mathcal{G}'_{S_1} \boxtimes \mathcal{G}'_{S_2} \arrow[r, "\sim"] & c^*(\mathcal{G}'_{S_1+S_2})
\end{tikzcd} \tag{5.8}$$

5.13 Example. In example 3.8 we have given a factorizing space $Y_I := \tilde{\mathcal{L}}(M)_{X^I}$, its cotangent bundle Ω_{Y_I} (defined in 2.69) is a Tate sheaf and have associated a gerbe $\mathcal{G}_I = \mathcal{D}et(\Omega_{Y_I})$. Since $\mathcal{D}et(T_1 + T_2) \simeq \mathcal{D}et(T_1) \otimes \mathcal{D}et(T_2)$, this defines a factorizing gerbe. This is the factorizing gerbe studied by Kapranov and Vasserot in [21].

5.14. From now on, we will restrict ourselves to factorizing gerbes over $\mathcal{D}iv(X, \Gamma)$.

Let Z be affine and Noetherian and for each $D \in \mathcal{D}iv(X, \Gamma)(Z)$, let $\mathcal{G}_D$ be a $\mathcal{A}$-gerbe over Z. Suppose that given $D_1, D_2 \in \mathcal{D}iv(X, \Gamma)(Z)$ disjoint divisors there are equivalences:

$$(D_1, D_2) : \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \xrightarrow{\sim} \mathcal{G}_{D_1+D_2}. \tag{5.9}$$

And for pairwise disjoint divisors $D_1, D_2, D_3 \in \mathcal{D}iv(X, \Gamma)(Z)$, isomorphisms:

$$\begin{tikzcd}
& \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \otimes \mathcal{G}_{D_3} \arrow[dl] \arrow[dr] & \\
\mathcal{G}_{D_1+D_2} \otimes \mathcal{G}_{D_3} \arrow[dr] \arrow[rr, phantom, "\Longrightarrow"] & & \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2+D_3} \arrow[dl] \\
& \mathcal{G}_{D_1+D_2+D_3}. &
\end{tikzcd} \tag{5.10}$$

Then, for $D_1, D_2, D_3, D_4 \in \mathcal{D}iv(X, \Gamma)(Z)$, the ways of composing the equivalences 5.9 are given by the paths of the cube:

$$\begin{tikzcd}
& \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \otimes \mathcal{G}_{D_3} \otimes \mathcal{G}_{D_4} \arrow[rrr] \arrow[dl] \arrow[dd] & & & \mathcal{G}_{D_1+D_2} \otimes \mathcal{G}_{D_3} \otimes \mathcal{G}_{D_4} \arrow[dl] \arrow[dd] \\
\mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2+D_3} \otimes \mathcal{G}_{D_4} \arrow[rrr] \arrow[dd] & & & \mathcal{G}_{D_1+D_2+D_3} \otimes \mathcal{G}_{D_4} \arrow[dd] & \\
& \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \otimes \mathcal{G}_{D_3+D_4} \arrow[rrr] \arrow[dl] & & & \mathcal{G}_{D_1+D_2} \otimes \mathcal{G}_{D_3+D_4} \arrow[dl] \\
\mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2+D_3+D_4} \arrow[rrr] & & & \mathcal{G}_{D_1+D_2+D_3+D_4}. &
\end{tikzcd}$$

Observe that the face on the back is a commutative diagram. Let $D_1, D_2, D_3 \in \mathcal{D}iv(X,\Gamma)(Z)$, we denote:

$$(D_1,(D_2,D_3)) : \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \otimes \mathcal{G}_{D_3} \to \mathcal{G}_{D_1+D_2+D_3}$$
$$(D_1,(D_2,D_3)) := (D_1, D_2+D_3) \circ (1 \otimes (D_2,D_3)).$$

5.15 Proposition. A factorizing $\mathcal{A}$-gerbe on $\mathcal{D}iv(X,\Gamma)$ is for each Z affine and Noetherian and $D \in \mathcal{D}iv(X,\Gamma)(Z)$ a gerbe $\mathcal{G}_D$ over Z, and given $D_1, D_2 \in \mathcal{D}iv(X,\Gamma)(Z)$ disjoint divisors an equivalence

$$(D_1,D_2) : \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \xrightarrow{\ \sim\ } \mathcal{G}_{D_1+D_2} \tag{5.11}$$

verifying the following properties.

