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Kissing Number of Hyperbolic Manifolds

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"If I have seen further than others, it is by standing upon the shoulders of giants."

Isaac Newton

Este trabalho é dedicado à mulher que mais admiro e que me inspira pela sua força, eu vi Deus e ele era uma mulher negra, minha querida mãe Maria.

CONTENTS

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ABSTRACT

In this thesis we study the relationship between kissing number and volume in hyperbolic arithmetic manifolds of the first type.

The first result presented is that we guarantee the existence of a sequence M_j of compact arithmetic hyperbolic 3-manifold whose volume tends to infinity and which satisfies the following inequality involving the kissing number and the volume

$$\mathrm{Kiss}(M_j) \geq C \frac{\mathrm{vol}(M_j)^{4/3}}{\log(\mathrm{vol}(M_j))}.$$

Here C is a universal constant. This result extends the result obtained in [15] in the sense that the quoted article only covers the non-compact case for manifolds of dimension 3, and generalizes [40] to dimension 3 keeping the value of the exponent associated with the geometric invariant.

Our second result is the existence of a sequence M_j of compact arithmetic hyperbolic n-manifold as in the previous result also with the volume tending to infinity, satisfying the following relation between kissing number, volume and dimension of the manifolds:

$$\mathrm{Kiss}(M_j) \geq C \; \frac{\mathrm{vol}(M_j)^{1+\frac{1}{3n(n+1)}}}{\log(\mathrm{vol}(M_j))}.$$

Here C is again a universal constant.

Exhibiting the systoles in these spaces is a much more delicate problem. We do this by constructing M_j containing a totally geodesic surface S_j whose systoles are

also systoles of M_j, that is,

$$\mathrm{sys}(S_j) = \mathrm{sys}(M_j).$$

CHAPTER 1

INTRODUCTION

1.1 Kissing number of Riemmanian manifolds and related problems

A *closed hyperbolic n-manifold M* is a compact manifold equipped with a Riemannian metric of constant curvature -1. We can identify M as a quotient space $\Gamma\backslash\mathbb{H}^n$, where $\mathbb{H}^n$ is the hyperbolic n-space and Γ is a torsion-free discrete group of isometries of $\mathbb{H}^n$. These geometric objets appear in many mathematical and physical theories.

Recent developments have brought attention to the search of hyperbolic manifolds with some extremal properties. For example, such spaces with minimal volume [3], minimal diameter [9], large systole [35], large kissing number [15], and combination of these geometrical invariants (see below the definitions of systole and kissing number). In this thesis we are interested in hyperbolic manifolds with large kissing number and in relation of kissing number with the systole and volume.

Recall that a natural geometric invariant associated to a closed manifold is its *volume*. Hence, if another invariant is related with the volume, the question about extreme problems becomes more interesting if we restrict it by considering some relation to the manifold. For example, it follows from simple geometric considerations that for any closed hyperbolic n-manifold M its diameter $\mathrm{diam(M)}$, volume $\mathrm{vol(M)}$ and injectivity radius $r(M)$, are related by (see [37] Lemmas 2.1.2 and 2.3.1 for a proof)

$$(n-1)r(M) - c_n \leq \mathrm{log(vol(M)) \leq (n-1)diam(M) + c_n},$$

for some constant c_n.

It is also useful to understand more intrisic invariants, for example, the Cheeger constant, spectral gap (of the Laplace-Beltrami operator), and the multiplicities in the bottom of the length spectrum and of the eigenvalue spectrum. The Cheeger constant, in particular, provides pertinent information about the measure of connectivity of a manifold (see [8]). The first eigenvalue of the Laplace-Beltrame operator relates to the Cheeger constant, by [10] and [13] the spectrum of the Laplace-Beltrami operator can also be considered as a connectivity measure.

Among the hyperbolic manifolds there is a class on which we will focus, which are the *arithmetic* manifolds. These are manifolds whose fundamental groups are arithmetic subgroups of the isometries of $\mathbb{H}^n$. The reason for this special attention is that in previous works these manifolds often appear in relation to extremal problems, see for example [40]. Not every hyperbolic manifold is arithmetic: in dimension ≥ 4 Gromov and Piatetski-Shapiro constructed in [21] a non-arithmetic manifold; several similar constructions are also known, see for example [4].

The asymptotic behavior of the above mentioned invariants is already known. Indeed, given an *arithmetic* hyperbolic manifold M, there exists a class of finite degree, but arbitrarily large, coverings $M_i \to M$ known as *congruence coverings* of M. It follows from recent results that, for any dimension n, there exist arithmetic closed hyperbolic manifolds such that their congruence coverings satisfy

$$d_n \mathrm{diam(M_i) \precsim log(vol(M_i)) \precsim \frac{n(n+1)}{4} r(M_i)},$$

for injectivity radius of manifold $r(M_i)$ and some constant d_n and which depends only on the dimension. See [35] for more details and our background reference to clarify the notation.

In [41], Schmutz started the investigation of extremal values of systole and kissing number of hyperbolic surfaces with a fixed area. The authors in [15] also contributed to the discussion on this problem, starting with a generalization to dimension 3.

Greater attention has been given to a discrete invariant called the *kissing number*. We start by defining what the kissing number is in the Euclidian space. The classical kissing number problem asks for maximal number of spheres that can touch another one, all of them with the same size in the n-dimensional space. For low dimensional cases such as n is $1, 2$ and 3 the solutions to the problem are well known and easily understood geometrically, as can be seen in Figure 1.1

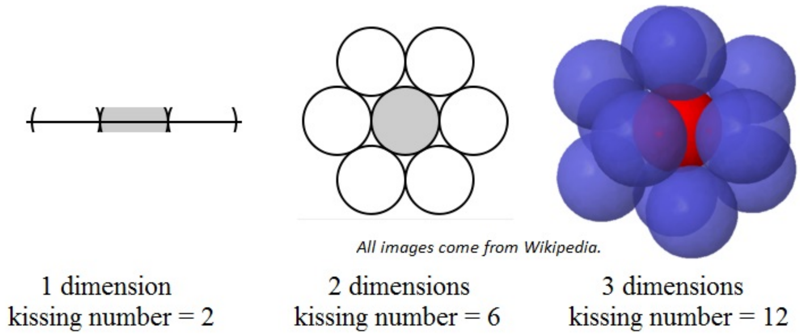


Figure 1.1

Inspired by the classical problem arising in sphere packings, Schmutz defined for an arbitrary Riemannian manifold M the *kissing number*, $\mathrm{Kiss}(M)$, of M as the number of closed geodesics on M of length $\mathrm{sys}(M)$ (see [40],[42], [43]). Recall that the *systole* of a manifold M, denoted by $\mathrm{sys}(M)$, is the minimum of the set of lengths of non-trivial closed geodesics of M. Any finite volume hyperbolic n-manifold M has well defined positive $\mathrm{sys}(M)$.

In general, it follows from a classical result of Anosov ([2]) that a generic Riemannian manifold has at most one systole. For a closed hyperbolic n-manifold M this number can be bigger. It is possible to bound $\mathrm{Kiss}(M)$ from above in terms of $\mathrm{sys}(M)$ and $\mathrm{vol}(M)$. Works by Buser [10] and Keen [24] show that

$$\mathrm{Kiss}(M) \leq A_n \mathrm{vol}(M) \mathrm{sys}(M)^{\lfloor \frac{n-1}{2} \rfloor / \lfloor \frac{n+1}{2} \rfloor}, \tag{1.1.1}$$

where $A_n > 0$ is a constant depending on n. In particular, if $\operatorname{sys}(M) \leq C_n$ for some $C_n > 0$, there exists $D_n > 0$ such that

$$\operatorname{Kiss}(M) \leq D_n \operatorname{vol}(M). \tag{1.1.2}$$

Recently, Bourque and Petri provided an upper bound for $\operatorname{Kiss}(M)$ independent of the size of $\operatorname{sys}(M)$ (see [7, Theorem 1]). More precisely, first they showed that

$$\operatorname{Kiss}(M) \leq A_n \operatorname{vol}(M) \frac{\exp\left(\frac{n-1}{2}\operatorname{sys}(M)\right)}{\operatorname{sys}(M)}. \tag{1.1.3}$$

Observe that Inequality (1.1.3) is weaker than Inequality (1.1.1) for small systole. However, by using a volume bound in terms of the systole applied to inequality (1.1.3), if $\operatorname{sys}(M)$ is large, the authors obtained that

$$\operatorname{Kiss}(M) \leq B_n \frac{\operatorname{vol}(M)^2}{\log(1+\operatorname{vol}(M))}, \tag{1.1.4}$$

(see [7, Corollary 1.2]). In dimension 2, this result had been previously established by Parlier ([36]). In [18], similar upper bounds were established for non-compact hyperbolic surfaces of finite area. It remained an open problem to establish some version of (1.1.3) and (1.1.4) for non-compact finite volume hyperbolic manifolds of dimension $n \geq 3$.

1.2 Results contained in the thesis

These restrictions for $\operatorname{Kiss}(M)$, and the aforementioned result by Anosov motivated us to study the following question formulated in [37]: Let $n \geq 2$ and

$$K_n(v) = \max\{\operatorname{Kiss}(M) \mid M \text{ is a hyperbolic } n\text{-manifold of } \operatorname{vol}(M) \leq v\}.$$

Question 1. *How does $K_n(v)$ grow as a function of v?*

Although this question is independent of the size of $\operatorname{sys}(M)$, it is interesting to understand $\operatorname{Kiss}(M)$ in relation to whether $\operatorname{sys}(M)$ is small or large. Recall that

Wang in [45], showed that the number of hyperbolic n-manifolds up to isometry of volume $\leq v$ is finite for $n \geq 4$.

Recalling the Question 1 mentioned in the beginning of this introduction, we are interested in the following: Given $n, V > 0,$ what is the maximal $K(n, V)$ that can be attained by the kissing number of a closed hyperbolic n-manifold of volume at most V?

Throughout the thesis we will focus on giving an answer to Question 1 (independently the size of the systoles.) For $n = 2$, it follows from results by Schmutz in [40] that

$$\limsup_{v \to \infty} \frac{\log K_2(v)}{\log v} \geq 1 + \frac{1}{3}. \tag{1.2.1}$$

To prove this result, the author constructed a sequence S_i of closed (also non-compact of finite area) hyperbolic surfaces with large kissing number obtained as congruence coverings of a fixed arithmetic hyperbolic surface. It is worth noting that the surfaces S_i also satisfy

$$\mathrm{sys}(S_i) \sim \frac{4}{3} \log(\mathrm{area}(S_i)) \xrightarrow{i \to \infty} \infty.$$

More generally, if a sequence M_i of non-diffeomorphic closed hyperbolic n-manifolds has $\mathrm{Kiss}(M_i)$ growing super linearly in $\mathrm{vol}(M_i)$, then $\mathrm{sys}(M_i)$ grows logarithmically in $\mathrm{vol}(M_i)$. Indeed, it follows from (1.1.2) that $\mathrm{sys}(M_i) \to \infty$ and the logarithmic growth follows from (1.1.3).

In [35], the author showed that some congruence coverings of closed arithmetic hyperbolic n-manifold of the first type have systole growing logarithmically with the volume and determined the precise growth ratio. It is then natural to investigate the kissing number of such manifolds asking whether they can provide a version of (1.2.1) in higher dimension and as the lower limitation depends on the dimension. In this direction we obtain that there exists a compact arithmetic hyperbolic n-manifold of the first type M and a sequence of congruence coverings M_j, such that

$$\mathrm{Kiss}(M_j) \geq C \, \frac{\mathrm{vol}(M_j)^{1 + \frac{1}{3n(n+1)}}}{\log(\mathrm{vol}(M_j))} \tag{1.2.2}$$

for some constant $C > 0$ independent of M_j.

On the other hand, as it has already been observed in the Appendix of [35], for any sequence of congruence covering N_i of a compact arithmetic hyperbolic manifold

N of the first type, we have a totally geodesic congruence arithmetic surface $\Sigma_i \subset N_i$ satisfying

$$\operatorname{sys}(N_i) \leq \operatorname{sys}(\Sigma_i) \sim \frac{8}{n(n+1)} \log(\operatorname{vol}(N_i)). \tag{1.2.3}$$

It follows from the authors constructions that the first inequality in (1.2.3) is optimal.

Moreover, when we consider specifically the case $n = 3$, we have the advantage of using matrices with complex entries, since the group of orientation preserving isometries of $\mathbb{H}^3$ is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. In analogy to the work of Schumtz on kissing number of arithmetic hyperbolic surfaces, in [15] Dória and Murillo constructed congruence coverings of some non-compact arithmetic hyperbolic 3-manifolds N_i such that

$$\frac{\log \operatorname{Kiss}(N_i)}{\log \operatorname{vol}(N_i)} \gtrsim 1 + \frac{4}{27}.$$

We are able to construct arithmetic hyperbolic 3-manifolds with a large number of systoles using the relation between length and trace of 2×2 matrices, and a result on equidistribution of closed geodesics with holonomy in prescribed intervals, proved by Margulis, Mohammadi and Oh (see [30]). In this way, we guarantee the existence of a sequence $\{M_j\}$ of compact arithmetic hyperbolic 3-manifolds, with $\operatorname{vol}(M_j)$ going to infinity, such that.

$$\operatorname{Kiss}(M_j) \gtrsim C \frac{\operatorname{vol}(M_j)^{4/3}}{\log(\operatorname{vol}(M_j))} \tag{1.2.4}$$

where $C > 0$ is a universal constant. The estimates (1.2.2) and (1.2.4) for the asymptotic growth of the kissing number are the main results of the thesis.

1.3 Structure of this work

We begin in Chapter 2 by recalling concepts and definitions from the theory of Riemannian manifolds and some relations with algebraic structures, focusing on the case of dimension 3. In the end of the chapter we discuss hyperbolic manifolds that arise from the study of quaternion algebras and study their systoles. With this we prove (1.2.4). Moreover, the discussion in this chapter motivates the investigation in higher dimensions that is developed in subsequent Chapters.

In Chapter 3 we construct n-dimensional arithmetic hyperbolic manifolds of the first type and their congruence coverings discuss the spin group that plays a fundamental role in the construction of sequences of manifolds in order to obtain result (1.2.2). Finally, in Chapter 4 we study the systoles of the sequence of manifolds through totally geodesic surfaces. In Chapter 5, we bring some basic results and definitions that were used in earlier the Thesis.

CHAPTER 2

PRELIMINARIES

2.1 Basic definitions

Definition 2.1.1. An *n-dimensional manifold* M is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$. This means that for each $x \in M$ there exists an open subset U containing x and a homeomorphism $\phi : U \to \mathbb{R}^n$, which we call a *local chart* around x. Two pairs (U_α, ϕ_α) and (U_β, ϕ_β) are said to be C^∞-related if $U_\alpha \cap U_\beta \neq \emptyset$ and the homeomorphisms $\phi_\beta \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi_\beta^{-1}$ are C^∞ (i.e., smooth). An *atlas* on a manifold M is a family of pairs $\{(U_\alpha, \phi_\alpha)\}$ such that $\cup U_\alpha = M$ and every couple of charts is C^∞-related. A *differentiable manifold* is a manifold with an atlas defined on it, in this situation we will usually call M the n-manifold.

A *smooth structure* on a manifold M is a collection of smoothly equivalent smooth atlas. Some basic examples of topological manifolds are points, lines and circles. There are a lot of more elaborate examples and deep results concerning the study of the topology of n-manifolds which is not the focus of this thesis.

Next, we would like to extend the idea of a tangent vector to differentiable manifolds, because then we will have, at each point, a linear approximation that is its tangent plane. On an n-manifold M given $p \in U \subset M$ and (U, ϕ) a pair, the tangent space $T_p(M)$ is defined as the equivalence classe of curves $\gamma : (-1, 1) \to M$

with $\gamma(0) = p$, where two curves γ_1 and γ_2 are equivalent if the usual derivative at 0 of $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$ coincide. This definition does not depend on ϕ. By definition, the derivative of the curve γ is the equivalence class in $T_p(M)$.

We can define in a natural way the concept of diffeomorphism between manifolds as follows. A map $f : M \to N$ we say is said to be *differentiable* at p if there exist pairs (U, ϕ) and (V, ψ) at p and $f(p)$ such that, in local charts, f is smooth as a map between open sets of Euclidean spaces. If f has an inverse which is also differentiable, then f is called a *diffeomorphism*.