a. Given disjoint $D_1, D_2, D_3 \in \mathcal{D}iv(X,\Gamma)(Z)$, there is an isomorphism of functors

$$\phi : (D_1,(D_2,D_3)) \xrightarrow{\ \sim\ } ((D_1,D_2),D_3) \tag{5.12}$$

such that given $D_1, D_2, D_3, D_4 \in \mathcal{D}iv(X,\Gamma)(Z)$ the diagram

$$\begin{tikzcd}
& (D_1,(D_2,(D_3,D_4))) \arrow[dl, "1\otimes\phi"'] \arrow[dr, "\phi"] & \\
(D_1,((D_2,D_3),D_4))) \arrow[d, "\phi"'] & & ((D_1,D_2),(D_3,D_4)) \arrow[d, "\phi"] \\
((D_1,(D_2,D_3)),D_4) \arrow[rr, "\phi\otimes 1"] & & (((D_1,D_2),D_3),D_4)
\end{tikzcd} \tag{5.13}$$

commutes.

b. The following diagram

$$\begin{tikzcd}
\mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2} \arrow[rr, "\sigma"] \arrow[dr] & & \mathcal{G}_{D_2} \otimes \mathcal{G}_{D_1} \arrow[dl] \\
& \mathcal{G}_{D_1+D_2} &
\end{tikzcd} \tag{5.14}$$

commutes.

c. All this data is compatible with base change.

Proof. By definition, a factorizing gerbe on $\mathcal{D}iv(X,\Gamma)$ is for each $S \in \mathcal{C}(X)_Z$, $y : Y \to Z$, $D \in \mathcal{D}iv(X,\Gamma)_S(Y)$ a gerbe $\mathcal{G}_{S,D}$ over Y. Then we obtain a structure as in the proposition considering the identity $id : Z \to Z$. And if we have a structure as in the proposition and $S \in \mathcal{C}(X)_Z$, $y : Y \to Z$, $D \in \mathcal{D}iv(X,\Gamma)_S(Y)$, since $\mathcal{D}iv(X,\Gamma)_S(Y) \subset \mathcal{D}iv(X,\Gamma)(Y)$, we just consider $\mathcal{G}_{S,D} := \mathcal{G}_D$. $\square$

5.16. Denote the category of factorizing $\mathcal{A}$-gerbes on $\mathcal{D}iv(X,\Gamma)$ by $\mathcal{BP}^f_{\mathcal{A}}(X,\Gamma)$. A factorizing $\mathcal{A}$-gerbe is called *commutative* if the isomorphisms (5.12) exist for every divisor (not necessarily disjoint).

5.17. Following Deligne [11], define an extension:

$$1 \to \{\mathcal{A} - torsors\} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ p\ } \mathcal{D}iv(X,\Gamma) \to 0. \tag{5.15}$$

to be a commutative factorizing $\mathcal{A}$-gerbe on $\mathcal{D}iv(X,\Gamma)$ and denote it $\mathrm{Ext}(\mathcal{D}iv(X,\Gamma), \mathcal{A}\text{-tors})$.

5.18 Theorem. There is an equivalence

$$\{\text{T-gerbes on } X\} \to \mathrm{Ext}(\mathcal{D}iv(X,\Gamma), \mathbb{G}_m\text{-tors})$$

where $T := Spec(\mathbb{C}[\Gamma]) \simeq \mathbb{G}_m \otimes \Gamma^\vee$.

Proof. First, consider the case $\Gamma = \mathbb{Z}$. Let $\mathcal{G} \in \mathcal{B}Pic(X)$ and $D \in \mathcal{D}iv(X)(Z)$. Denote $p_X : X \times Z \to X$ the projection. If D is effective, we define

$$\mathcal{G}_D := N_{D/Z}(p_X^*(\mathcal{G})).$$

If $D = D_1 - D_2$, D_1 and D_2 effective,

$$\mathcal{G}_D := \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2}^{-1}.$$

Observe that if $p_X^*(\mathcal{G})$ is defined by $\mathcal{L}_{ij} \in \check{H}^1(X, \mathcal{A}\text{-tors})$ then $\mathcal{G}_D$ is given by $\langle \mathcal{O}(D), \mathcal{L}_{ij}\rangle$. Then by 5.3,

$$\mathcal{G}_{D_1+D_2} \simeq \mathcal{G}_{D_1} \otimes \mathcal{G}_{D_2}.$$

For the inverse, given an extension consider

$$\mathcal{G} := \mathcal{G}_\Delta \in \mathcal{B}Pic(X)$$

where $\Delta \subset X \times X$ is the diagonal divisor.

Now, for any lattice, let $\mathcal{G}$ be a T-gerbe. An element $\gamma \in \Gamma$ induces a morphism $H^2(X,T) \to H^2(X,\mathbb{G}_m)$. Denote by $\mathcal{G}^\gamma$ the image of $\mathcal{G}$ through this morphism. Then, if D is effective

$$\mathcal{G}_{D\otimes\gamma} := N_{D/Z}(p_X^*(\mathcal{G}^\gamma)).$$

And for general D is defined as before. For the inverse, consider the isomorphism

$$\phi : \prod_{i=1}^n H^1(X,\mathbb{G}_m) \to H^1(X,T)$$

given by $\gamma_i^\vee \in \Gamma^\vee$. And define

$$\mathcal{G} := \phi(\mathcal{G}_{\Delta\otimes\gamma_1}, ..., \mathcal{G}_{\Delta\otimes\gamma_n}).$$

□

Analogously, we have an equivalence

$$\{\text{T-gerbes on } X\} \to Ext(\mathcal{D}iv(X,\Gamma), \mathbb{C}^*\text{-tors})$$

where $T := \mathbb{C}^* \otimes \Gamma$.

5.19 Definition. Let $q : \Gamma \times \Gamma \to \mathbb{C}^*$ a symmetric pairing, a *q-twisted T-gerbe* on X is:

a. for each $\gamma \in \Gamma$, $\mathcal{G}^\gamma$ a $\mathbb{C}^*$-gerbe;

b. given $\gamma_1, \gamma_2 \in \Gamma$ a isomorphism

$$\mathcal{G}^{\gamma_1} \otimes \mathcal{G}^{\gamma_2} \xrightarrow{(\gamma_1\gamma_2)} \mathcal{G}^{\gamma_1+\gamma_2} \otimes \omega_X^{2log(q(\gamma_1,\gamma_2))}$$

that commute with the symmetry constraint;

c. given $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, a isomorphism of functors

$$(\gamma_1, (\gamma_2, \gamma_3)) \simeq ((\gamma_1, \gamma_2), \gamma_3)$$

such that given four elements of the lattice the corresponding pentagonal diagram commutes.

5.20 Remark. Let's denote $\mathcal{BP}^t(X,\Gamma)^q$ the set of *q-twisted T-gerbes* on X and $\mathcal{BP}^t(X,\Gamma)$ the union of the $\mathcal{BP}^t(X,\Gamma)^q$. The set $\mathcal{BP}^t(X,\Gamma)$ is a Picard groupoid. Observe that $\mathcal{BP}^t(X,\Gamma)^1 \simeq T$-gerbes.

5.21 Theorem. There is an equivalence of categories

$$\mathcal{BP}^f_{\mathbb{C}^*}(X,\Gamma) \to \mathcal{BP}^\theta(X,\Gamma).$$

Proof. Here, we sketch a proof using the result of proposition 5.18, for a complete proof see [27]. The functor given by

$$\mathcal{G}^\gamma := \mathcal{G}_{\Delta\otimes\gamma}$$

restricts to the isomorphism of the proposition 5.18 and is faithful. By factorization, we have that

$$\mathcal{G}_{\Delta_{13}\otimes\gamma_1+\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta} \simeq \mathcal{G}_{\Delta_{13}\otimes\gamma_1} \otimes \mathcal{G}_{\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta}$$

then

$$\mathcal{G}_{\Delta_{13}\otimes\gamma_1+\Delta_{23}\otimes\gamma_2} \simeq \mathcal{G}_{\Delta_{13}\otimes\gamma_1} \otimes \mathcal{G}_{\Delta_{23}\otimes\gamma_2} \otimes \mathcal{O}(\Delta)^{\frac{\log(q(\gamma_1,\gamma_2))}{2}}.$$

Restricting to the diagonal we obtain an element of $\mathcal{BP}^\theta(X,\Gamma)$. Let's see that this assignment is surjective. A factorizing T- gerbe, is a rule for each $n \in \mathbb{N}$ and $\gamma_1, ..., \gamma_n \in \Gamma$ a $\mathbb{G}_m$-gerbe

$$\mathcal{G}_n^{\gamma_1,\dots,\gamma_n} := \mathcal{G}_{\Delta_{1,n+1}\otimes\gamma_1+\dots\Delta_{n,n+1}\otimes\gamma_n}$$

compatible with restrictions. Given $q : \Gamma \times \Gamma \to \mathbb{C}^*$ and a T-gerbe $\mathcal{G}$ on X define

$$\mathcal{G}_1^{\gamma} = \mathcal{G}^{\gamma} \otimes \omega_X^{\frac{log(q(\gamma,\gamma)}{2}}$$

$$\mathcal{G}_n^{\gamma_1,\dots,\gamma_n} := \mathcal{G}_1^{\gamma_1} \boxtimes \mathcal{G}_1^{\gamma_2} \dots \boxtimes \mathcal{G}_1^{\gamma_n} \bigotimes_{i \le j} \mathcal{O}(\Delta)^{log(q(\gamma_i,\gamma_j))}$$

.