The derivative of a smooth map at a point represents a linear approximation of the map near that point. The derivative of f at a point $p \in M$ is the map $(df)_p : T_p(M) \to T_{f(p)}(N)$ defined by

$$df_p([\gamma]) = [f \circ \gamma].$$

The derivative plays an important role in the study of geometric objects to better understand the geometry of manifolds. For this we introduce the following definitions:

The map f is said to be an *immersion*, *submersion* or an *embedding* if $(df)_p$ is, respectively, injective, surjective or if f is an immersion which is also a homeomorphism over its image, at each point p, respectively.

Example 2.1.2.
The real coordinate space $\mathbb{R}^n$ is an n-manifold. Note also that the subspaces $\mathbb{R}^k$ with $1 \leq k < n$ have the k-manifold structure. These spaces are prototype of a *submanifold* of a manifold.

For a smooth map between manifolds, $f : M \to N$, a point $q \in N$ is called a *regular value* of f if $df_p : T_p(M) \to T_q(N)$ is surjective at every point p such that $f(p) = q$. Otherwise, we say q is *critical value* of f.

Definition 2.1.3. Let M be an n-manifold and N be a subset of M. Then N is called an *m-submanifold* of M if, for every $p \in N$, there exists a smooth chart (U, ϕ) in M such that $p \in U$ and $\phi(N \cap U) = \mathbb{R}^m \cap \phi(U)$, where $\mathbb{R}^m$ is embedded into $\mathbb{R}^n$ as the subspace $\{x_{m+1} = 0, \ldots, x_n = 0\}$.

Loosely speaking, a manifold is a topological space which, locally, looks like an Euclidian space. Similarly, a submanifold is a subset of a manifold which, locally, looks like a subspace of an Euclidian space. One of the most useful ways to construct submanifolds is given by the following theorems, whose details can be seen in ([5, Section 3.5]).

Theorem 2.1.4. *If $f : M \to N$ is an embedding, then the image $f(M)$ with the smooth structure induced by f is a submanifold of N.*

Theorem 2.1.5. *If q is a regular value of a smooth map $f : M \to N$, then the preimage $f^{-1}(q)$ is a submanifold of M, with dim$(f^{-1}(q)) =$ dim$(M) -$ dim(N).*

Example 2.1.6.
For $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$, $(df)_{(x_0,y_0)} = (2x_0, 2y_0)$, hence if $q \neq 0, f^{-1}(q)$ consists of a pair of hyperbolas. However, $f^{-1}(0)$ consists of two intersecting lines, $y = \pm x$, so it is not a submanifold of $\mathbb{R}^2$. That is, if q is a critical value of f, $f^{-1}(q)$ need not be in general a submanifold.

2.2 Riemannian manifolds and geodesics

Definition 2.2.1. *A Riemannian metric* on a differentiable manifold M is a correspondence that associates to each point p of M an inner product $\langle , \rangle_p$ on the tangent space $T_p(M)$, which varies diferrentiably. A manifold M endowed with a Riemannian metric is called a *Riemannian manifold*. From the metric we can also define

1. The *length* of a tangent vector $v \in T_pM$ by $||v||_p = \langle v, v\rangle_p^{1/2}$;

2. Given a piecewise smooth curve $\gamma : [0, 1] \to M$, the *arc-length* of γ between $a = \gamma(0)$ and $b = \gamma(1)$ is

$$l(\gamma) = \int_0^1 \Big|\Big|\frac{d\gamma}{dt}(t)\Big|\Big|_{\gamma(t)} dt.$$

From the Riemannian metric, we define a distance function over the manifold as follows:

$$d(x,y) = \inf\{l(\gamma)|\gamma : [0,1] \to M \text{ is piecewise smooth}, \gamma(0) = x, \gamma(1) = y\}.$$

It is immediate that the function d satisfies the following

$$\begin{cases} d(x,x) = & 0, \\ d(x,y) > & 0 \text{ if } x \neq y, \\ d(x,y) = & d(y,x), \\ d(x,z) \geq & d(x,y) + d(y,z). \end{cases}$$

With this notion of metric, it is natural to ask how to find a curve whose length realizes the distance in the manifold, and what kind of curves on a given manifold should be the analogues of straight lines in the plane to answer these question we define geodesics, a geodesic is a locally length-minimizing curve.

For any smooth curve $\gamma(t)$ in a Riemannian manifold M, it is possible to define the "acceleration" of γ as the second derivative of $\gamma(t)$, extending the concept from the Euclidean geometry. In this sense, a smooth curve $\gamma(t)$ is called *geodesic* if $\gamma''(t) = 0$ for all t. For more details see [11].

Example 2.2.2.
In the Euclidean space $\mathbb{R}^k$, its only geodesics are the straight lines. Moreover, if we consider $\mathbb{R}^k$ as a vector subspace of $\mathbb{R}^n$ the geodesics of $\mathbb{R}^k$ will also be geodesics in $\mathbb{R}^n$. This property is intriguing and guides us to the following definition.

Definition 2.2.3. A submanifold N of a Riemannian manifold (M, g) is called *totally geodesic* if any geodesic on the submanifold N, with the induced Riemannian metric, is also a geodesic on the Riemannian manifold (M, g).

Example 2.2.4.
An example of a submanifold that is not totally geodesic, is the sphere $\mathbb{S}^{n-1}$ embedded in the Euclidean space $\mathbb{R}^n$, in the natural way. Because the geodesics on the sphere are not geodesics, in $\mathbb{R}^n$, as can be seen in Figure 2.1

The following result allows us to build a rich variety of examples.

Theorem 2.2.5. *Let $f : (M, g) \to (M, g)$ be an isometry of the Riemannian manifold (M, g). Then every connected component of the fixed point set*

$$\{y \in M; f(y) = y\},$$

with the induced Riemannian metric, is a totally geodesic submanifold.

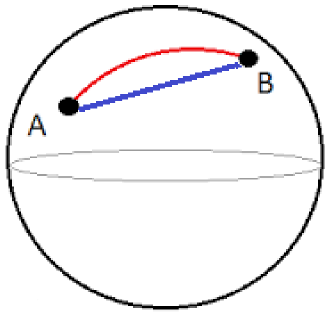
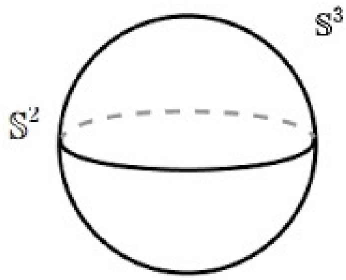


Figure 2.1

(See [25] for more details.)

Example 2.2.6.
Consider the standard sphere $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1};\ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$. For $1 \leq k < n$, the k-sphere $\mathbb{S}^k = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n;\ x_{k+1} = \cdots = x_{n+1} = 0\}$ is a totally geodesic submanifold of $\mathbb{S}^n$. It is the fixed point set of the isometry $f : \mathbb{S}^n \to \mathbb{S}^n$ given by

$$f(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_{n+1}).$$



We end this subsection by summarizing the discussion on differentiable curves. Since the reparameterization of differentiable curves is an equivalence relation, it is natural to consider a parametrized closed geodesic as an equivalence class of closed differentiable curves under a reparameterization. Moreover, it is well known (see [31]) that for any class $[c]$ of non-trivial and not bounding a cusp of closed differentiable curves in M there exists a representative c that is a smooth geodesic. We now consider more closely at hyperbolic manifolds.

Any closed geodesic on a hyperbolic manifold M is parametrized by constant speed from the circle to M, and we can identify the geodesic with its equivalence class under reparametrization. Let $\gamma : \mathbf{S}^1 = \mathbb{R}/[t \mapsto t+1] \to M$ be a closed geodesic. We say that γ is *primitive* if γ is injective, i.e., if γ is an embedding. Any closed geodesic δ is an n-fold iterate of some primitive geodesic γ, i.e., there exists $n \in \mathrm{N}$ such that $\delta(t) = \gamma(nt)$ (up to reparametrizations of δ and γ). We note that n is uniquely determined by the relation $\ell(\delta) = n\ell(\gamma)$ and, because of this, we call it the *order* of δ.

Any hyperbolic manifold is isometric to a quotient space $M = \Gamma\backslash\mathbb{H}^n$, where $\mathbb{H}^n$ is the hyperbolic n-space and Γ is a torsion-free discrete group of isometries of $\mathbb{H}^n$. When Γ is not torsion-free the quotient space is called *n-orbifold*.

Let $\pi : M \to N$ be a covering map between two hypebolic n-orbifolds M and N. A closed geodesic $c : \mathbf{S}^1 \to N$ *lifts* to M if there is a closed geodesic $\tilde{c} : \mathbf{S}^1 \to M$ such that $c = \pi \circ \tilde{c}$. In this case, we say that any such $\tilde{c}$ is a lift of c.

We note that the deck group $\mathrm{Deck}(\pi) = \{\mathrm{g} \in \mathrm{Isom(M)} \mid \pi \circ \mathrm{g} = \pi\}$ is always finite whenever M and N have finite volume. In the sequel, we consider the natural action of $\mathrm{Deck}(\pi)$ on the set of closed geodesics of M.

Lemma 2.2.7. *Let $\pi : M \to N$ be a covering map between the hypebolic n-orbifolds M and N of finite volume, and let $G = \mathrm{Deck}(\pi)$.*

1. *If $\tilde{\gamma_1}, \tilde{\gamma_2}$ are closed geodesics on M which are liftings of two distinct closed geodesics γ_1, γ_2 on N respectively, then the orbits $G \cdot \tilde{\gamma_1}$ and $G \cdot \tilde{\gamma_2}$ are disjoint.*

2. *If γ is a closed geodesic on N of order n that lifts, then for any lift $\tilde{\gamma}$, its isotropy group $G_{\tilde{\gamma}}$ has at most n elements.*

Proof. If $\tilde{\gamma_1} = g \circ \tilde{\gamma_2}$ (up to reparametrization of $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$), then $\gamma_1 = \pi \circ \tilde{\gamma_1} = \pi \circ g \circ \tilde{\gamma_2} = \pi \circ \tilde{\gamma_2} = \gamma_2$, which proves (1). For (2) we can suppose that $M = \Lambda'\backslash\mathbb{H}^n$ and $N = \Lambda\backslash\mathbb{H}^n$ where $\Lambda' < \Lambda$. With this identification, the group G can be considered as $\mathrm{N}_\Lambda(\Lambda')/\Lambda'$ and a closed geodesic on M can be associated with a conjugacy class $[\gamma']$ of a loxodromic element $\gamma' \in \Lambda'$. Moreover, the action of G on the set of closed geodesics is given by $\lambda\Lambda' \cdot [\gamma'] = [\lambda^{-1}\gamma'\lambda]$. If $[\gamma']$ denotes a closed geodesic of order n on N, we can use the same notation for its lift on M since $\gamma \in \Lambda'$. Hence $\lambda\Lambda' \cdot [\gamma'] = [\gamma']$ means that $\lambda^{-1}\gamma'\lambda = \lambda_1^{-1}\gamma'\lambda_1$ for some $\lambda_1 \in \Lambda'$, then $\lambda_1\lambda^{-1}$ comutes with γ'. By hyphotesis, $\gamma' = \eta_0^n$, and by the results in hyperbolic geometry we have that the centralizer of γ' is the cyclic group generated by η_0. Therefore, $\lambda\Lambda' \in \{\eta_0^i\Lambda' \mid 0 \leq i \leq n-1\}$. $\square$

Remark 2.2.8. Let M be a closed hyperbolic n-manifold and let $\Sigma \subset M$ be a totally geodesic submanifold. If α, β are distinct primitive closed geodesics on Σ, then the same fact remains true on M. Indeed, if α is an n-folded iterate of α_0 for some primitive $\alpha_0 : \mathbf{S}^1 \to M$, we have $\alpha_0(0) \in \Sigma$ and $\alpha_0'(0) \in T_{\alpha_0(0)}\Sigma$, thus α_0 is a closed geodesic on Σ and then $\alpha = \alpha_0$. In particular, if $\mathrm{sys}(\Sigma) = \mathrm{sys}(M)$, then $\mathrm{Kiss}(M) \geq \mathrm{Kiss}(\Sigma)$.

2.3 Arithmetic hyperbolic 3-manifolds

When working with hyperbolic manifolds we need to introduce the hyperbolic space. The upper-half space model of the hyperbolic 3-space is given by

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R}; t > 0\},$$

endowed with the Riemannian metric $ds^2 = \frac{dz^2+dt^2}{t^2}$.

The group $G = \mathrm{SL}_2(\mathbb{C})$ acts by isometries on $\mathbb{H}^3$. This action is described as follows: First, we realize $\mathbb{H}^3$ as a subset of the Hamilton's quaternion algebra

$$\mathscr{H} = \{a + bi + cj + dk |\ a, b, c, d \in \mathbb{R}, i^2 = j^2 = -1, k = ij\},$$

where we represent a point $P \in \mathbb{H}^3$ as a Hamiltonian quaternion $P = (z,t) := x + yi + tj$, where $z = x + iy$ and $t > 0$. Then, the action of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, is given by

$$P \mapsto MP := (aP + b)(cP + d)^{-1},$$

where the inverse is taken in the skew field of Hamilton's quaternions. This action is not faithful since $-\mathrm{I}$ acts trivially, but the finite quotient $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{I}\}$ is isomorphic to $\mathrm{Isom}^+(\mathbb{H}^3)$ (see [17, Chapter. 1]). Identifying elements in $\mathrm{SL}_2(\mathbb{C})$ with their projection in $\mathrm{PSL}_2(\mathbb{C})$, an element $\gamma \in \mathrm{SL}_2(\mathbb{C})$ is said to be:

- *Parabolic* if γ is conjugate to $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{C},\ z \neq 0$.
- *Elliptic* if γ is conjugate to $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$, $|\eta| = 1, \eta \neq \pm 1$.

- *Loxodromic* if γ is conjugate to $\begin{pmatrix} re^{i\theta} & 0 \\ 0 & r^{-1}e^{-i\theta} \end{pmatrix}$, $r > 1, r \in \mathbb{R}$.

The trace of $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ is well-defined up to a sign. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$ we set

$$\mathrm{tr}(\gamma) := \pm(a+d),$$

where the sign is chosen so that $\mathrm{tr}(\gamma) = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, \pi)$. In this way we can categorise the elements according to their respective traces: if they are in $(-2, 2)$, equal to ± 2, or otherwise . We shall abuse notation and consider the eigenvalues of γ as the eigenvalues of a lift to $\mathrm{SL}_2(\mathbb{C})$. Hence the roots of the characteristic polynomial associated to γ are

$$\lambda_\gamma^{\pm} = \frac{\mathrm{tr}(\gamma) \pm \sqrt{(\mathrm{tr}(\gamma))^2 - 4}}{2}.$$

In the rest of this thesis, when γ is loxodromic we represent by λ_γ the root with norm greater than one. It is well-known that λ_γ determines the traslation length of γ. More precisely

$$l(\gamma) = 2\log(|\lambda_\gamma|) \tag{2.3.1}$$

is the translation length of γ.

We can consider the branch of the argument function $\mathrm{Arg}(z)$ on $V = \mathbb{C} \setminus (-\infty, 0]$ with $\mathrm{Arg}(z) \in (-\pi, \pi)$. The *holonomy* of γ is defined as

$$\theta(\gamma) := 2\mathrm{Arg}(\lambda_\gamma). \tag{2.3.2}$$

The complex number $l(\gamma) + i\theta(\gamma)$ is usually called the *complex translation length* of γ. Let $T : \mathbb{C}^* \to \mathbb{C}$ be the holomorphic map given by $T(z) = z + z^{-1}$. Hence, $\mathrm{Arg}(T(z))$ is a continuous map from $T^{-1}(V)$ to $(-\pi, \pi)$. This leads us to the following technical lemma which will be of great importance to better understand the relationship between the holonomy of an element and the holonomy of the corresponding trace.