5.22 Remark. Observe that the gerbes on X^n are product of (pullbacks of) gerbes in X^2.

5.23 Examples

Let $\Delta_{ij} \subset X^n$ be the diagonal $\{(x_1, ..., x_n) \in X^n / x_i = x_j\}$. Since the effective divisors are represented by the symmetric power $Sym^n(X)$, by theorem C.1 of the appendix, to give a factorizing gerbe over $Div(X, \Gamma)$ is equivalent to the following data:

a. A gerbe $\mathcal{G}_{\Delta_{in+1} \otimes \gamma}$ on X^n for every $n \in \mathbb{N}$, $i \le n$, $\gamma \in \Gamma$,

b. Isomorphisms

$$\mathcal{G}_{\Delta_{in+1} \otimes \gamma_i} \otimes \mathcal{G}_{\Delta_{jn+1} \otimes \gamma_j}|_{X^n \setminus \Delta_{ij}} \xrightarrow{\phi(\gamma_i \gamma_j)} \mathcal{G}_{\Delta_{in+1} \otimes \gamma_i + \Delta_{jn+1} \otimes \gamma_j}|_{X^n \setminus \Delta_{ij}}$$

such that $\sigma_{ij}^*(\phi(\gamma_i \gamma_j)) = \phi(\gamma_j, \gamma_i)$ for σ_{ij} the permutation $i \leftrightarrow j$

c. isomorphisms of functors defined in the complement of the diagonals where the divisors intersect:

$$\begin{tikzcd}
& \mathcal{G}_{\Delta_{in+1} \otimes \gamma_1} + \mathcal{G}_{\Delta_{jn+1} \otimes \gamma_2} + \mathcal{G}_{\Delta_{kn+1} \otimes \gamma_3} \arrow[dl] \arrow[dr] & \\
\mathcal{G}_{\Delta_{in+1} \otimes \gamma_1} + \mathcal{G}_{\Delta_{jn+1} \otimes \gamma_2 + \Delta_{kn+1} \otimes \gamma_3} \arrow[dr] & \Rightarrow & \mathcal{G}_{\Delta_{in+1} \otimes \gamma_1 + \Delta_{jn+1} \otimes \gamma_2} + \mathcal{G}_{\Delta_{kn+1} \otimes \gamma_3} \arrow[dl] \\
& \mathcal{G}_{\Delta_{in+1} \otimes \gamma_1 + \Delta_{jn+1} \otimes \gamma_2 + \Delta_{kn+1} \otimes \gamma_3} &
\end{tikzcd} \tag{5.16}$$

S_n-invariant as before.

Such that given four diagonals the induced pentagon commutes.

Again, we will identify $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}$ via the exponential, so our $\mathcal{O}^*$-torsors (and gerbes) are seen as $\mathcal{O}/\mathbb{Z}$-torsors (gerbes) and use the additive notation. Also, denote $det(\gamma_i, \gamma_j, \gamma_k)$ by $det(\gamma_{ijk})$. We define a factorizing gerbe over $Div(\mathbb{C}^*, \Gamma)$. For each $n \in \mathbb{N}$, define the gerbes

$$\mathcal{G}_{X^n} = \mathcal{G}_{\sum \Delta_{in+1} \otimes \omega_i} := \sum_{i<j} \mathcal{O}(\Delta_{ij})^{log(\frac{x_i}{x_j})}.$$

Observe that all these gerbes are trivial because they are (sums of) pullbacks of the gerbe $\mathcal{G}_X := \mathcal{O}(1)^{log(x)}$ on X, but this description will guide us through our definitions. Also, the isomorphisms of gerbes will be determined by $\mathcal{O}/\mathbb{Z}$-torsors globally.
The factorizing structure is defined by

a. For X^2, the isomorphism

$$\mathcal{G}_{\Delta_{13}\otimes\gamma_1} + \mathcal{G}_{\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta} \to \mathcal{G}_{\Delta_{13}\otimes\gamma_1+\Delta_{23}\otimes\gamma_2}|_{X^2\setminus\Delta}$$

is the identity.

b. For X^3,

$$\begin{tikzcd}
& \mathcal{G}_{\Delta_{14}\otimes\gamma_1} + \mathcal{G}_{\Delta_{24}\otimes\gamma_2} + \mathcal{G}_{\Delta_{34}\otimes\gamma_3} \arrow[dl, "0"'] \arrow[dr, "0"] & \\
\mathcal{G}_{\Delta_{14}\otimes\gamma_1} + \mathcal{G}_{\Delta_{24}\otimes\gamma_2+\Delta_{34}\otimes\gamma_3} \arrow[dr, "det(\gamma_{123})(\mathcal{L}(\frac{x}{y}))-\mathcal{L}(\frac{x}{z})"] & & \mathcal{G}_{\Delta_{14}\otimes\gamma_1+\Delta_{24}\otimes\gamma_2} + \mathcal{G}_{\Delta_{34}\otimes\gamma_3} \arrow[dl, "det(\gamma_{123})(\mathcal{L}(\frac{x}{z})-\mathcal{L}(\frac{y}{z}))"] \\
& \mathcal{G}_{\Delta_{14}\otimes\gamma_1+\Delta_{24}\otimes\gamma_2+\Delta_{34}\otimes\gamma_3} &
\end{tikzcd} \tag{5.17}$$