Lemma 2.3.1. *Let $T : V \to \mathbb{C}$ be the homeomorphism given by $T(z) = z + z^{-1}$ and consider the continuous function $\mathrm{Arg} : \mathbb{C} \setminus (-\infty, 0] \to (-\pi, \pi)$ given by $z = |z|e^{i\mathrm{Arg}(z)}$. Then we have the following properties:*

i) For any $c \in (1, 2)$, with $|w| > 1, |z| > \frac{1}{c-1}$ we have that

$$|T(w)| > c|z| \textit{ implies } |w| > |z|. \tag{2.3.3}$$

ii) If $z \in T^{-1}(V)$ with $|z| > 1$ and $|\mathrm{Arg}(z)| \neq \frac{\pi}{2}$, then $|\mathrm{Arg}(T(z)| \neq \frac{\pi}{2}$. Moreover, $\tan(\mathrm{Arg}(T(z))) > 0$ if and only if $\tan(\mathrm{Arg}(z)) > 0$, and it holds that:

$$|\tan(\mathrm{Arg}(T(z)))| \leq |\tan(\mathrm{Arg}(z))|.$$

Proof. $i)$ Indeed, since $|\omega| - 1 \leq |T(\omega)| \leq |\omega| + 1$ for all ω with $|\omega| > 1$, we have

$$|w| \geq |T(w)| - 1 > c|z| - 1.$$

Since $|z| > \frac{1}{c-1} > 1$ is equivalent to $c|z| - 1 > |z|$, we obtain

$$|w| > c|z| - 1 > |z|.$$

$ii)$ If we write $z = |z|\cos(\mathrm{Arg}(z)) + i|z|\sin(\mathrm{Arg}(z))$, where $|\mathrm{Arg}(z)| < \pi$, then

$$T(z) = (|z| + |z|^{-1})\cos(\mathrm{Arg}(z)) + i(|z| - |z|^{-1})\sin(\mathrm{Arg}(z)).$$

Note that when $|z| \neq 1$, $|\mathrm{Arg}(T(z))| = \frac{\pi}{2}$ if, and only if, $|\mathrm{Arg}(z)| = \frac{\pi}{2}$. Hence, if $|z| > 1$ and $|\mathrm{Arg}(z)| \neq \frac{\pi}{2}$ we obtain

$$\tan(\mathrm{Arg}(T(z))) = \frac{|z| - |z|^{-1}}{|z| + |z|^{-1}}\tan(\mathrm{Arg}(z)).$$

The result follows directly from this equality. □

We end this section by determining $l(\gamma)$ from $\mathrm{tr}(\gamma)$, a result that can be found in [15, Proposition. 2.1] (*c.f.* [19, Lemma. 5.1]), but first we will prove Lemma 2.3.3 that gives an interesting property of loxodromic elements.

Lemma 2.3.2. *Let γ be a loxodromic element. Then,*

$$\cosh((l(\gamma) + i\theta(\gamma))/2) = \mathrm{tr}(\gamma)/2.$$

(See [28, Section 12] for the details.)

Lemma 2.3.3. *Let $\gamma \in \Gamma$ be an arbitrary loxodromic element. We have that $l(\gamma^2) = 2l(\gamma)$.*

Proof. First note that, for any $B \in \mathrm{SL}_2(\mathbb{C})$, $B^2 = \mathrm{tr}(B)B - I$. It follows directly that $\mathrm{tr}(B^2) = \mathrm{tr}(B)^2 - 2$. In addition, it is worth noting that $\cosh^{-1}(s) = \log(s + \sqrt{s^2 - 1})$.

Therefore:

$$\begin{aligned} l(\gamma^2) &= 2\cosh^{-1}\left(\frac{\mathrm{tr}(\gamma)^2}{2} - 1\right) - i\theta(\gamma^2) \\ &= 2\log\left(\frac{\mathrm{tr}(\gamma)^2}{2} - 1 + \mathrm{tr}(\gamma)\sqrt{\frac{\mathrm{tr}(\gamma)^2}{4} - 1}\right) - 2i\theta(\gamma) \\ &= 4\log\left(\frac{\mathrm{tr}(\gamma)}{2} + \sqrt{\frac{\mathrm{tr}(\gamma)^2}{4} - 1}\right) - 2i\theta(\gamma) \\ &= 2l(\gamma). \end{aligned}$$

□

Proposition 2.3.4. *For any loxodromic element $\gamma \in \mathrm{SL}_2(\mathbb{C})$ we have*

$$4\cosh\left(\frac{l(\gamma)}{2}\right) = |\mathrm{tr}(\gamma) - 2| + |\mathrm{tr}(\gamma) + 2|.$$

In particular,

$$4\cosh(l(\gamma)) = |\mathrm{tr}(\gamma)^2| + |\mathrm{tr}(\gamma)^2 - 4|.$$

Proof. Let $X \in \mathrm{SL}_2(\mathbb{C})$ be any loxodromic element. Denoting $\mathrm{tr}(X) = x + iy$, by Lemma 2.3.2, we have that $x = \pm 2\cosh\left(\frac{l(X)}{2}\right)\cos\left(\frac{\theta(X)}{2}\right)$ and $y = \pm 2\sinh\left(\frac{l(X)}{2}\right)\sin\left(\frac{\theta(X)}{2}\right)$. With these equalites, the pair (x, y) satisfies the following equation:

$$\frac{x^2}{\left(2\cosh\left(\frac{l(X)}{2}\right)\right)^2} + \frac{y^2}{\left(2\sinh\left(\frac{l(X)}{2}\right)\right)^2} = 1.$$

The points (x, y) form an ellipse that intersects the real axis at $\pm 2\cosh\left(\frac{l(X)}{2}\right)$, hence

$$|\mathrm{tr}(X) - 2| + |\mathrm{tr}(X) + 2| = 4\cosh\left(\frac{l(X)}{2}\right).$$

Considering $X = B^2$ and using Lemma 2.3.3, we obtain the second formula.

□

2.4 Quaternion algebras

Let k be a field with characteristic other than 2, we denote by $k^\times$ the invertible elements in k.

Definition 2.4.1. A ring D with unit is a k-algebra if D satisfies the following condition

$$\lambda(xy) = (\lambda x)y = x(\lambda y), \text{ for all } \lambda \in k \text{ and } x, y \in D.$$

Example 2.4.2.
The space $M_n(k)$ of $n \times n$ matrices with entries in k is a k-algebra.

Take $a, b \in k^\times$. A quaternion algebra $\mathscr{A} = \left(\frac{a,b}{k}\right)$ is defined to be the k-algebra with two generators i, j, which satisfy the following relations

$$i^2 = a, \ j^2 = b, \ ij = -ji.$$

Consider $t = ij \in \mathscr{A}$. Then $t^2 = -ab \in k^\times$.

Example 2.4.3.
Take the case where $k = \mathbb{R}$ and $a = b = -1$. Then $\mathscr{A}$ coincides with the usual Hamiltonian quaternions, denoted by $\mathscr{H}$.

Proposition 2.4.4. *For any $a, b, x, y \in k^\times$ we have*

$$\left(\frac{a, b}{k}\right) \cong \left(\frac{ax^2, by^2}{k}\right).$$

Proof. Let $\mathscr{A} = \left(\dfrac{a, b}{k}\right)$, with basis $1, i, j, t$ as in the general construction, and let $\mathscr{A}' = \left(\dfrac{ax^2, by^2}{k}\right)$, with basis $1, i', j', t'$ such that $(i')^2 = ax^2$, $(j')^2 = by^2$. Consider the elements xi and yj in $\mathscr{A}$, for which we have

$$(xi)^2 = x^2 i^2 = ax^2, \ (yj^2) = y^2 j^2 = by^2 \text{ and } (xi)(yi) = -(yi)(xi).$$

Thus, $\phi : \mathscr{A}' \to \mathscr{A}$, induced by mapping $i' \mapsto xi, j' \mapsto yj$, provides a k-algebra isomorphism between $\mathscr{A}'$ and $\mathscr{A}$. □

Definition 2.4.5. An element $x = \alpha + \beta i + \gamma j + \delta t \in \mathscr{A}$ is called a *pure quaternion* if $\alpha = 0$. The k-space of pure quaternions is denote by $\mathscr{A}_0$.

Proposition 2.4.6. *Let $x \in \mathscr{A}$ be different from zero. Then $x \in \mathscr{A}_0$ if, and only if, $x \notin k$ and $x^2 \in k$.*

Proof. In general, if $x = \alpha + \beta i + \gamma j + \delta t$, then

$$x^2 = (\alpha^2 + a\beta^2 + b\gamma^2 - ab\delta^2) + 2\alpha(\beta i + \gamma j + \delta t).$$

Thus, if x is pure, we have $x^2 \in k$. Conversely, if $x \notin k$ and $x^2 \in k$, then the equation above implies that $\alpha = 0$; this means that x is pure. □

The following corollary guarantees us the invariance of pure quaternions under isomorfisms.

Corollary 2.4.7. *Let* $\mathscr{A} = \left(\frac{a,b}{k}\right)$, $\mathscr{A}' = \left(\frac{a',b'}{k}\right)$ *be quaternion algebras and let* $\phi : \mathscr{A} \to \mathscr{A}'$ *be a k-algebra isomorphism. Then* $\phi(\mathscr{A}_0) = \mathscr{A}'_0$.

2.4.1 Quaternion algebras as quadratic spaces

For an arbitrary quaternion $x = \alpha + \beta i + \gamma j + \delta t$, we define the *conjugate* of x to be $\overline{x} = \alpha - (\beta i + \gamma j + \delta t)$. A direct computation shows that

$$\overline{x+y} = \overline{x} + \overline{y},\ \overline{xy} = \overline{y}\,\overline{x},\ \overline{\overline{x}} = x,$$

and $\overline{rx} = r\overline{x}\ \ (r \in k)$.

Definition 2.4.8. The map $x \mapsto \overline{x}$ is called the *bar involution* on $\mathscr{A}$. For $x \in \mathscr{A}$ as above, we define the *norm* of x as $\mathrm{N}(x) = x\overline{x}$, and $\mathscr{T}(x) = x + \overline{x}$ is the *trace* of x.

Example 2.4.9.
Let x be an element in the quaternion algebra $\mathscr{A} = \left(\frac{a,b}{k}\right)$, represented in coordinates by $\alpha + \beta i + \gamma j + \delta t$, then

$$\mathrm{N}(x) = \alpha^2 - \beta^2 a - \gamma^2 b + \delta^2 ab.$$

Now consider the following function:

$$B(x,y) := (x\overline{y} + y\overline{x})/2 = \mathscr{T}(x\overline{y})/2.$$

This is a bilinear form on $\mathscr{A} = \left(\frac{a,b}{k}\right)$, so $(\mathscr{A}, B)$ becomes a quadratic space over k. The quadratic form associated with this bilinear form B sends

$$x \mapsto B(x,x) = \mathrm{N}(x).$$

Note that i, j, t form an orthogonal basis for the quadratic subspace $\mathscr{A}_0 \subset \mathscr{A}$. Futher, if x is pure, then $B(x,1) = \mathscr{T}(x)/2 = 0$, so k is orthogonal to the entire subspace $\mathscr{A}_0$, the following theorem can be found in [28].

Theorem 2.4.10. *For algebras of quaternions* $\mathscr{A} = \left(\frac{a,b}{k}\right)$ *and* $\mathscr{A}' = \left(\frac{a',b'}{k}\right)$, *the following statements are equivalent:*

1. *$\mathscr{A}$ and $\mathscr{A}'$ are isomorphic as k-algebras.*

2. *$\mathscr{A}$ and $\mathscr{A}'$ are isometric as quadratic spaces.*

3. *$\mathscr{A}_0$ and $\mathscr{A}_0'$ are isometric as quadratic spaces.*

Proof. The equivalence $(2) \Leftrightarrow (3)$ is clear from Witt's Cancellation Theorem [12]. Let us show now $(1) \Rightarrow (2)$. Suppose $\phi : \mathscr{A} \to \mathscr{A}'$ is an algebra isomorphism. Then, Corollary 2.4.7 implies that $\phi(\mathscr{A}_0) = \mathscr{A}_0'$. If $x = \alpha + x_0$, where $\alpha \in k$ and $x_0 \in \mathscr{A}_0$, then $\overline{x} = \alpha - x_0$, and hence $\phi(x) = \alpha + \phi(x_0)$ and $\phi(\overline{x}) = \alpha - \phi(x_0)$. Since $\phi(x_0) \in \mathscr{A}_0'$, we have $\overline{\phi(x)} = \phi(\overline{x})$. Therefore,

$$\mathrm{N}(\phi(x)) = \phi(x) \cdot \overline{\phi(x)} = \phi(x)\phi(\overline{x}) = \phi(\mathrm{N}(x)) = \mathrm{N}(x)$$

so ϕ is an isometry from $\mathscr{A}$ to $\mathscr{A}'$. Finally, let us show that $(3) \Rightarrow (1)$. Start with an isometry $\sigma : \mathscr{A}_0 \to \mathscr{A}_0'$. Then,

$$\mathrm{N}(\sigma(i)) = \mathrm{N}(i) = -a, \text{ and also } \mathrm{N}(\sigma(i)) = \sigma(i)\overline{\sigma(i)} = -\sigma(i)^2.$$

Therefore, $\sigma(i)^2 = a$, and similarly, $\sigma(j)^2 = b$. Lastly,

$$i \perp j \Rightarrow \sigma(i) \perp \sigma(j) \Rightarrow \sigma(i)\sigma(j) = -\sigma(j)\sigma(i).$$

All of these put together imply that $\mathscr{A}' \cong \left(\dfrac{a,b}{k}\right) = \mathscr{A}$, proving (1). $\square$

To finish this section we discuss congruence subgroups in quaternion algebras. For this, we briefly introduce orders, which are the analogues in quaternion algebras of rings of integers in number fields.

Throughout the end this section, k denotes a totally real number field. Let $\mathscr{A}$ be a quaternion algebra over k and denote the ring of integers of k by $\mathscr{O}_k$. An *ideal* I in $\mathscr{A}$ is a finitely generated $\mathscr{O}_k$-module of rank 4 such that any $\mathscr{O}_k$-basis of I is a k-basis of $\mathscr{A}$. An *order* $\mathscr{O}$ in $\mathscr{A}$ is an ideal which is also a subring of $\mathscr{A}$ containing 1.

Example 2.4.11.
Let $k = \mathbb{Q}(\sqrt{5})$ with $\mathscr{O}_k = \mathbb{Z}[\gamma]$ where $\gamma = \frac{1+\sqrt{5}}{2}$. We can take $\mathscr{A} = \left(\frac{-1,-1}{k}\right)$ so that $\mathscr{A}$ has basis $1, i, j, ij$ over k and $i^2 = -1 = j^2$. Then $\mathscr{O} = \mathscr{O}_k \oplus \mathscr{O}_k i \oplus \mathscr{O}_k j \oplus \mathscr{O}_k ij$ is an $\mathscr{O}_k$-order.

For any ideal $I \subset \mathcal{O}$ we define

$$I\mathcal{O} = \left\{ \sum_j t_j w_j | \; t_j \in I \text{ and } w_j \in \mathcal{O} \right\}.$$

We denote by $\mathcal{O}^1$ the elements belonging to an order in the quaternion algebra with norm equal to 1. The *congruence subgroup* of $\mathcal{O}^1$ of level I is given by

$$\mathcal{O}^1(I) = \{\gamma \in \mathcal{O}^1 \mid \gamma - 1 \in I\mathcal{O}\}.$$

Lemma 2.4.12. *For any $\gamma \in \mathcal{O}^1(I)$ it holds that $\mathcal{T}(\gamma) \equiv 2 \bmod I^2$.*

Proof. Let $\gamma \in \mathcal{O}^1(I)$. By definition we can write $\gamma = 1 + \eta$, with $\eta \in I\mathcal{O}$. Since $\mathcal{T}(\eta) = \eta + \overline{\eta} \in I$ and $\mathrm{N}(\gamma) \in I^2$ (see [23, Lemma 3.3]), we have

$$1 = \gamma\overline{\gamma} = 1 + \mathcal{T}(\eta) + \mathrm{N}(\gamma).$$

Therefore $\mathcal{T}(\eta) \in I^2$, and then $\mathcal{T}(\gamma) \equiv 2 \bmod I^2$. $\square$

We will now quote a proposition that can be found in [22, Chapter 5] that is very useful in the construction of objects that will be studied later.

Proposition 2.4.13. *Let $\mathcal{A}$ be quaternion algebra and $\mathcal{O}$ an order in $\mathcal{A}$. Then there is a discrete embedding from $\mathcal{O}^1$ in $\mathrm{SL}_2(\mathbb{C})$ given by*

$$\alpha + \beta i + \gamma j + \delta t \mapsto \begin{pmatrix} \alpha + \beta\sqrt{a} & \gamma + \delta\sqrt{a} \\ b(\gamma - \delta\sqrt{a}) & \alpha - \beta\sqrt{a} \end{pmatrix},$$

which has cocompact image if and only if $\mathcal{A}$ is a division algebra.

2.5 Systole of hyperbolic 3-manifolds

A *hyperbolic n-manifold M* is a complete Riemannian manifold of dimension n with constant sectional curvature equal to -1. If M is an orientable hyperbolic 3-manifold, then M is isometric to $\Gamma\backslash\mathbb{H}^3$, where Γ is a specific group called a Kleinian group [22, Section 8]. So it is natural to study this class of groups to understand hyperbolic 3-manifolds.