The isomorphisms (line bundles) are defined where the divisors are disjoint and they have trivializations given by the dilogarithms. Outside the three diagonals, the isomorphism of functors is defined by the section of $det(\gamma_{123})(\mathcal{L}(\frac{x}{z}) - \mathcal{L}(\frac{y}{z})) - det(\gamma_{123})(\mathcal{L}(\frac{x}{y}) - \mathcal{L}(\frac{x}{z}))$:

$$det(\gamma_{123})(L(\frac{x}{z})-L(\frac{x-y}{x-z})-L(\frac{x(z-y)}{y(z-x)})) = det(\gamma_{123})(2(L(\frac{x}{z})-L(\frac{y}{z}))-L(\frac{x}{y})).$$

c. In general, let $D_i = \Delta_{i,n+1}\otimes\gamma_i$, and L_1, L_2 sets of ordered indices with $L_1 < L_2$. Then, the isomorphism

$$\mathcal{G}_{\sum_{i\in L_1} D_i} + \mathcal{G}_{\sum_{i\in L_2} D_i} \to \mathcal{G}_{\sum_{i\in L_1\cup L_2} D_i}$$

is defined in $X^n \setminus \sum_{\substack{i\in L_1\\ j\in L_2}} \Delta_{ij}$ by the line bundle:

$$\sum_{\substack{i_1\in L_1\\ i_2,i_3\in L_2}} det(\gamma_{i_1i_2i_3})(\frac{x_{i_1}}{x_{i_2}} - \frac{x_{i_1}}{x_{i_3}}) + \sum_{\substack{i_1,i_2\in L_1\\ i_3\in L_2}} det(\gamma_{i_1i_2i_3})(\frac{x_{i_1}}{x_{i_3}} - \frac{x_{i_2}}{x_{i_3}}).$$

d. The pentagonal diagram is verified by the following straightforward lemma.

5.24 Lemma. Consider the following diagram where each vertex is a gerbe:

$$\begin{tikzcd}
& & \bullet \arrow[ddll, bend right, "A_1"'] \arrow[d, "A_3"] \arrow[ddrr, bend left, "C_3"] & & \\
& (A) & \bullet \arrow[dl, "A_4"'] \arrow[dr, "B_3"] & (C) & \\
\bullet \arrow[r, "A_2"] \arrow[ddrr, bend right, "C_3"'] & \bullet \arrow[dr, "B_2"'] & (B) & \bullet \arrow[dl, "B_4"] & \bullet \arrow[l, "C_4"'] \arrow[ddll, bend left, "A_1"] \\
& (D) & \bullet & (E) & \\
& & \bullet \arrow[u, "D_4"] & &
\end{tikzcd}$$

and suppose each morphism is given by a line bundle with a global section such that the morphisms of functors are determined by these sections. Then the pentagonal equality is verified.

5.25. Second Example
We define the gerbes to be trivial, and the morphisms also but we force the functions to be dilogarithms

$$\begin{tikzcd}
& & \mathcal{G}_{D_1} + \mathcal{G}_{D_2} + \mathcal{G}_{D_3} + \mathcal{G}_{D_4} \arrow[ddll, bend right] \arrow[d] \arrow[ddrr, bend left] & & \\
& L(\frac{y-t}{y-z}) & \mathcal{G}_{D_1} + \mathcal{G}_{D_2+D_3} + \mathcal{G}_{D_4} \arrow[dl] \arrow[dr] & L(\frac{x-y}{x-z}) & \\
\mathcal{G}_{D_1} + \mathcal{G}_{D_2} + \mathcal{G}_{D_3+D_4} \arrow[r] \arrow[ddrr, bend right] & \mathcal{G}_{D_1} + \mathcal{G}_{D_2+D_3+D_4} \arrow[dr] & -L(\frac{x-y}{x-t}) - L(\frac{x-t}{x-z}) & \mathcal{G}_{D_1+D_2+D_3} + \mathcal{G}_{D_4} \arrow[dl] & \mathcal{G}_{D_1+D_2} + \mathcal{G}_{D_3} + \mathcal{G}_{D_4} \arrow[l] \arrow[ddll, bend left] \\
& L(\frac{y(t-x)}{x(t-y)}) - L(\frac{y(z-x)}{x(z-y)}) & \mathcal{G}_{D_1+D_2+D_3+D_4} & L(\frac{x(t-z)}{z(t-x)}) - L(\frac{y(t-z)}{z(t-y)}) & \\
& & \mathcal{G}_{D_1+D_2} + \mathcal{G}_{D_3+D_4} \arrow[u] & &
\end{tikzcd}$$

A Groethendieck sites

In this appendix, we give some definitions for the sake of completeness.