Definition 2.5.1. A *Kleinian* group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. A hyperbolic 3-orbifold is a quotient $\Gamma\backslash\mathbb{H}^3$, where Γ is a Klenian group. In the case when Γ is torsion free, the quotient is an orientable hyperbolic 3-manifold.

For more details on the notation used below, see Chapter 5. Let k be a number field with exactly 1 complex place and let $\mathscr{A}$ be a quaternion algebra over k which is ramified at all real places. A Kleinian group is called *arithmetic* if it is commensurable with some $P\rho(\mathcal{O}^1)$, where ρ is a k-embedding of $\mathscr{A}$ into $M_2(\mathbb{C})$. Futhermore, when the group is a finite index subgroup of the image of such an embedding we say that this *arithmetic group is derived from a quaternion algebra*.

Hyperbolic 3-manifolds or 3-orbifolds will be referred to as arithmetic when their uniformising groups Γ are arithmetic Kleinian groups. As examples of arithmetic Kleinian groups we have the well-known Bianchi group which are Kleinian group of the form $\mathrm{PSL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is a ring of integers of an imaginary quadratic field k.

The shortest length of a nontrivial closed geodesic of a Riemannian manifold M is called *systole*, and is denoted $\mathrm{sys}(M)$. From this we can define the *kissing number* $\mathrm{Kiss}(M)$ as the number of distinct free homotopy classes of closed geodesics in M of length $\mathrm{sys}(M)$. Now, we use the construction made above to investigate possible systole candidates. Systoles and kissing numbers are the main characters of this thesis.

Proposition 2.5.2. *For any arithmetic Kleinian group Γ derived from a quaternion algebra over an imaginary quadratic field, there exist $L, \varepsilon > 0$ such that if $\gamma \in \Gamma$ is a loxodromic element with $\ell(\gamma) > L$ and $0 \leq \theta(\gamma) < \varepsilon$, then γ^2 realizes the systole of the congruence hyperbolic orbifold $\Gamma(I)\backslash\mathbb{H}^3$, where the ideal $I = (\mathrm{tr}(\gamma))$ is generated by $\mathrm{tr}(\gamma)$.*

Proof. Consider a quaternion algebra $\mathscr{A}$ over an imaginary quadratic number field k and $\tilde{\Gamma}$ is it the preimage of Γ under a natural projection. By definition, $\tilde{\Gamma} < \rho(\mathcal{O}^1)$ where $\mathcal{O} \subset \mathscr{A}$ is a maximal order and $\rho : \mathscr{A} \to \mathrm{SL}_2(\mathbb{C})$ is a k-monomorphism of algebras. Hence, we can suppose that $\tilde{\Gamma} < \mathcal{O}^1$.

Now we consider $\gamma \in \Gamma$ to be an arbitrary loxodromic element and a representative $\tilde{\gamma} \in \tilde{\Gamma}$ with trace $t \in \mathcal{O}_k$. If we define $\tilde{\Gamma}(t) = \langle \tilde{\Gamma} \cap \mathcal{O}^1(t), -1 \rangle$, then $\Gamma(t) = \tilde{\Gamma}(t)/\{\pm 1\} \trianglelefteq \Gamma$ is a congruence subgroup of Γ and $-\tilde{\gamma}^2 \in \tilde{\Gamma}(t)$ since $\tilde{\gamma}^2 - t\tilde{\gamma} + 1 = 0$ in $\mathcal{O}$. Thus $\gamma^2 \in \Gamma(t)$ for any γ with $\mathrm{tr}(\gamma) = \pm t$.

Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ such that $T(\lambda) = t$. By (2.3.1) and Lemma 2.3.3 we have $\ell(\gamma^2) = 2\ell(\gamma) = 4\log(|\lambda|)$. Hence $\mathrm{sys}(\Gamma(\mathrm{tr}(\gamma))\backslash\mathbb{H}^3) \leq 4\log(\lambda)$. We find now the conditions under which the converse inequality holds.

Let $\eta \in \Gamma(t)$ be any loxodromic element. There exists a representative $\tilde{\eta}$ of η such that $\tilde{\eta} \in \tilde{\Gamma} \cap \mathcal{O}^1(t)$. Hence, it follows from Lemma 2.4.12 that, if τ denotes the trace of $\tilde{\eta}$, then $\tau = 2 + t^2\zeta$, for some $\zeta \in \mathcal{O}_k$ and $\zeta \neq 0$.

If $\tau = T(\mu)$ with $|\mu| > 1$, since $t^2 = \lambda^2 + \lambda^{-2} + 2$, equality $\tau = 2 + t^2\zeta$ can be rewritten as

$$T(\mu) = \zeta T(\lambda^2) + 2(\zeta + 1). \tag{2.5.1}$$

By (2.3.1) it is sufficient to show that $|\mu| > |\lambda|^2$. We will divide our analysis into two cases:

Case 1. $\zeta \notin \mathcal{O}_k^*$: Since k is a quadratic field, we have $|\zeta|^2 \geq 2$. First, we can rewrite (2.5.1) as

$$T(\mu) = \zeta\lambda^2(1 + (\lambda^2)^{-2} + 2(\lambda^2)^{-1} + 2(\lambda^{-2}\zeta^{-1})) = \zeta\lambda^2 R(\lambda^2, \zeta) \tag{2.5.2}$$

where $R(z, \theta) = 1 + z^{-2} + 2z^{-1} + 2z^{-1}\theta^{-1}$ is defined on $\mathbb{C}^* \times (\mathcal{O} - \{0\})$. Since $|\theta| \geq 1$ for any $\theta \in \mathcal{O} - \{0\}$, it follows that for any $\delta > 0$ there exists $N > 0$ such that if (z, θ) is in $\mathbb{C}^* \times (\mathcal{O} - \{0\})$ and $|z| > N$, then $|R(z, \theta)| > 1 - \delta$. In particular, we can choose $N_0 > 2$ such that $|z| > N_0$ implies $|R(z, \theta)| > \dfrac{3\sqrt{2}}{4}$.

Hence, if $|\lambda|^2 > N_0 > 2$, since $|\zeta| \geq \sqrt{2}$, by (2.5.2) we have that

$$|\mu| \geq |T(\mu)| - 1 > \frac{3}{2}|\lambda|^2 - 1 > |\lambda|^2.$$

Case 2. If $\zeta \in \mathcal{O}_k^*$: It is well known that ζ is contained in the set $J = \{\pm 1, \pm i, \pm\omega, \pm\omega^2\}$, where ω is the primitive cubic root of unity.

By Proposition 2.3.4, if $\ell(\eta)$ denotes the displacement of η, then

$$4\cosh\left(\frac{\ell(\eta)}{2}\right) = |T(\mu) - 2| + |T(\mu) + 2|.$$

However, since $\zeta \in J$, we have $|\zeta| = 1$. Equation (2.5.1) implies that

$$|T(\mu) - 2| + |T(\mu) + 2| = |T(\lambda^2) + 2| + |T(\lambda^2) + 2 + 4\zeta^{-1}|.$$

Therefore, $\ell(\eta) \geq 2\ell(\gamma)$ whenever

$$|T(\lambda^2) + 2 + 4\zeta^{-1}| \geq |T(\lambda^2) - 2| \tag{2.5.3}$$

for any choice of $\zeta \in J$. When $\zeta = -1$ we have $|T(\lambda^2) + 2 + 4\zeta^{-1}| = |T(\lambda^2) - 2|$, hence it is enough to prove (2.5.3) for $\zeta \neq -1$.

To prove that the inequality holds for the other choices of ζ we construct auxiliary functions. For any $P > 1$ and $\zeta \in J \backslash \{-1\}$ consider the map defined on $(-\pi/2, \pi/2)$ by

$$h_{P,\zeta}(\phi) = |Pe^{i\phi} + 2 + 4\zeta|^2 - |Pe^{i\phi} - 2|^2.$$

It is straightforward to check that

$$h_{P,\zeta}(\phi) = 16(1 + \Re(\zeta)) + 8P\cos(\phi)[1 + \Re(\zeta) + \Im(\zeta)\tan(\phi)]$$

where $P > 1$ and $\zeta \in \mathbb{C}$ are fixed. In this way, inequality (2.5.3) is equivalent to

$$h_{|T(\lambda^2)|,\zeta^{-1}}(\mathrm{Arg}(T(\lambda^2))) \geq 0 \tag{2.5.4}$$

for any $\zeta \in J$. We then look for conditions on $\mathrm{Arg}(\lambda)$ for which (2.5.4) holds for any $\zeta \in J$.

If $\zeta = 1$, since $\cos(\phi) > 0$ we have $h_{P,1}(\phi) > 0$ for all $\phi \in (-\pi/2, \pi/2)$.

It follows from $\zeta \in J \setminus \{\pm 1\}$ that

$$1 + \Re(\zeta) \geq \frac{1}{2} \tag{2.5.5}$$

and

$$\Im(\zeta) \geq -1\,,\, \Im(\zeta) \neq 0. \tag{2.5.6}$$

Therefore $h_{P,\zeta} > 0$ whenever

$$1 + \Re(\zeta) + \Im(\zeta) \tan \phi > 0. \tag{2.5.7}$$

Suppose now that $0 < \tan(\phi) < \frac{1}{2}$. If $\Im(\zeta) \geq 0$, then (2.5.7) follows from (2.5.5). On the other hand, if $\Im(\zeta) < 0$, by (2.5.6) we get $0 < -\Im(\zeta) \leq 1$, and together with (2.5.5) we obtain that

$$\tan(\phi) < \frac{1}{2} \leq \frac{1 + \Re(\zeta)}{-\Im(\zeta)},$$

from which (2.5.7) follows. So, if $0 < \mathrm{Arg}(\lambda) < \frac{1}{2} \arctan\left(\frac{1}{2}\right)$, then $0 < \tan(\mathrm{Arg}(T(\lambda^2))) < \frac{1}{2}$, by Lemma 2.3.1 and the fact that $\mathrm{Arg}(\lambda^2) = 2\mathrm{Arg}(\lambda)$. Therefore (2.5.4) follows as desired.

We conclude the analysis of the two cases that for $L = 4\log(N_0) > 0$ (with N_0 given in Case 1), and $\epsilon = \frac{1}{2}\arctan(\frac{1}{2})$, if $\ell(\gamma) > L$ and $0 \leq \theta(\gamma) < \epsilon$, then γ^2 minimizes the set of displacements of $\Gamma(\mathrm{tr}(\gamma))$, and:

$$\mathrm{sys}(\Gamma(t)\backslash\mathbb{H}^3) = 2\ell(\gamma).$$

□

It follows from Proposition 2.5.2 that we have necessary conditions for the candidate γ^2 to be a systole, but it also raises the question of how the multiple primitive conjugation classes behave asymptotically, and that is precisely what the next proposition addresses.

We need to fix notation. Let $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ be a loxodromic element, we can associate to γ the complex number $z(\gamma) = e^{\frac{\ell(\gamma)}{2}} e^{i\frac{\theta(\gamma)}{2}}$. Thus, from Section 2.3 we can view $T(z(\gamma))$ as the trace of some lifting of γ in $\mathrm{SL}_2(\mathbb{C})$. In what follows, we call by *trace* of γ the complex number $T(\gamma) = T(z(\gamma))$.

Note that this definition of trace remains invariant for conjugation and we extent the definition of trace for a conjugacy class of any subgroup of $\mathrm{PSL}_2(\mathbb{C})$. For a complex number z we will define the *norm of z* as the nonnegative real number $|z|^2$. If $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is a Kleinian group, we define $\sigma(N, I)$ (resp. $\tau(N, I)$) as the number of primitive conjugacy classes of Γ with norm of trace at most N and holonomy in I, counted with multiplicity (resp. counted without multiplicity). By definition, the *mean multiplicity* is given by

$$\mu_0(N, I) = \frac{\sigma(N, I)}{\tau(N, I)}.$$

These definitions will be convenient for presenting the following proposition, consider $k = \mathbb{Q}(\sqrt{d}), d < 0$, be an imaginary quadratic field, then.

Proposition 2.5.3. *Let Γ be an arithmetic Kleinian group derived from a quaternion algebra over an imaginary quadratic field k. For any subinterval $I \subset [0, 2\pi]$, let $\mu_0(N, I)$ be the mean multiplicity of primitive conjugacy classes of Γ with trace of norm at most N and holonomy contained in I. Then there exists a constant $c > 0$ depending only on k and I such that*

$$\mu_0(N, I) \gtrsim c\frac{N}{\log(N)} \text{ when } N \to \infty.$$

Proof. Let $\mathcal{O}_k$ be the ring of integers of k. For any conjugacy class $[\gamma] \subset \Gamma$ we have $T(\gamma) \in \mathcal{O}_k$. Moreover, as $|z(\gamma)| > 1$, we have

$$|T(\gamma)|^2 \leq |z(\gamma)|^2 + 3. \tag{2.5.8}$$

On the other hand, for any $L > 0$ and subinterval $I \subset [0, 2\pi]$, consider

$$\mathcal{N}(L, I) = \#\{[\gamma] \subset \Gamma \mid \gamma \text{ is primitive, } |z(\gamma)| \leq L \text{ and } \theta(\gamma) \in I\}.$$

By [30, Theorem 1.3], there exists a constant c_1 which depends only on I such that

$$\mathcal{N}(L, I) \sim c_1\frac{L^4}{\log(L)} \text{ when } L \to \infty.$$

Hence by (2.5.8), $\sigma(N, I)$ is at least $\mathcal{N}(\sqrt{N-3}, I)$ implying that

$$\sigma(N, I) \gtrsim c_1'\frac{N^2}{\log(N)} \tag{2.5.9}$$

for some constant c_1' depending only on I, and for N sufficiently large.

Moreover, since $\mathcal{O}_k$ is a lattice in $\mathbb{C}$, there exists a constant $c_2 > 0$ depending only on k such that (see [27, Chapter V, Theorem 2]).

$$\#(\mathcal{O}_k \cap B(0, R)) \sim c_2R^2,$$

Hence, when N is big enough we have

$$\tau(N, I) \precsim c_2 N. \tag{2.5.10}$$

Thus, if we combine both growths given in (2.5.9) and (2.5.10), by definition of mean multiplicity, there exists a constant $c > 0$ which depends only in I and k, such that

$$\mu_0(N, I) \gtrsim c\frac{N}{\log(N)} \text{ when } N \to \infty.$$

□

We state a more precise result which implies that every commensurability class of arithmetic hyperbolic 3-manifolds with imaginary quadratic invariant trace field contains a sequence of manifolds with kissing number as we desired. It is by the Classification Theorem of Quaternions Algebras over number fields (see [28, Theorem 7.3.6] and [28, Theorem 8.2.3]) that there exist compact and non compact arithmetic hyperbolic $3-$manifolds with this property. As a consequence, we get the following result.

Theorem 2.5.4. *There exists a sequence $\{M_j\}$ of compact (resp. noncompact) arithmetic hyperbolic 3-manifolds with $\mathrm{vol}(M_j)$ going to infinity such that*

$$\mathrm{Kiss}(M_j) \geq C\frac{\mathrm{vol}(M_j)^{\frac{4}{3}}}{\log(\mathrm{vol}(M_j))},$$

for some constant $C > 0$ which does not depend on j.

Proof. We can suppose that Γ is derived from a quaternion algebra since $\Gamma^{(2)} = \langle\gamma^2 \mid \gamma \in \Gamma\rangle$, has finite index in Γ (see [28, Corollary 8.3.5]).

Consider the constants L and ε given in Proposition 2.5.2 and let k be the invariant trace field of Γ. If we set $I = [0, \varepsilon]$, then by Proposition 2.5.3 there exists a sequence of traces $t_j \in \mathcal{O}_k$ with $|t_j|^2 = N_j \to \infty$ such that the number n_j of primitive conjugacy classes in Γ with trace t_j satisfy

$$n_j \geq c\frac{N_j}{\log(N_j)},$$

where $c = c_{k,\varepsilon} > 0$.

By Lemma (2.4.12), if N_j is large enough, $\mathrm{tr}(\gamma) = \pm 2$ or $|\mathrm{tr}(\gamma)| > 2$. Thus we can assume that for j sufficiently large $\Gamma(t_j)$ is torsion-free and $N_j > L$ for all j.

Let $\gamma_1, \ldots, \gamma_{n_j} \in \Gamma$ be a set of representatives of all primitive conjugacy classes in Γ of trace t_j, and let $G_j = \Gamma/\Gamma(t_j)$ be a group of isometries of $M_j = \Gamma(t_j)\backslash\mathbb{H}^3$. By Proposition 2.5.2, each $\gamma_i^2 \in \Gamma(t_j)$ and when we identify these elements with their induced closed geodesics in M_j, they are systoles of M_j. Since γ_j^2 has order 2, we have by Lemma 2.2.7

$$\mathrm{Kiss}(M_j) \geq \sum_{i=1}^{n_j} \#(G_j \cdot \gamma_i^2) \geq \frac{1}{2} n_j \# G_j.$$

It is a well known fact that $\# G_j = [\Gamma : \Gamma(t_j)] \leq C N_j^3$ for some constant $C > 0$ which does not depend on j (see [23], [26] or [29]). Moreover, $\mathrm{vol}(M_j) = \mu[\Gamma : \Gamma(t_j)]$ for all j, where $\mu = \mathrm{vol}(\Gamma\backslash\mathbb{H}^3)$. Therefore, when we put all the above information together, we get

$$\mathrm{Kiss}(M_j) \geq \frac{1}{2} n_j \# G_j \geq c\,\mathrm{vol}(M_j) \frac{\mathrm{vol}(M_j)^{1/3}}{\log(\mathrm{vol}(M_j))},$$

for any j, where c does not depend on j. $\square$

Theorem (2.5.4) gives a sequence of compact and non-compacts 3-manifolds whose kissing number grows at least as $\mathrm{vol}(M_j)^{\frac{4}{3}-\epsilon}$ for any $\epsilon > 0$. This result is analogous to the main result obtained in [40]: Schmutz showed that the sequence of principal congruence subgroups $\Gamma(N)$ of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ produce hyperbolic surfaces $S(N) := \Gamma(N)\backslash\mathbb{H}^2$ of finite area satisfying

$$\mathrm{Kiss}(S(N)) \geq c\,\mathrm{area}(S(N))^{\frac{4}{3}-\epsilon},\ N \to \infty,$$

for any $\epsilon > 0$ and a universal constant $c > 0$. This rises the question whether it is possible to increase the dimension of the hyperbolic manifolds and how this might influence the exponent of the manifolds volume. These questions lead us to study a possible generalization in the following chapters and for that it is natural to study discrete subgroups of the group $\mathrm{SO}(1,n)^{\circ}$. We do that by first considering the group $\mathrm{Spin}_n(k, Q)$.

CHAPTER 3

ARITHMETIC HYPERBOLIC N-MANIFOLDS

3.1 Hyperbolic manifolds in high dimension

The hyperbolic n-space is the complete simply connected n-dimensional Riemannian manifold with constant sectional curvature equal to -1. A model of the hyperbolic n-space is given by

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1};\ -x_0^2 + x_1^2 + ... + x_n^2 = -1,\ x_0 > 0\}$$

with the metric $ds^2 = -dx_0^2 + dx_1^2 + ... + dx_{n-1}^2 + dx_n^2$.

Consider the Lie group $\mathrm{SO}(1,n)$. Its identity component $\mathrm{SO}(1,n)^\circ$ is isomorphic to the group of orientation preserving isometries $\mathrm{Isom}^+(\mathbb{H}^n)$. Given a lattice $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$, i.e., a discrete subgroup having finite covolume with respect to the Haar measure, the associated quotient space $M = \Gamma\backslash\mathbb{H}^n$ is a finite volume *hyperbolic orbifold*. It is a manifold when Γ is torsion-free.

The classification of the elements isometry group for the case $n = 3$ can also be carried out for higher dimensions. Indeed, γ in $\mathrm{SO}(1,n)^\circ$ is called *elliptic* if it has a fixed point in $\mathbb{H}^n$, *parabolic*, (resp. *loxodromic*) if it has exactly one, (resp. two) fixed points on $\partial\mathbb{H}^n$. For a loxodromic isometry γ its displacement at $x \in \mathbb{H}^n$ is defined by $l(\gamma, x) = d(x, \gamma x)$ and the *displacement* of γ (also called *translation length*) is

$$l(\gamma) = \inf_{x \in \mathbb{H}^n} l(\gamma, x).$$

From these concepts, we can keep the definition of systole and kissing number already discussed in Section (2.5), but now for the case of n-dimensional manifolds.

3.2 Clifford algebras

Assume (E, Q) is a quadratic space of dimension n over k, which is a field with $\mathbf{char}(k) \neq 2$. Denote its associated symmetric bilinear form by Φ, represent by $T(E)$ the algebra of contravariant tensors of (E, Q), and consider the ideal $\mathfrak{I}_Q$ of $T(E)$, generated by the elements $x \otimes y + y \otimes x - 2\Phi(x, y)$. The *Clifford algebra* of Q is defined as

$$\mathscr{C}(Q) := T(E)/\mathfrak{I}_Q.$$

Let $a_1, \ldots, a_n$ be an orthogonal basis of E with respect to Q and denote by e_i the class of $a_i \bmod \mathfrak{I}_Q$ in the algebra $\mathscr{C}(Q)$. Let $\mathscr{P}_n$ be the set of subsets of $\{1, ..., n\}$. Given $M \in \mathscr{P}_n$ represented by $\{\mu_1, \ldots, \mu_\nu\}$ with $\mu_1 < \cdots < \mu_\nu$, we define $e_M = e_{\mu_1} e_{\mu_2} ... e_{\mu_\nu}$, where we adopt the convention $e_\emptyset = 1$. By (see [14, Section 3]) we have that the elements e_M form a 2^n-element basis of $\mathscr{C}(Q)$ over k, so any element s of $\mathscr{C}(Q)$ may be written as $\sum_{M \in \mathscr{P}_n} s_M e_M$. In addition we define the *real part* as the coefficient that accompanies $e_\emptyset$, this is usually denoted by $s_{\mathbb{R}}$.

Consider $A, B \in \mathscr{P}_n$. We can define the following product rule:

$$e_A e_B = \gamma_{A,B} e_{A \triangle B},$$

where $A \triangle B$ is the symmetric difference (in other words, the characteristic function of $A \triangle B$ is the sum of the characteristic functions of A and $B \bmod 2$), and

$$\gamma_{A,B} = (-1)^{\rho(A,B)} \prod_{i \in A \cap B} \Phi(a_i, a_i),$$

Here, $\rho(A, B)$ is the number of inversions obtained in the juxtaposition of A and B, that is:

$$\rho(A, B) = \sum_{j \in B} \rho(A, j).$$

Where, $\rho(A, j)$ denotes the number of elements in A greater than j. In order to simplify the notation we identify a_i with e_i.

Example 3.2.1.
For each μ and ν in $\{1, \cdots, n\}$ with $\mu \neq \nu$, we have $e_\nu^2 = Q(e_\nu)$ and $e_\nu e_\mu = -e_\mu e_\nu$.

Example 3.2.2.
When we consider $k = \mathbb{R}$ and $Q(x_1, ..., x_n) = -x_1^2 - x_2^2 - \cdots - x_n^2$, it follows that:

(i) If $n = 0$ or $n = 1$, $\mathscr{C}(Q)$ is simply $\mathbb{R}$ or $\mathbb{C}$, respectively;

(ii) If $n = 2$, we can identify the sets $\{e_1, e_2\}$ and $\{i, j\}$ and get an identification between $\mathscr{C}(Q)$ and the quaternion algebra $\mathscr{H}$;

(iii) In the case where $n = 4$, consider $e_M = e_2e_3e_4$ and $e_N = e_1e_2$, so we have

$$e_M e_N = \gamma_{M,N} e_1e_3e_4, \text{with } \gamma_{M,N} = (-1)^5(-1) = 1.$$

3.3 Spin group

The Clifford algebra $\mathscr{C}(Q)$, as defined above, has an important anti-involution and an involution which are denoted respectively by $(\ )^*$ and $(\ )'$. They commute with each other and act as follows:

$$* : \mathscr{C}(Q) \to \mathscr{C}(Q)$$
$$(e_{v_1}e_{v_2} \cdots\ e_{v_r})^* = (e_{v_r} \cdots\ e_{v_2}e_{v_1})$$

While

$$' : \mathscr{C}(Q) \to \mathscr{C}(Q)$$
$$(e_{v_1}e_{v_2} \cdots\ e_{v_r})' = (-1)^r(e_{v_1}e_{v_2} \cdots\ e_{v_r}).$$

Example 3.3.1.
Under the same conditions of Example 3.2.2 (ii), we have

$$(e_1e_2)^* = e_2e_1 \text{ and } (e_1e_2)' = e_1e_2.$$

The following result can be found in [46].

Proposition 3.3.2. *For all e_M, e_N in $\mathscr{C}(Q)$, the following relations hold:*

(i) $e_M^* = (-1)^{r(r-1)/2}\, e_M$,

(ii) $(e_M e_N)^* = e_N^* e_M^*$,

(iii) $(e_M + e_N)^* = e_M^* + e_N^*$,

(iv) $(e_M e_N)^{'} = e_M^{'} e_N^{'}$,

(v) $(e_M + e_N)^{'} = e_M^{'} + e_N^{'}$.

Proof. We have

$$\begin{aligned} e_M^* &= (e_{v_1} e_{v_2} \cdots e_{v_{r-1}} e_{v_r})^* \\ &= e_{v_r} e_{v_{r-1}} \cdots e_{v_2} e_{v_1} \\ &= (-1)^{r-1} e_{v_1} e_{v_r} e_{v_{r-1}} \cdots e_{v_3} e_{v_2} \\ &= (-1)^{r-1} (-1)^{r-2} e_{v_1} e_{v_2} e_{v_r} e_{v_{r-1}} \cdots e_{v_4} e_{v_3} \\ &= (-1)^{r-1} (-1)^{r-2} \cdots (-1)^2 (-1) (e_{v_1} e_{v_2} \cdots e_{v_{r-1}} e_{v_r}) \\ &= (-1)^{r(r-1)/2} e_M. \end{aligned}$$

Furthermore,

$$\begin{aligned} (e_M e_N)^* &= (e_{v_1} e_{v_2} \cdots e_{v_{r-1}} e_{v_r} e_{s_1} e_{s_2} \cdots e_{s_{t-1}} e_{s_t})^* \\ &= e_{s_t} e_{s_{t-1}} \cdots e_{s_2} e_{s_1} e_{v_r} e_{v_{r-1}} \cdots e_{v_2} e_{v_1} \\ &= e_N^* e_M^*. \end{aligned} \tag{3.3.1}$$

The formulas for the sum and the properties of the involution $'$ follow directly from the definitions. □

From these definitions and properties, the anti-involution "$(^-)$" defined by $\bar{x} := (x^{'})^*$ satisfies $\bar{e}_M = (-1)^{r(r+1)/2}\, e_M$. The span of the elements e_M with $M \in \mathscr{P}_n$ and $|M| \equiv 0 \bmod 2$ forms a subalgebra of $\mathscr{C}(Q)$, denoted by $\mathscr{C}^+(Q)$.

Now that we are more familiar with these concepts of Clifford algebra we will define the spin group. Let (E, Q) be a quadratic n-dimensional space. Then the spin group of Q is defined as

$$\mathrm{Spin}_n(k, Q) := \left\{ s \in \mathscr{C}^+(Q) \;\middle|\; sEs^* \subseteq E,\ ss^* = 1 \right\}.$$

We omit the lower index when the dimension of the vector space is clear. In the case $k = \mathbb{R}$, $E = \mathbb{R}^{n+1}$ and $Q(x_0, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2$, the corresponding spin group is usually denoted by $\mathrm{Spin}(n, 1)$. For an element $s \in \mathrm{Spin}(n, 1)$ the linear map $\varphi_s : \mathbb{R}^n \to \mathbb{R}^n$ given by $\varphi_s(x) = sxs^*$ lies in $\mathrm{SO}(n,1)^\circ$ and the map $s \to \varphi_s$ is a two-sheeted covering of $\mathrm{SO}(n,1)^\circ$ with kernel $\{\pm 1\}$.

Since the image of a lattice under a finite covering map is a lattice, in order to produce hyperbolic orbifolds we contruct lattices in $\mathrm{Spin}(n,1)$ and project them to $\mathrm{SO}(n,1)^\circ$. Furthermore, we abuse notation and say that an element $s \in \mathrm{Spin}(n,1)$ is elliptic, parabolic or loxodromic if φ_s is elliptic, parabolic or loxodromic, respectively.

In the 3-dimensional case, the Möbius transformations can be represented by 2x2 matrices and they play an important role in understanding the orientation preserving isometries of $\mathbb{H}^3$. It makes us wonder if there is any similar relationship in higher dimensions.

In 1902 Vahlen showed the existence of such a relationship for higher dimensions by defining the *Vahlen group* [44]. The elements of this group, denoted by $\mathrm{SV}_n(k,Q)$, are 2x2 matrices with entries in the real Clifford algebra $\mathscr{C}(Q)$ satisfying certain conditions (see [16] for details). A survey of related results can be found, for example, in [33]. Futhermore, there is an isomorphism between the Spin group and the Vahlen group, defined as follows:

$$\psi : \mathrm{SV}_{(n-2)}(k,Q) \to \mathrm{Spin}_{n+1}(k,\widetilde{Q}) \tag{3.3.2}$$

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \dot{a}\frac{1}{2}(1+f_0f_1) + \dot{b}\frac{1}{2}(f_0f_2 - f_1f_2) + \dot{c}\frac{1}{2}(f_0f_2+f_1f_2) + \dot{d}\frac{1}{2}(1-f_0f_1).$$

Here the quadratic form $\tilde{Q}$ decomposes as

$$\tilde{Q} = Q_0 \perp Q,$$

where:

$$Q_0(y_0,y_1,y_2) = y_0^2 - y_1^2 - y_2^2,$$
$$Q(x_1,x_2,\ldots,x_{n-2}) = -x_1^2 - \ldots - x_{n-2}^2,$$

and $\{f_0,f_1,f_2\}$ is an orthogonal basis for Q_0. Additionally, the $(\dot{\ })$ map denotes a k-algebra homomorfism from $\mathscr{C}(Q)$ to $\mathscr{C}(\tilde{Q})$ that acts as follows

$$\cdot : E \to \mathscr{C}^+(\widetilde{Q}), \quad \dot{x} = f_0f_1f_2x. \tag{3.3.3}$$

For more details on the notation and proof of this isomorphism see [16, Section 2]. We state below a result in [46] that will be useful to get more information on the real part of loxodromics elements.

Lemma 3.3.3. *Let* $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathscr{C}(Q))$ *be a loxodromic, with rotation angles* $2\theta_0, \ldots, 2\theta_{(n-3)/2}$ *and multiplier* λ*, i.e,* $\lambda = e^{\frac{l(T)}{2}}$*. Then,*

$$(a+d)_{\mathbb{R}} = \left(\lambda + \lambda^{-1}\right) \prod_{i=1}^{[\frac{n-3}{2}]} \cos(\theta_i).$$

In order to obtain explicitly the systole of a closed manifold $M = \Gamma \backslash \mathbb{H}^n$, we need to find a hyperbolic element $g_0 \in \Gamma$ such that $\ell(g_0) \leq \ell(g)$, for any nontrivial element $g \in \Gamma$. As observed in [34], it is useful to estimate the displacement of loxodromic elements using information about the real parts of elements in the Spin group. We do this using the relation established between Spin and Vahlen groups.