A.1. Let $\mathcal{C}$ be a category. A *Groethendieck topology* on $\mathcal{C}$ is a set of families of morphisms $\{U_i \to U\}_{i \in I}$ called *coverings* verifying:

- **a**. For every $V \to U$, $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.
- **b**. If $\{U_{i,j} \to U_i\}_{j \in J_i}$ are coverings then $\{U_{i,j} \to U\}_{i,j}$ is a covering.
- **c**. For every object $U \in \mathcal{C}$ the identity $U \to U$ is a covering.

A *Groethendieck site* is a category with a Groethendieck topology.

When it doesn't lead to confusion we will denote $V \cap V' := V \times_U V'$.

A.2 Examples. The main two examples are:

a. Let X be a topological space consider $\mathcal{U}_X$ the category of open subsets of X whose morphisms are inclusions. The coverings are the families of open subsets $\{U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$. This defines *the topological site* X_{top}. In the case X is a scheme with the Zariski topology, we call this site the Zariski site.

b. Let X be a scheme, consider the category Et/X the subcategory of Sch/X whose objects are $\{U \to X\}$ étale morphism of finite type. The étale site X_{et} is the category Et/X and the coverings are families $\{f_i : U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} f_i(U_i)$.

A.3 Definition. A *sheaf* (of sets) on a site $\mathcal{E}$ is a contravariant functor $\mathcal{F} : \mathcal{E} \to \mathcal{S}ets$ such that the induced sequence:

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \tag{A.1}$$

is exact for every covering $\{U_i \to U\}$.

In the case of a topological site, this definition coincides with the usual one. A contravariant functor $\mathcal{F} : \mathcal{E} \to \mathcal{S}ets$ is just a presheaf of sets and the exactness means that, if $U = \bigcup_{i\in I} U_i$, given $\{s_i \in \mathcal{F}(U_i)\}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$.

Observe that in the previous definition, the category of $\mathcal{S}ets$ can be replaced by any category $\mathcal{C}$ where the exactness of (A.1) make sense.

B Ind and Pro-objects

Let Y be an object of a category $\mathcal{E}$, the functor of point of Y is the contravariant functor $h_Y : \mathcal{E} \to \mathcal{S}ets$

$$h_Y(X) = Hom(X, Y).$$

The Yoneda lemma states that any object in a category is determined by its functor of points. Moreover, let $\mathcal{E}$ be a category define $\mathcal{S}ets^{\mathcal{E}^{op}}$ as the category of contravariant functors $\mathcal{E} \to \mathcal{S}ets$. The Yoneda lemma states that the functor defined by:

$$\mathcal{E} \xrightarrow{h} \mathcal{S}ets^{\mathcal{E}^{op}}$$

$$\begin{tikzcd} Y \arrow[d, "f"'] & h_Y \arrow[d, "f\circ -"] \\ Y' \arrow[r, phantom, "\longmapsto"] & h_{Y'} \end{tikzcd}$$

is a fully faithful functor.

Then, we can consider any category $\mathcal{E}$ embedded in the category $\mathcal{S}ets^{\mathcal{E}^{op}}$.

A functor $F \in \mathcal{S}ets^{\mathcal{E}^{op}}$ is *Pro-representable* (*Ind-representable*) if there exists a cofiltered (filtered) diagram X_i (Y_j) such that $F(X) = \varprojlim Hom(X, X_i)$ ($F(X) = \varinjlim Hom(X, Y^j)$). A *Pro-object* will refer to the Pro-representable functor or the cofiltered diagram indistinctly. An *Ind-object* is a Pro-object in the opposite category. In this setting, the category of schemes is a subcategory of $\mathcal{S}ets^{\mathcal{R}ings}$. An *Ind-scheme* is an *Ind*-object in $\mathcal{S}ets^{\mathcal{R}ings}$ such that the morphisms $Y^j \to Y^{j'}$ are closed embeddings. Morphisms between *Ind*-schemes are natural transformations between its functors.