Proposition 3.3.4. *For any loxodromic element* $r \in \mathrm{Spin}(1,n)$ *we have that*

$$\ell(r) \geq 2\cosh^{-1}(|r_{\mathbb{R}}|).$$

Proof. It is clear that $\mathrm{Spin}(\mathbb{R}, \tilde{Q}) = \mathrm{Spin}(1,n)$, thus from isomorphism (3.3.2) it follows that for a given r in $\mathrm{Spin}(1,n)$ there is a matrix in $\mathrm{M}_2(\mathscr{C}(Q))$ which is its. We have

$$\begin{aligned} r_{\mathbb{R}} = \Big(& \dot{a}\frac{1}{2}(1 + f_0 f_1) + \dot{b}\frac{1}{2}(f_0 f_2 - f_1 f_2) \\ & + \dot{c}\frac{1}{2}(f_0 f_2 + f_1 f_2) + \dot{d}\frac{1}{2}(1 - f_0 f_1) \Big)_{\mathbb{R}} \end{aligned}$$

On the other hand, the map defined in (3.3.3) restricted to any element $a \in \mathscr{C}(Q)$, can be explicitly rewritten as

$$\dot{a} = \left(\sum_{M \in \mathscr{P}_n} a_M e_M \right)^{\cdot} = \sum_{M \in \mathscr{P}_n} a_M (f_0 f_1 f_2)^{\xi_M} e_M,$$

where

$$\xi_M = \begin{cases} 0 \text{ for } |M| \equiv 0 \bmod 2, \\ 1 \text{ for } |M| \equiv 1 \bmod 2. \end{cases}$$

Therefore, the real part of the expression is determined by the real part of the sum of a and d. More precisely

$$r_{\mathbb{R}} = (a+d)_{\mathbb{R}}/2.$$

On the other hand, if we take $r_{\mathbb{R}}$ large enough we are able to apply Lemma 3.3.3 and get

$$r_{\mathbb{R}} = \cosh\left(\frac{l(r)}{2}\right)\prod_{i=1}^{[\frac{n-3}{2}]}\cos(\theta_i).$$

Hence,

$$|r_{\mathbb{R}}| = \cosh\left(\frac{l(r)}{2}\right)\left|\prod_{i=1}^{[\frac{n-3}{2}]}\cos(\theta_i)\right| \leq \cosh\left(\frac{l(r)}{2}\right).$$

It follows directly that

$$l(r) \geq 2\cosh^{-1}(|r_{\mathbb{R}}|).$$

$\square$

3.4 Congruence coverings of arithmetic hyperbolic manifolds

3.4.1 Arithmetic subgroups of Spin group

Definition 3.4.1. A discrete subgroup $\Gamma \subset \mathrm{Spin}(1,n)$ is called *arithmetic* if there exist a number field k, a k-algebraic group $\mathbf{H}$ and an epimorphism $\varphi : \mathbf{H}(k\otimes_{\mathbb{Q}}\mathbb{R}) \to \mathrm{Spin}(1,n)$ with compact kernel such that $\varphi(\mathbf{H}(\mathscr{O}_k))$ is commensurable to Γ. We denoting by $\mathscr{O}_k$ the ring of integers of k and $\mathbf{H}(\mathscr{O}_k) = \mathbf{H}\cap \mathrm{GL}_n(\mathscr{O}_k)$ for some fixed embedding of $\mathbf{H}$ into GL_n.

If Γ is an arithmetic subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$ then the hyperbolic orbifold $M = \Gamma\backslash\mathbb{H}^n$ is called an *arithmetic hyperbolic orbifold*. The Borel-Harish–Chandra Theorem [6] implies that any arithmetic hyperbolic orbifold has finite volume. We refer to Chapter 5 for definitions.

The arithmetic groups commensurable to $\mathrm{Spin}_f(\mathscr{O}_k)$, denoting the restriction of $\mathrm{Spin}(k, f)$ in the ring of integers, with f admissible are called *arithmetic groups of the first type*.

Under these conditions, by restriction of scalars $\mathrm{Spin}_f(\mathscr{O}_k)$ and $\mathrm{SO}_f(\mathscr{O}_k)$ embed as arithmetic subgroups of $\mathrm{Spin}(1, n)$ and $\mathrm{SO}_f(\mathbb{R})$, respectively. Intersecting with $\mathrm{SO}(1, n)^\circ$ we obtain an arithmetic lattice in $\mathrm{Isom}^+(\mathbb{H}^n)$. The subgroups Γ of $\mathrm{Spin}(1, n)$ and $\mathrm{SO}(1, n)^\circ$ constructed in this way and subgroups commensurable to them are called *arithmetic lattices of the first type*. If Γ is torsion-free, $M = \Gamma\backslash\mathbb{H}^n$ is called an *arithmetic hyperbolic manifold of the first type*.

3.4.2 Congruence subgroups of Spin group

Let Γ be an arithmetic subgroup of $\mathrm{Spin}(n, 1)$ commensurable with $\varphi(\mathbf{H}(\mathscr{O}_k))$, and $M = \Gamma\backslash\mathbb{H}^n$ the corresponding hyperbolic arithmetic orbifold. If $I \subset \mathscr{O}_k$ is a non-zero ideal of $\mathscr{O}_k$, the *principal congruence subgroup* of Γ associated to I is the subgroup $\Gamma(I) := \Gamma \cap \varphi(\mathbf{H}(I))$, where

$$\mathbf{H}(I) := \ker\left(\mathbf{H}(\mathscr{O}_k) \xrightarrow{\pi_I} \mathbf{H}(\mathscr{O}_k/I)\right)$$

and π_I denotes the reduction modulo I map. Any ideal $I \subset \mathscr{O}_k$ defines a *principal congruence covering* $M_I = \Gamma(I)\backslash\mathbb{H}^n \to M$. Since $\Gamma(I)$ is a normal finite index subgroup of Γ, the covering $M_I \to M$ is a regular finite sheeted covering map. More generally, a discrete subgroup Λ in $\mathrm{Spin}(n, 1)$ is called a *congruence subgroup* if $\Gamma(I) \subset \Lambda$ for some ideal $I \subset \mathscr{O}_k$.

Let f be an admissible quadratic form over a totally real number field k. We can describe the group $\Gamma = \mathrm{Spin}_f(\mathscr{O}_k)$ and its pricipal congruence subgroups $\Gamma(I)$ in the following way. Denote by $e_1, e_2, \ldots, e_{n+1}$ an orthogonal basis with respect to f. Then under the linear representation given by left multiplication in $\mathscr{C}^+(f, \mathbb{R})$ we get

$$\Gamma = \{s = \sum_{|M| \text{ even}} s_M e_M | s_M \in \mathscr{O}_k \text{ and } ss^* = 1\}$$

and

$$\Gamma(I) = \{s = \sum_{|M| \text{ even}} s_M e_M \in \Gamma | s_M \in I \text{ for } M \neq \emptyset \text{ and } s_{\mathbb{R}} - 1 \in I\}$$

(*see* [35, Sec. 2.4]). To simplify notation, we denote by $\mathcal{Q}$ the $\mathcal{O}_k$-order in $\mathscr{C}^+(f,\mathbb{R})$ given by

$$\mathcal{Q} = \left\{ s = \sum_{|M| \text{ even}} s_M e_M | s_M \in \mathcal{O}_k \right\}.$$

For a principal ideal $I = (\alpha)$ in $\mathcal{O}_k$, we denote $\Gamma(I)$ simply by $\Gamma(\alpha)$.

An important example of congruence subgroup which will play a special role in this work is the following. Fix an element $\alpha \in \mathcal{O}_k$. Let $\tau \in (\mathcal{O}_k/\alpha\mathcal{O}_k)^\times$ be an element of order 2 and define

$$\Gamma_\tau(\alpha) = \{\gamma \in \Gamma \mid \gamma \in \Gamma(\alpha) \text{ or } \gamma \equiv \tau (\operatorname{mod} \alpha\mathcal{Q})\}.$$

The group $\Gamma_\tau(\alpha)$ is a normal subgroup of Γ since it is the preimage under the natural projection map $\Gamma \to (\mathcal{Q}/\alpha\mathcal{Q})^\times$ of the normal subgroup $\{Id, \tau(\operatorname{mod} \alpha\mathcal{Q})\}$.

We present a series of technical lemmas relating the real part of elements in $\Gamma(\alpha)$ with α. These lemmas will play an important role in the proof of the length inequalities in the next chapter.

Lemma 3.4.2 (Compare to Lemma 2.4.12)**.** *Let $\alpha \in \mathcal{O}_k$ be a nonzero element. For any $s \in \Gamma(\alpha)$ we have the equality*

$$s_{\mathbb{R}} = 1 + \frac{1}{2}\alpha^2\zeta$$

for some $\zeta \in \mathcal{O}_k$.

Proof. By definition, we can write $s = 1 + \alpha t$ for some $t \in \mathcal{Q}$. Since $s^* = 1 + \alpha t^*$ we have

$$1 = ss^* = 1 + \alpha(t + t^*) + \alpha^2 tt^*$$

Taking the equality of real parts and observing that $2t_{\mathbb{R}} = (t + t^*)_{\mathbb{R}}$, the result follows for $\zeta = -(tt^*)_{\mathbb{R}}$. ☐

It is useful to have the following complement of Lemma 3.4.2.

Lemma 3.4.3. *Let $\alpha \in \mathcal{O}_k$ be a nonzero element and $s \in \Gamma$ such that $s - s_{\mathbb{R}} \in \alpha\mathcal{Q}$. For any $r \in \Gamma(\alpha)$ we have the equality*

$$(sr)_{\mathbb{R}} = s_{\mathbb{R}} + \frac{1}{2}\alpha^2\zeta$$

for some $\zeta \in \mathcal{O}_k$.

Proof. By previous lemma, we have $r = 1 + \alpha t$ for some $t \in \mathcal{Q}$, and $2t_{\mathbb{R}} = \alpha\xi$, for some $\xi \in \mathcal{O}_k$. Since $s = s_{\mathbb{R}} + \alpha u$, for some $u \in \mathcal{Q}$, we can write

$$sr = s(1 + \alpha t) = s + \alpha(s_{\mathbb{R}} + \alpha u)t = s + \alpha s_{\mathbb{R}} t + \alpha^2 ut.$$

When we take the real parts in the last equality we finish the proof for $\zeta = s_{\mathbb{R}}\xi + 2(ut)_{\mathbb{R}}$. $\square$

Assume that the following diagonal form $f = -a_0 x^2 + a_1 x_1^2 + \cdots a_n x_n^2$ is an admissible quadratic form, defined over a totally real number field of degree d over $\mathbb{Q}$. From this, consider $f' = -a_0 x^2 + a_1 x_1^2 + a_2 x_2^2$ and the following group $\Gamma' = \mathrm{Spin}_{f'}(\mathcal{O}_k)$, we present now some facts which give systolic inequalities for congruenge coverings as in [34].

Proposition 3.4.4. *Let $\alpha \in \mathcal{O}_k$ be a nonzero element and $s \in \Gamma'$ such that $s - s_{\mathbb{R}} \in \alpha\mathcal{Q}$. Then $\tau = \overline{s_{\mathbb{R}}}$ has order two in $(\mathcal{O}_k/\alpha\mathcal{O}_k)^{\times}$. Furthermore, for any loxodromic element $\gamma \in \Gamma_\tau(\alpha) \setminus \Gamma(\alpha)$ we have*

$$|\gamma_{\mathbb{R}}| > \frac{1}{2^{2d-1}}\mathrm{N}(\alpha)^2 - |\mathrm{s}_{\mathbb{R}}|.$$

Proof. Since s is contained in a quaternion algebra we have $s = s_{\mathbb{R}} + \alpha u$ for some $u \in \mathcal{Q}$ with $u^* + u = 0$. Hence, the equalities $1 = ss^* = s_{\mathbb{R}}^2 + \alpha^2 uu^*$ implie that $s_{\mathbb{R}}^2 = 1(\mathrm{mod}\ \alpha)$. Since the index $[\Gamma_\tau(\alpha) : \Gamma(\alpha)] = 2$ and $\Gamma_\tau(\alpha) = \Gamma(\alpha) \cup s\Gamma(\alpha)$, we need to estimate the real part of any product $\gamma = sr$ with $r \in \Gamma(\alpha)$. In this case, by Lemma 3.4.3, we get

$$\gamma_{\mathbb{R}} = s_{\mathbb{R}} + \frac{1}{2}\alpha^2\zeta. \tag{3.4.1}$$

Now, for any non-trivial archimedean place σ of k we know that $|\sigma(\gamma_{\mathbb{R}})| \leq 1$, and $|\sigma(s_{\mathbb{R}})| \leq 1$ ([35, Equation 8]). Therefore, by applying σ to (3.4.1) we get

$$|\sigma(\alpha^2\zeta)| = 2|\sigma(\gamma_{\mathbb{R}}) - \sigma(s_{\mathbb{R}})| \leq 4.$$

Again, by (3.4.1) and the fact that $\zeta \in \mathcal{O}_k$, we obtain that

$$\begin{aligned}
|\gamma_{\mathbb{R}}| &\geq \frac{1}{2}|\alpha|^2|\zeta| - |s_{\mathbb{R}}| \\
&= \frac{1}{2\prod_{\sigma \neq id}|\sigma(\alpha)^2\sigma(\zeta)|}\mathrm{N}(\alpha)^2|\mathrm{N}(\zeta)| - |s_{\mathbb{R}}| \\
&\geq \frac{1}{2^{2d-1}}\mathrm{N}(\alpha)^2 - |s_{\mathbb{R}}|.
\end{aligned}$$

$\square$

Proposition 3.4.5. *Let $\alpha \in \mathcal{O}_k$ be a nonzero element. For any loxodromic element $r \in \Gamma(\alpha)$ we have*

$$|r_{\mathbb{R}}| > \frac{1}{2^{2d-1}}\mathrm{N}(\alpha)^2 - 1.$$

Proof. See [35, Lemma 4.1]. $\square$

CHAPTER 4

HYPERBOLIC MANIFOLDS WITH A SYSTOLE IN A SURFACE

4.1 Construction of congruence coverings

The goal of this Chapter is to show that, under certain conditions, the manifold $\Gamma_\tau(\alpha)\backslash\mathbb{H}^n$ has a systole contained in a totally geodesic surface. We do this in the reverse order, namely, we start considering loxodromic elements in Fuchsian groups and provide conditions for τ and α to satisfy the requirements.

The construction of arithmetic Fuchsian groups is very similar to that of Klenian groups. Let k be a totally real field and $\mathscr{A}$ a quaternion algebra over k which is unramified at a unique real place σ of k. We therefore have an identification $\mathscr{A}_\sigma = \mathscr{A} \otimes_k k_\sigma \cong \mathrm{M}_2(\mathbb{R})$. Keeping the previously established notation, $\mathscr{O}$ denotes a maximal order of $\mathscr{A}$ and $\mathscr{O}^1$ the multiplicative group consisting of those elements which have reduced norm one. Denote by $\Gamma^1_{\mathscr{O}}$ the image of $\mathscr{O}^1$ in $\mathrm{PSL}_2(\mathbb{R})$. A subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ is an arithmetic Fuchsian group if it is commensurable with a group of the form $\Gamma^1_{\mathscr{O}}$.

Since we have a general estimate for displacements in terms of the real part of loxodromic elements, the next step is to construct groups and to exibit candidates for realizing the minimal displacement. In order to do this, we need more definitions and notation.

4.1.1 A totally geodesic surface embedded in $\Gamma_\tau(\alpha)\backslash\mathbb{H}^n$

Let k be a totally real number field, and (E, f) be an admissible n-dimensional quadratic space over k. We assume that, with respect to the orthogonal basis $\{e_0, e_1, \ldots, e_n\}$, f has the diagonal form $f = -a_0x^2 + a_1x_1^2 + \cdots a_nx_n^2$, with $a_i > 0$ in $\mathscr{O}_k$, such that for any non-trivial Galois embedding $\sigma : k \to \mathbb{R}$ we have $\sigma(a_0) < 0$ and $\sigma(a_i) > 0$ for all $i = 1, \ldots, n$.

Let E' be the subspace generated by $\{e_0, e_1, e_2\}$, and $f' : E' \to k$ the restriction of f to E'. The inclusion $E' \to E$ defines a natural inclusion $\Gamma' = \mathrm{Spin}_{f'}(\mathscr{O}_k) \hookrightarrow \mathrm{Spin}_f(\mathscr{O}_k) = \Gamma$. For any $\alpha \in \mathscr{O}_k$ and $\tau \in (\mathscr{O}_k\backslash\alpha\mathscr{O}_k)^\times$ of order two, by definition, we get an inclusion

$$\Gamma'_\tau(\alpha) \hookrightarrow \Gamma_\tau(\alpha).$$

Consider an isometric embedding of $\mathbb{H}^2$ into $\mathbb{H}^n$ equivariant with respect to the actions of Γ' and Γ. For any α and τ as before, we obtain a totally geodesic embedding

$$S_{\alpha,\tau} \hookrightarrow M_{\alpha,\tau} \tag{4.1.1}$$

where $S_{\alpha,\tau} = \Gamma'_\tau(\alpha)\backslash\mathbb{H}^2$ and $M_{\alpha,\tau} = \Gamma_\tau(\alpha)\backslash\mathbb{H}^n$. This implies, in particular, that $\mathrm{sys}(M_{\alpha,\tau}) \leq \mathrm{sys}(S_{\alpha,\tau})$.

4.1.2 Embeddings of quadratic fields in $\mathscr{C}^+(f', k)$

A direct computation shows that the Clifford algebra $\mathscr{C}^+(f', k)$ is a quaternion algebra. In fact, it coincides with the invariant quaternion algebra. It is well known that closed geodesics in $S' = \Gamma'\backslash\mathbb{H}^2$ are related with quadratic extensions of k that embed in $\mathscr{C}^+(f', k)$. In this subsection we recall the properties of this conection that will be important in the sequel.

Definition 4.1.1. For any $s \in \mathscr{C}^+(f', k)$ we define

- The application $s + s^*$, called the *reduced trace* of s;

- The operation ss^*, called the *reduced norm* of s.

Example 4.1.2.
Let E be a vector space of dimension $n+1$ over $k = \mathbb{Q}(\sqrt{5})$ and $f(x_0, ..., x_n) = x_0^2 - \sqrt{5}x_1^2 - ... - \sqrt{5}x_n^2$ be a quadratic form defined on E, keeping the previously established notation we have by definition $f'(x_0, x_1, x_2) = x_0^2 - \sqrt{5}x_1^2 - \sqrt{5}x_2^2$.

$$\mathscr{C}^+(f', k) = \left\{ s = \sum_{|M| \text{ even}} s_M e_M | s_M \in k \text{ and } M \in \mathscr{P}(\{0,1,2\}) \right\}.$$

Using the notation established in Section 3.3, if we denote $s = s_{\mathbb{R}} + s_{01}e_{01} + s_{02}e_{02} + s_{12}e_{12}$, we have that

$$s + s^* = 2s_{\mathbb{R}} \text{ while } ss^* = s_{\mathbb{R}}^2 - s_{01}^2\sqrt{5} - s_{02}^2\sqrt{5} + 5s_{12}^2.$$

We compare it with the definitions from the algebra of quaternions.

Hence s is a root of the quadratic polynomial $g(x) = x^2 - (s+s^*)x + (ss^*) \in k[x]$. If g is irreducible over k, and $L \mid k$ is the quadratic extension where g splits, then for any fixed root α of g in L, there exists a unique monomorphism $\phi : L \to \mathscr{C}^+(f', k)$ such that $\phi(\alpha) = s$, $\phi|_k$ is the identity, and $\phi(\sigma(x)) = \phi(x)^*$ for the non-trivial Galois automorphism $\sigma : L \to L$ of L over k. In particular, via the identification of L with $\phi(L)$, the map σ coincides with the restriction of * to L.

The following proposition is a fact about quaternion algebras that we recall here for completion.

Proposition 4.1.3. *Let $s \in \Gamma'$ be a loxodromic element. There exist a quadratic extension $L = k(\sqrt{D})$ for some positive $D \in k$, and a k-homomorphism $\psi : L \to \mathscr{C}^+(f')$ such that $s = s_{\mathbb{R}} + \psi(\sqrt{D})$.*

Proof. Consider the irreducible polynomial

$$g(x) = x^2 - (s + s^*)x + 1 \tag{4.1.2}$$

over k. Since s is loxodromic, g has two distinct real roots λ and λ^{-1}. Let L be the quadratic extension $k(\lambda)$. Without loss of generality we can suppose that $|\lambda| > 1$. Since λ and s satisfy $g(x) = 0$, there is a unique k-homomorphism $\phi : L \to \mathscr{C}^+(f')$

with $\phi(\lambda) = s$. Let $\theta = \lambda - s_{\mathbb{R}} \in L$, the equality $\lambda + \sigma(\lambda) = 2s_{\mathbb{R}}$ implies $\theta + \sigma(\theta) = 0$, and then $\theta^2 = s_{\mathbb{R}}^2 - 1 = D$. Moreover, s loxodromic implies that $|2s_{\mathbb{R}}| > 2$, thus $D > 0$. To finish the proof, if $\theta > 0$ we take $\psi = \phi$, otherwise we consider $\psi = \phi^*$. □

We are interested in primitive elements in the subgroup Γ' and their relation with primitive units in quadratic extensions of k. By Proposition 4.1.3, we can write s as $s = \psi(\lambda_0)$ where $\lambda_0 = s_{\mathbb{R}} + \sqrt{D}$, and s is primitive if and only if λ_0 is. Thus we have an isomorphism between the cyclic group generated by λ_0 in $L = k(\sqrt{D})$, and the cyclic group generated by s in Γ'. For each $n \in \mathbb{N}$ we can write $\lambda_0^{n+1} = t_n + u_n\sqrt{D}$ with $t_n, u_n \in \mathcal{O}_k$. In the next result we obtain asymptotic relations between u_n, t_n and $s_{\mathbb{R}}$.

Lemma 4.1.4. *If $\lambda = x_0 + \sqrt{D}$ is a unit in $\mathcal{O}_k[\sqrt{D}]$ and $\lambda^{n+1} = t_n + u_n\sqrt{D}$, then for each $n \geq 1$ fixed, we have*

$$t_n = 2^n x_0^{n+1} + O(x_0^n) \tag{4.1.3}$$

and

$$u_n = 2^n x_0^n + O(x_0^{n-1}). \tag{4.1.4}$$

Proof. Since $D = x_0^2 - 1$, we have the following relations

$$t_n = (x_0^2 - 1)u_{n-1} + x_0 t_{n-1} \text{ and } u_n = x_0 u_{n-1} + t_{n-1}.$$

Hence,

$$t_n = x_0 u_n - u_{n-1} \text{ and } u_n = 2x_0 u_{n-1} - u_{n-2} \text{ for all } n \geq 2.$$

We prove (4.1.4) by induction. For $n = 0, 1$ the relation is trivial. Assuming it is valid for any $1 \leq k < n$, it follows that

$$u_n = 2x_0(2^{n-1}x_0^{n-1} + O(x_0^{n-2})) - 2^{n-2}x_0^{n-2} + O(x^{n-3}) = 2^n x_0^n + O(x_0^{n-1}).$$

Now, we obtain (4.1.3) using the relation $t_n = x_0 u_n - u_{n-1}$. □

4.1.3 Salem numbers of degree four

A complex number is an algebraic integer if it is the zero of a polynomial with integer coefficients and leading coefficient equal to 1. See Section 5.2 for more details. Then its Galois conjugates are the zeros of its minimal polynomial these polynomials are monic polynomials with integer coefficients. The galois conjugates are the then roots of minimal polynomial, the *degree* of an algebraic integer is the degree of its minimal polynomial.

A Salem number is an algebraic integer $\lambda > 1$ of degree at least 4, conjugate to λ^{-1}, all of whose conjugates, excluding λ and λ^{-1}, lie on $|z| = 1$. Hence $\lambda + \lambda^{-1}$ is a real algebraic integer greater than 2. The sum of the other conjugates different from λ and λ^{-1}, belong to the real interval $(-2, 2)$. Such numbers are easy to find. An example is $\lambda + \lambda^{-1} = 1 + \sqrt{5}$, giving $(\lambda + \lambda^{-1} - 1)^2 = 5$, so that $\lambda^4 - 2\lambda^3 - 2\lambda^2 - 2\lambda + 1 = 0$ and $\lambda = 2.8901...$ The Salem numbers play an important role in many areas of mathematics, such as number theory, algebra and dynamical systems. For more examples see [1] where the authors establish a method for constructing Salem numbers. In addition, the largest root of the characteristic polynomial associated to loxodromic elements in Γ' are Salem numbers (*see* [20]).

Salem numbers may appear in the literature with another definition that appears to be more general, but this is not case as can be shown by the proposition belows see [38] and [39] for more details.

Proposition 4.1.5. *Suppose that $\lambda > 1$ is an algebraic integer whose conjugates belong to the closed unit disc $|z| \leq 1$, with at least one conjugate on the boundary $|z| = 1$. Then λ is a Salem number.*

By a result of Kronecker we have that an algebraic integer lying with all its conjugates on the unit circle must be a root of unity. A similar results holds for Salem numbers.

Proposition 4.1.6. *If λ is a Salem number, then λ^n is also a Salem number for all $n \in \mathbb{N}$.*

The proof follows directly from the definition of a Salem number given above.

For the interest of this thesis, it is important to develop some results about Salem numbers of low degree. Let μ be a Salem number of degree four. The field $K = \mathbb{Q}(\mu + \mu^{-1})$ is a totally real number field that is a subfield of $\mathbb{Q}(\mu)$, with a nontrivial $\mathbb{Q}$-isomorphism $\tau : K \to K$. Since $[\mathbb{Q}(\mu) : K] = 2$, there exists a unique nontrivial K-isomorphism $\sigma : \mathbb{Q}(\mu) \to \mathbb{Q}(\mu)$ such that $\sigma(\mu) = \mu^{-1}$. Hence, the four embeddings of $\mathbb{Q}(\mu)$ into $\mathbb{C}$ are the inclusion, σ, τ and $\overline{\tau}$, where τ is the extension of the nontrivial $\mathbb{Q}$-morphism of K into $\mathbb{Q}(\mu)$. In particular, we can suppose that $\tau(\mu) = e^{i\nu}$ for some $\nu \in (0, \pi)$.

Now, we may take $D \in \mathscr{O}_k$ such that $\mathbb{Q}(\mu) = K(\sqrt{D})$ and $\mu = t + u\sqrt{D}$ for some $t, u \in \mathscr{O}_K$. Since σ is the non-trivial K-automorphism of $\mathbb{Q}(\mu)$ we have $\sigma(\sqrt{D}) = -\sqrt{D}$, and

$$1 = \mu \cdot \mu^{-1} = \mu \cdot \sigma(\mu) = (t + u\sqrt{D}) \cdot (t - u\sqrt{D}) = t^2 - u^2 D; \tag{4.1.5}$$

$$2\tau(t) = \tau(2t) = \tau(\mu + \sigma(\mu)) = \tau(\mu) + \tau(\mu^{-1}) = 2\cos(\nu). \tag{4.1.6}$$

Thus $t^2 - u^2 D = 1$ and $|\tau(t)| < 1$. For geometric reasons, it is important to get Salem numbers of this form for which $\tau(t)$ is not very small. The next lemma shows that we can assume that this property is always true, up to a small power of μ.

Proposition 4.1.7. *Let $\mu > 1$ be a Salem number of degree four. With the previous notation, if $\mu = t + u\sqrt{D}$, $t, u \in \mathscr{O}_K$, then there exists $m \in \{0, 1, 2\}$ such that $\mu^{m+1} = t_m + u_m\sqrt{D}$ with $\tau(t_m)^2 > \frac{1}{2}$.*

Proof. For each m, let $\mu^{m+1} = t_m + u_m\sqrt{D}$ with $t_m, u_m \in \mathscr{O}_K$. Then

$$\mu^{2(m+1)} = (t_m^2 + Du_m^2) + 2t_m u_m\sqrt{D} = (2t_m^2 - 1) + 2t_m u_m\sqrt{D}.$$

Hence, $t_{2m+1} = 2t_m^2 - 1$ and $\tau(t_{2m+1}) = 2\tau(t_m)^2 - 1$. It follows that $\tau(t_m)^2 > \frac{1}{2}$ if and only if $\tau(t_{2m+1}) > 0$. On the other hand,

$$\tau(2t_{2m+1}) = \tau(\mu^{2(m+1)} + \mu^{-2(m+1)}) = 2\cos(2(m+1)\nu),$$

Then, it remains to show that $\cos(2k\nu) > 0$, for this consider $k \in \{1, 2, 3\}$. Indeed,

$$S_1 = (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi), S_2 = (\frac{3\pi}{8}, \frac{5\pi}{8}), S_3 = (\frac{\pi}{4}, \frac{3\pi}{8}) \cup (\frac{5\pi}{8}, \frac{3\pi}{4}).$$

For each $\nu \in S_j$, we have $\cos(2j\nu) > 0$. The lemma is now proved since $[0, \pi] - (S_1 \cup S_2 \cup S_3)$ only contains rational multiples of π and Salem numbers cannot have conjugates of finite order, note that the restriction on k implies that $m \in \{0, 1, 2\}$. $\square$

4.2 Relation between the kissing numbers

We construct in this section hyperbolic manifolds with systole lying in a totally geodesic surface. More specifically, we are seek conditions on τ and α such that the manifold $M_{\tau,\alpha}$ has a systole in $S_{\tau,\alpha}$ (see Section 4.1.1). Since $\mathrm{sys}(M_{\alpha,\tau}) \leq \mathrm{sys}(S_{\alpha,\tau})$, it is necessary to bound $\mathrm{sys}(M_{\alpha,\tau})$ from below. Proposition 3.4.4 and Proposition 3.4.5 show that this requires a lower bound for the norm of α in the base field k, which at the same time implies that the Galois conjugates of α cannot be very small. We are able to find such α when k is a *real quadratic number field*.

In the sequel, we consider the definitions of Γ, Γ', τ, α to be as in Section 4.1.1, and Q as in Section 3.4.2. Let k be a real quadratic field, with $\sigma : k \to \mathbb{R}$ the nontrivial embedding of k into $\mathbb{R}$.

Lemma 4.2.1. *Let $s \in \Gamma'$ be a primitive loxodromic element with $s_{\mathbb{R}} > 0$. There exists $l \in \{2, 5, 8\}$, which depends only on $s_{\mathbb{R}}$, such that $s^{l+1} = (s^{l+1})_{\mathbb{R}} + \alpha_l u_l$ with $u_l \in \mathcal{Q}$, $\alpha_l \in \mathcal{O}_k$ and*

$$1 \leq |\sigma(\alpha_l)| \leq 5.$$

Moreover, if $t = s_{\mathbb{R}}$ the following asymptotic relation

$$\alpha_l = C_l t^{\frac{2}{3}(l+1)} + O\left(t^{\frac{2}{3}(l+1)-2}\right)$$

holds for some constant $C_l > 0$ depending only on l.

Proof. By Proposition 4.1.3 and the discussion in Section 4.1.1, the element s corresponds to a Salem number $\lambda_0 = t_0 + \sqrt{D}$ (see [20]) and $L = k(\lambda_0)$ is a quadratic extension of k. Since k is a real quadratic number field, λ_0 is a Salem number of degree four. By Proposition 4.1.7, there exists $m \in \{0, 1, 2\}$ such that $\lambda_0^{m+1} = t_m + u_m\sqrt{D}$ with $1 > |\sigma(t_m)|^2 > \frac{1}{2}$. For convenience, we can rewrite

$$\lambda = \lambda_0^{m+1} = t + \sqrt{E}$$

where $t = t_m$ and $E = u_m^2 D$. By (4.1.5), it is straightforward to check that

$$\lambda^3 = (4t^3 - 3t) + (4t^2 - 1)\sqrt{E}.$$

If l is given by $l = 3(m+1) - 1$, then

$$\lambda_0^{l+1} = t_l + u_l\sqrt{D} = t_l + \alpha_l\sqrt{E},$$

where $\alpha_l = 4t_m^2 - 1$ and $E = u_m^2 D$. Since $\frac{1}{2} < |\sigma(t_m)|^2 < 1$ we conclude that $1 < |\sigma(\alpha_l)| < 5$. The asymptotic behaviour of α_l follows directly from (4.1.3) and the equality $m+1 = \frac{1}{3}(l+1)$. $\square$

We prove now that, for any primitive element $s \in \Gamma'$ producing a closed geodesic with length sufficiently large, some power s^l, with l *uniformly bounded*, realizes the systole of some congruence hyperbolic n-manifold.

Proposition 4.2.2. *There exists a universal constant $L > 0$ such that, for any primitive loxodromic element $s \in \Gamma'$, with $s_{\mathbb{R}} > L$, we can find $l \in \{2, 5, 8\}$ depending only on $s_{\mathbb{R}}$ with $s^{l+1} - (s^{l+1})_{\mathbb{R}} \in \alpha_l \mathcal{Q}$, for some $\alpha_l \in \mathcal{O}_k$ and*

$$\ell(s^{l+1}) \leq \ell(r) \text{ for all loxodromic elements } r \in \Gamma_{\tau_l}(\alpha_l),$$

where τ_l is the class of $(s^{l+1})_{\mathbb{R}}$ modulo α_l.

Proof. Fixe $s \in \Gamma'$ primitive and loxodromic with real part $s_{\mathbb{R}}$. By Lemma 4.2.1 there exists $l \in \{2, 5, 8\}$ depending only on $s_{\mathbb{R}}$ such that $s^{l+1} - (s^{l+1})_{\mathbb{R}} \in \alpha_l \mathcal{Q}$ for some $\alpha_l \in \mathcal{O}_k$ with $1 \leq |\sigma(\alpha_l)| \leq 5$. Let $t_l = (s^{l+1})_{\mathbb{R}}$, it follows from Proposition 3.4.4 that $t_l^2 \equiv 1 (\operatorname{mod} \alpha_l)$. Hence, if we denote by τ_l the class of t_l in $\mathcal{O}_k / \alpha_l \mathcal{O}_k$, we have

$$\Gamma_{\tau_l}(\alpha_l) = \Gamma(\alpha_l) \cup s^{l+1} \Gamma(\alpha_l).$$

If $r \in \Gamma(\alpha_l)$, by Proposition 3.4.5 and since $|\sigma(\alpha_l)| \geq 1$, we have

$$|r_{\mathbb{R}}| \geq \frac{1}{8} |\sigma(\alpha_l)|^2 \alpha_l^2 - 1 \geq \frac{1}{8} \alpha_l^2 - 1.$$

Since $\ell(s^{l+1}) = 2 \cosh^{-1}(|t_l|)$, by Proposition 3.3.4, in order to show that $\ell(r) \geq \ell(s^{l+1})$ it is sufficient to guarantee that $|r_{\mathbb{R}}| \geq t_l$. On the other hand, by Lemma 4.2.1 and Lemma 4.1.4 we have

$$\frac{1}{8} \alpha_l^2 - 1 = C (s_{\mathbb{R}})^{\frac{4}{3}(l+1)} + O\left((s_{\mathbb{R}})^{\frac{4}{3}(l+1) - 4} \right) \text{ and } t_l \sim 2^l (s_{\mathbb{R}})^{l+1}.$$

Hence, $\frac{1}{8} \alpha_l^2 - 1 \geq t_l$ whenever $s_{\mathbb{R}}$ sufficiently large.