B.1 Example. Let $\mathcal{V}ect$ be the category of finitely generated vector spaces over $\mathbb{C}$. Then the space of series $\mathbb{C}[[t]]$ is *Pro*-object of $\mathcal{V}ect$. It Pro-represents the functor $V \mapsto Hom_{\mathbb{C}}(V, \mathbb{C}[[t]])$. The cofiltered diagram is for $n \leq m$ the projection:

$$\mathbb{C}[t]/(t^m) \to \mathbb{C}[t]/(t^n).$$

And, a morphism of arbitrary vector spaces $\phi : V \to \mathbb{C}[[t]]$ is equivalent to the data of a morphism $\phi_m : V \to \mathbb{C}[t]/(t^m)$ in $\mathcal{V}ect$ for each $m \in \mathbb{N}$ such that the following diagrams commute for all $n \leq m$,

$$\begin{tikzcd} \mathbb{C}[t]/(t^m) \arrow[rr] & & \mathbb{C}[t]/(t^n) \\ & V \arrow[ul, "\phi_m"] \arrow[ur, "\phi_n"'] & \end{tikzcd} .$$

B.2 Examples. The loop spaces. Let X be a scheme over $\mathbb{C}$ of finite type. And let $R((t))^{\surd}$ be the Laurent series of the form $\sum_{i \geq n_0} a_i t^i$ with a_i nilpotent for $i < 0$. Consider the following functors from $\mathbb{C}$-algebras:

$$\tilde{\lambda}_X(R) = Hom(Spec(R((t))), X)$$

$$\lambda_X(R) = Hom(Spec(R((t))^{\surd}), X)$$

$$\lambda_X^0(R) = Hom(Spec(R[[t]], X).$$

In [24], the authors prove that:

a. λ_X^0 is represented by a scheme $\mathcal{L}^0(X)$ of infinite type that is a projective limit of the schemes that represent the functors:

$$\lambda_X^{0,n}(R) = Hom(Spec(R[t]/t^{n+1}), X).$$

b. When X is affine, $\tilde{\lambda}_X$ is represented by an Ind-scheme $\tilde{\mathcal{L}}(X)$ that is an inductive limit of schemes of infinite type.

c. For any X, λ_X is represented by an Ind-scheme $\mathcal{L}(X)$.

Now, we introduce differential geometry on this setting.

B.3 Definition. An Ind-scheme Y is *formally smooth* if and only if, for every ring R and $I \subset R$ nilpotent, $h_Y(R) \to h_Y(R/I)$ is surjective.

Consider the following baby example for an intuitive insight of this definition. Let $Y = Spec(\mathbb{C}[x_1, ..., x_n]/\langle f_1, ..., f_n\rangle)$, $R = \mathbb{C}[\epsilon]$ and $I = \langle \epsilon^2 \rangle$. A morphism $\mathbb{C}[x_1, ..., x_n]/\langle f\rangle \to \mathbb{C}[\epsilon]/\langle \epsilon^2\rangle$ is a n-tuple $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ such that $f(\alpha) = 0$ and a n-tuple $\beta = (\beta_1, ..., \beta_n)$ such that $\sum_i \frac{\partial f}{\partial x_i}(\alpha).\beta_i = 0$. Then, the definition says that every tangent vector in a point can be integrated to a line.

B.4 Example. If $X = Spec(A)$ is affine and smooth, then $\tilde{\mathcal{L}}(X)$ is formally smooth. We want to see that for every ring R and $I \subset R$ nilpotent, $\tilde{\lambda}_X(R) \to \tilde{\lambda}_X(R/I)$ is surjective. Namely, that $Hom(A, R((t))) \to Hom(A, R/I((t)))$ is surjective. But, $R/I((t)) \simeq R((t))/I((t))$, $I((t))$ is nilpotent and X is smooth. Analogously, if X is smooth then $\mathcal{L}(X)$ is formally smooth.

Let X be a scheme, an *Ind-scheme over X* is an Ind-scheme with a morphism to X. Denote by $\Delta \subset X \times X$ the diagonal. If $U \subset X$ is an open subscheme such that Δ is defined by a sheaf of ideals $\mathcal{I}$, recall that the first infinitesimal neighbourhood $\Delta^{(2)}$ is defined locally by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

B.5 Definition. Let $p_1, p_2 : \Delta^{(2)} \to X$ the projections of the first order infinitesimal neighbourhood of the diagonal in $X \times X$ and $\Delta : X \to X \times X$ the diagonal morphism. A *connection* on a Ind-scheme Y over X is

$$\alpha : p_1^*(Y) \to p_2^*(Y)$$

such that $\Delta^*(\alpha) = Id$.

An *integrable connection* is a connection such that $p_{13}^*(\alpha) = p_{23}^*(\alpha) \circ p_{12}^*(\alpha)$ where $p_{ij} : X \times X \times X \to X \times X$ are the projections to the i, j-factors.