Analogously, if $r \in s^{l+1} \Gamma(\alpha_l)$ and $r \neq s^{l+1}$, we have by Proposition 3.4.4 that

$$|r_{\mathbb{R}}| \geq \frac{1}{8} |\sigma(\alpha_l)|^2 \alpha_l^2 - t_l \geq \frac{1}{8} \alpha_l^2 - t_l.$$

Again, the right hand side is larger than t_l whenever $s_{\mathbb{R}}$ is large enough. $\square$

Finally, from the construction above we obtain the following theorem.

Theorem 4.2.3. *For any $n \geq 2$, there exists a compact arithmetic hyperbolic n-manifold of first-type M and a sequence of congruence coverings $M_j \to M$ of arbitrarily large degree such that*

$$\mathrm{Kiss}(M_j) \geq C\,\frac{\mathrm{vol}(M_j)^{1+\frac{1}{3n(n+1)}}}{\log(\mathrm{vol}(M_j))} \tag{4.2.1}$$

for some constant $C > 0$ independent of j.

Proof. Let $\phi'(x)$ denote the number of conjugacy classes of loxodromic elements in Γ' with reduced trace equal to x. By the Prime Geodesic Theorem (see [40]), there is a sequence $x_i \to \infty$ such that

$$\phi'(x_i) \geq \frac{x_i}{\log(x_i)}.$$

For each i with x_i large enough, let $x_i = 2(s_i)_{\mathbb{R}}$ for primitive loxodromic elements $s_i \in \Gamma'$ and consider

$$m_i = \min\{l \mid l \in \{2,5,8\} \text{ satisfies Proposition 4.2.2 for } (s_i)_{\mathbb{R}}\}.$$

Futhermore, take τ_{m_i} and α_{m_i} as given in Lemma 4.2.1 and Proposition 4.2.2. Then the manifold $M_i = \Gamma_{\tau_{m_i}}(\alpha_{m_i})\backslash\mathbb{H}^n$ and the totally geodesic surface $S_i = \Gamma'_{\tau_{m_i}}(\alpha_{m_i})\backslash\mathbb{H}^2$ satisfy

$$\mathrm{Kiss}(M_i) \geq \mathrm{Kiss}(S_i) \geq \frac{x_i}{\log(x_i)}. \tag{4.2.2}$$

In addition, take the isometry group $G_i = \Gamma/\Gamma_{\tau_{m_i}}(\alpha_{m_i})$ acting on the set of closed geodesics of M_i. By item (1) of Lemma 2.2.7, if we denote by $\gamma_1, \ldots, \gamma_{\mathrm{Kiss}(S_i)}$ the systoles of S_i embeded in M_i, the orbit sets $G_i\gamma_j$, $j \in \{1, \ldots, \mathrm{Kiss}(S_i)\}$ are pairwise disjoint. It follows that

$$\mathrm{Kiss}(M_i) \geq \sum_{j=1}^{\mathrm{Kiss}(S_i)} |G_i\gamma_j| \tag{4.2.3}$$

$$= \sum_{j=1}^{\mathrm{Kiss}(S_i)} \frac{|G_i|}{|(G_i)_{\gamma_j}|}, \tag{4.2.4}$$

where $(G_i)_{\gamma_j}$ denotes the isotropy group of γ_j under the action of G_i. Using item (2) of Lemma 2.2.7 we have $|(G_i)_{\gamma_j}|$ is at most the order of γ_j, which is bounded by a fixed constant, since $m_i \leq 8$. Therefore, we get from (4.2.2) and (4.2.4) that

$$\mathrm{Kiss}(M_i) \geq C\mathrm{Kiss}(S_i) \cdot |G_i| \geq C\frac{x_i}{\log(x_i)}|G_i|.$$

Since $\mathrm{vol}(M_i) = [\Gamma : \Gamma_{\tau_{m_i}}(\alpha_{m_i})]\mathrm{vol}(\Gamma\backslash\mathbb{H}^n) = |G_i|\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)$, the above inequality becomes

$$\mathrm{Kiss}(M_i) \geq C\frac{x_i}{\log(x_i)}\mathrm{vol}(M_i). \tag{4.2.5}$$

Our goal now is to bound $\frac{x_i}{\log(x_i)}$ from below in terms of $\mathrm{vol}(M_i)$. Since $\Gamma_{\tau_{m_i}}(\alpha_{m_i})$ has index two in $\Gamma(\alpha_{m_i})$, we have

$$\mathrm{vol}(M_i) = [\Gamma : \Gamma_{\tau_{m_i}}(\alpha_{m_i})]\mathrm{vol}(\Gamma\backslash\mathbb{H}^n) = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^n)}{2}[\Gamma : \Gamma(\alpha_{m_i})]. \tag{4.2.6}$$

Note that the norm of the ideal (α_{m_i}) goes to infinity (see Lemma 4.2.1), thus we can apply the results in [35, Section 5] along with the fact that $m_i \leq 8$, to obtain the following

$$\begin{aligned}
[\Gamma : \Gamma(\alpha_{m_i})] \leq \mathrm{N}(\alpha_{m_i})^{\frac{n(n+1)}{2}} &= O\left(|\alpha_{m_i}|^{\frac{n(n+1)}{2}}\right) \\
&= O\left(\left(x_i^{\frac{2(m_i+1)}{3}}\right)^{\frac{n(n+1)}{2}}\right) \\
&= O\left(x_i^{3n(n+1)}\right).
\end{aligned}$$

By joining the estimate obtained above with equation (4.2.6), we find the lower bound

$$x_i \geq C_1 \cdot \mathrm{vol}(M_i)^{\frac{1}{3n(n+1)}} \tag{4.2.7}$$

for some constant $C_1 > 0$. Finally, since the function $x \mapsto \frac{x}{\log(x)}$ is increasing for large x, by (4.2.5) we get that

$$\mathrm{Kiss}(M_i) \geq C_2\,\frac{\mathrm{vol}(M_i)^{1+\frac{1}{3n(n+1)}}}{\log(\mathrm{vol}(M_i))},$$

for a constant $C_2 > 0$. $\square$

The Proposition 4.2.2 has a geometric counterpart.

Corollary 4.2.4. *For any $n \geq 3$, the manifold M obtained in Theorem 4.2.3 contains a closed totally geodesic surface S such that for any j, the congruence coverings $M_j \to M$ contain a congruence covering $S_j \to S$ satisfying* $\mathrm{sys}(S_j) = \mathrm{sys}(M_j)$.

4.3 Conclusion

We discuss now what we have proved during this work, what had already been proved, and some questions that were left open for future research.

Theorem 2.5.4 gives an improvement on the exponent that had been given in [15] for the non-compact case, Furthermore, we also manage to encompass the compact case, that is, the result we find allows us to obtain a sequence of compact hyperbolic manifolds that have a lower bound for the kissing number in terms of the volume.

On the other hand, Theorem 4.2.3 gives us, in any dimension $n \geq 2$, a sequence of hyperbolic arithmetic manifolds of the first type, with kissing number boundes below, in terms of their volume. However, now the exponent associated with the volume decreases as the dimension of the manifolds increases.

A natural question arising from Theorem 4.2.3 is the following: *Is there a universal constant $\varepsilon > 0$ such that for any $n \geq 2$, there is a sequence of closed hyperbolic n-manifolds M_j with $\mathrm{vol}(M_j) \to \infty$ and*

$$\mathrm{Kiss}(M_j) \gtrsim \frac{\mathrm{vol}(M_j)^{1+\varepsilon}}{\log(\mathrm{vol}(M_j))} \ ?$$

This would be a natural question to pursue. Furthermore, we have already noticed that this would imply

$$\mathrm{sys}(M_j) \gtrsim \frac{2\varepsilon}{n-1} \log(\mathrm{vol}(M_j)). \tag{4.3.1}$$

From the Appendix of [35], there is a sequence of compact arithmetic hyperbolic manifolds M_i such that

$$\mathrm{sys}(M_i) \sim \frac{8}{n(n+1)} \log(\mathrm{vol}(M_i)). \tag{4.3.2}$$

and the bound in (4.3.1) grows considerably faster than the one in (4.3.2).

Another possible way forward is to extend Theorem 4.2.3 or to find an equivalent for noncompact manifolds. We could use the properties of totally geodesic surfaces embedded in arithmetic hyperbolic manifolds of the first kind to estimate how large the kissing number can be, in order to improve the upper bound given in [7].

CHAPTER 5

BACKGROUND REFERENCE

Before we recall some basic facts used throughout the text, we will explain here a notation used that appears in some passages of the work.

We recall that two positive sequences (a_j) and (b_j) satisfy the relation $a_j \gtrsim b_j$ (resp. $a_j \lesssim b_j$) when for any $\delta > 0$ there exists j_0 such that $\frac{a_j}{b_j} \geq (1-\delta)$ (resp. $\frac{a_j}{b_j} \leq 1-\delta$) for $j > j_0$. Hence, the sequences satisfy $a_j \sim b_j$ if and only if $a_j \lesssim b_j$ and $a_j \gtrsim b_j$.

In the following we will deal with groups and their actions on topological spaces.

5.1 Groups and actions

Definition 5.1.1. A *topological group* is a group G equipped with a topology such that the operations are continuous maps, i.e. the maps

$$\begin{array}{cc} G \times G \to G & G \to G \\ (g,h) \mapsto gh & g \mapsto g^{-1} \end{array}$$

are continuous. A subset of a group could inherit the group structure, when this occurs these subsets are called a *subgroup* of G. A group is called *torsion* free if the only element of finite order is the identity.

Now, in order to create functions that preserve the group structure, an application $h : (G, \cdot) \to (\mathscr{G}, \times)$ between groups is called a *homomorphism* if h satisfies

$$g_1 \cdot g_2 = h(g_1) \times h(g_2).$$

Example 5.1.2.
Let $n \in \mathbb{Z}$ be fixed. Then,

$$\phi_n : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$$
$$\phi_n(z) = nz$$

is a homomorphism.

A surjective homomorphism is called an *epimorphism*, while an injective homomorphism is called a *monomorphism*. Finally, when a homomorphism is injective and surjective it is called an *isomorphism*.

Let G be a topological group and Y be a topological space.

Definition 5.1.3. We call by *action* of G on Y a continuous map $\mu : G \times Y \to Y$ which satisfies the following conditions:

- For all $y \in Y$, we have $\mu(1, y) = y$.
- For any $g, h \in G$ and $y \in Y$, $\mu(gh, y) = \mu(g, \mu(h, y))$.

Throughout the text we denote $\mu(g, y)$ by $g \cdot y$.

Example 5.1.4.
Let G be a group and consider any element y in a topological space Y, then

$$G_y = \{g \in G |\ g \cdot y = y\}$$

is an example of a subgroup and this is called the *isotropy subgroup*.

An interesting group theory concept that is used in this thesis is commensurability, which is equivalent to saying that two subgroups correspond if they are distinct only by a finite quantity.

Definition 5.1.5. Two subgroups Γ_1 and Γ_2 of a group G are said to be *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2

Example 5.1.6.
Any finite group G is commensurable with the trivial group.

Definition 5.1.7. In the case where G is a topological group and the maps defined above are smooth, we call G a *Lie group*. A common example is the group of real invertible matrices with the multiplication operation.

Proposition 5.1.8. *Let Γ be a topological group and M be a manifold. If Γ acts on M in such a way that, for every compact $K \subset M$ the set $\{g \in \Gamma | \; g \cdot K \cap K \neq \emptyset\}$ is finite, then the quotient space $\Gamma \backslash M$ is a manifold, and the natural projection $M \to \Gamma \backslash M$ is a covering map.*

See [32, Subsection 1.5] for details of proof and other results related to n-manifolds.

5.2 Number fields

Recall that a *field* is a set endowed with an addition and multiplication which behave as the corresponding operations on rational and real numbers do. (Finite fields are not exactly like this).

Definition 5.2.1. A complex number z is an *algebraic integer*, if there exists a monic polynomial $P \in \mathbb{Z}[X]$, i.e., a nonzero polynomial with integer coefficients and the leading coefficient equal to 1, such that $P(z) = 0$.

Definition 5.2.2. A number field is a subfield $k \subset \mathbb{C}$ such that k has finite dimension as a $\mathbb{Q}$-vector space. In addition we define the ring of integers of k by the set

$$\mathcal{O}_k = \{x \in k | \; x \text{ is an algebraic integer}\}.$$

Definition 5.2.3. Given a number field k, we define a Galois embedding as any embedding of fields $\sigma : k \to \mathbb{C}$. We say that a Galois embedding σ is *real* if $\sigma(k) \subset \mathbb{R}$, otherwise, we say that σ is *complex*.

Let d be the *degree* of the extension, i.e. $[k : \mathbb{Q}] = d$. If we let r_1 denote the number of real embeddings and r_2 the number of complex conjugate pairs, then

$$d = r_1 + 2r_2.$$

We say that k has r_1 real places and r_2 complex places. Furthermore, we refer to k as being *totally real* if $r_2 = 0$. An *Archimedean place* of k is either a real place or a pair of complex-conjugated places.

Futhermore, for L/k a finite extension of number fields, a real place σ of k is said to *ramify* in L if it extends to an embedding of L into $\mathbb{C}$ with non-real image. If all extensions of σ to places of L are real (the associated embeddings have real image), then σ is *unramified* (also said to be *split*) in L.

5.3 Quadratic forms

Definition 5.3.1. Let V be a finite dimensional vector space over a field k and let $B : V \times V \to k$ be a symmetric bilinear map. Then the pair (V, B) is a *quadratic space*. From this the *quadratic form* q associated to B is obtained by $q(x) = B(x, x)$, this map satisfies:

$$q(\lambda v) = \lambda^2 q(v), \text{ for any } \lambda \in k \text{ and } v \in V.$$

The bilinear map determines a quadratic form $q : V \to k$ by $q(v) = B(v, v)$. A quadratic form is said to be *irreducible* if it is not the product of two other distinct linear forms. In addition, q is said to be *non-degenerate* if $B(v, w) = 0$ for any $w \in V$ implies $v = 0$.

Example 5.3.2.
A diagonal form dot-product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x_1, ..., x_n) \cdot (y_1, ..., y_n) = x_1y_1 + ... + x_ny_n$, is a bilinear map in $\mathbb{R}^n$ and its associated quadratic form is called the *diagonal form*.

Proposition 5.3.3 ([28], Lemma 0.9.4)**.** *Let (V, B) be a quadratic space over k. Then V has an orthogonal basis such that every quadratic form is equivalent to a diagonal form.*

From the above proposition, any quadratic form q defined over a field k is equivalent to a diagonal form. If $k = \mathbb{R}$ we can conclude that any real quadratic form is equivalent to

$$\sum_{i=1}^{r} x_i^2 - \sum_{i=r+1}^{n} x_i^2.$$

The pair $(r, n - r)$ is called the *signature* of q.

Definition 5.3.4. Let k be a totally real number field. Suppose that f is a quadratic form with signature $(1, n)$ over $\mathbb{R}$, and for any non-trivial embedding $\sigma : k \to \mathbb{R}$ the quadratic form f^σ (that denotes the quadratic form obtained by applying σ to the coefficients of f) is positive definite. Then the quadratic form f is called *admissible*.

Example 5.3.5.
Consider $k = \mathbb{Q}(\sqrt{5})$ and $f(x_1, ..., x_{n+1}) = x_1^2 - \sqrt{5}x_2^2 - ... - \sqrt{5}x_{n+1}^2$. For the non-trivial place $\sigma : k \to \mathbb{R}$, we have $f^\sigma(x_1, ..., x_{n+1}) = x_1^2 + \sqrt{5}x_2^2 + ... + \sqrt{5}x_{n+1}^2$. Thus f is an admissible quadratic form

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