Finally, we define the double limit:

B.6 Definition. Given an exact category $\mathcal{E}$, the double limit

$$\lim_{\leftrightarrow} \mathcal{E} \subset IndPro(\mathcal{E})$$

is the subcategory of objects $X_i^j \in IndPro(\mathcal{E})$ such that given $i \leq i'$ and $j \leq j'$ the corresponding diagram is a cartesian square.

B.7 Proposition. Let $\mathcal{E}$ be an exact category. Then, the inclusion

$$\mathcal{E} \hookrightarrow Ind(\mathcal{E}) \cap Pro(\mathcal{E})$$

is an isomorphism.

Proof. Let $f\colon \varprojlim_i X_i \to \varinjlim_j Y^j$ be an isomorphism. Since,

$$Hom(\varprojlim_i X_i, \varinjlim_j Y^j) \simeq \varinjlim_i \varinjlim_j Hom(X_i, Y_j) \simeq \coprod_{i,j} Hom(X_i, Y^j)$$

there exists $f_{ij}\colon X_i \to Y^j$ such that the diagram

$$\begin{tikzcd} \varprojlim_i X_i \arrow[r, "f"] \arrow[d, two heads] & \varinjlim_j Y^j \\ X_i \arrow[r, "f_{ij}"] & Y^j \arrow[u, hook] \end{tikzcd}$$

commutes. Then, the map from Y^j (also the one to X_i) is an isomorphism. Then, the inverse of the inclusion in the statement is defined.

C The symmetric power of a curve and relative divisors

In this appendix, we introduce the definition of the symmetric power of a curve X and show how it represents the contravariant functor $\mathcal{D}iv_n^{ef}(X)$ of relative effective Cartier divisors of degree n. Let S_n denote the group of symmetries of lenght n. The symmetric power of X is the quotient:

$$Sym^n(X) := X^n/S_n.$$

C.1 Theorem. Let X be a curve. The symmetric power $Sym^n(X)$ represents the functor $\mathcal{D}iv_n^{ef}(X)$. And there exists a universal divisor $D_{univ} \in Sym^n(X) \times X$ such that, if $D \in \mathcal{D}iv_n^{ef}(X)(Z)$ induces a map $\phi_D : Z \to Sym^n(X)$, then

$$(\phi_D \times id)^{-1}(D_{univ}) = D.$$

Proof. Let $D \in \mathcal{D}iv_n^{ef}(X)(Z)$. First, consider the case where there exist sections $s_i : Z \to Z \times X$ such that $D = \sum n_i s_i(Z)$, $n = \sum n_i$. Then, let

$$\phi_D : Z \to Sym^n(X)$$

be the projection to $Sym^n(X)$ of the morphism:

$$t \mapsto (p_X \circ s_1(t), ..., p_X \circ s_1(t), ..., p_X \circ s_r(t)),$$

where each $p_X \circ s_i(t)$ appears n_i times.

Now, define

$$D_{univ} = \sum_{i=1}^{n} \Delta_{i,n+1} \Big/ S_n.$$

And for $\phi : T \to Sym^n(X)$ define

$$D_\phi := (\phi \times id)^{-1}(D_{univ}).$$

Observe that if D_ϕ is given by sections, ϕ factors through $X^n \to Sym^n X$ and

$$\phi_{D_\phi} = \phi.$$

And, also,

$$D_{\phi_D} = D.$$

In the general case, let $\pi : Z' \to Z$ be faithfully flat such that $D' = (\pi \times id)^{-1}(D)$ is in the conditions of the previous case. Then, there exists $\phi_{D'} : Z' \to Sym^n(X)$ such that $D' = \phi^{-1}(D_{univ})$. By faithfully flat descent, $\pi : Z' \to Z$ is a strict epimorphism. Then, to see that there exist ϕ_D such that $\phi_{D'} = \phi_d \circ \pi$ it is enough to prove that $\phi_{D'} \circ p_1 = \phi_{D'} \circ p_2$. Where,

$$p_i : Z' \times_Z Z' \to Z'$$

are the projections.

Since,

$$\begin{aligned}(\phi_{D'} \circ p_1 \times id)^{-1}(D_{univ}) &= (p_1 \times id)^{-1}(D') = (p_1 \times id)^{-1}(\pi^{-1}(D)) \\ &= (p_2 \times id)^{-1}(\pi^{-1}(D)) = (\phi_{D'} \circ p_2 \times id)^{-1}(D_{univ})\end{aligned}$$

and, $(\phi_{D'} \circ p_1 \times id)^{-1}(D_{univ})$ is defined by sections, the conclusion follows.

□

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