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Continuity of fractal dimensions in the Markov and Lagrange spectra

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Abstract

In this work, we establish some continuity statements on the classical and dynamical Markov and Lagrange spectra:

Let φ_0 be a smooth conservative diffeomorphism of a compact surface S and let Λ_0 be a mixing horseshoe of φ_0. Given a smooth real function f defined in S and a small smooth conservative perturbation φ of φ_0, let $L_{\varphi,f}$ and $M_{\varphi,f}$ be respectively the Lagrange and Markov spectra associated to the hyperbolic continuation $\Lambda(\varphi)$ of the horseshoe Λ_0 and f. We show that for generic choices of φ and f, the Hausdorff dimension of the sets $L_{\varphi,f} \cap (-\infty, t)$ and $M_{\varphi,f} \cap (-\infty, t)$ are equal and determine a continuous function as $t \in \mathbb{R}$ varies; generalizing then the Cerqueira-Matheus-Moreira theorem to horseshoes with arbitrary Hausdorff dimension.

Moreover, as before, if φ_0 is a conservative diffeomorphism and Λ_0 is a mixing horseshoe of φ_0 with Hausdorff dimension strictly smaller than one, we prove that, for generic choices of φ and f (φ not necessarily conservative), if L is the map that gives the Hausdorff dimension of the set $L_{\varphi,f} \cap (-\infty, t)$ for $t \in \mathbb{R}$, then the minimum accumulation point of $L_{\varphi,f}$ is the only possible limit of an infinite sequence of discontinuities of L.

Finally, we prove in the classical setting that, for $t \geq 6$, the sets $k^{-1}((-\infty, t])$ and $k^{-1}(t)$, which are the sets of irrational numbers with best constant of Diophantine approximation bounded by t and exactly t respectively, have the same Hausdorff dimension. We also show that, as $t \geq 6$ varies, this Hausdorff dimension is a strictly increasing function.

Keywords: Fractal geometry, Markov Dynamical Spectrum, Lagrange Dynamical Spectrum, Regular Cantor sets, Horseshoes, Hyperbolic Dynamics, Diophantine Approximation.

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Contents

List of symbols

$\mathrm{Leb}(X)$	–	The Lebesgue measure of the set X.
$\mathrm{int}(X)$	–	The interior of the set X.
$HD(X)$	–	Hausdorff dimension of the set X.
$C^r(S, \mathbb{R})$	–	The space of C^r functions $f : S \to \mathbb{R}$.
$\mathrm{Diff}^r(S)$	–	Set of diffeomorphisms of S of class C^r.
$\mathrm{Diff}^r_\omega(S)$	–	The set of conservative diffeomorphisms of S of class C^r with respect to a volume form ω.
$C^{1+\alpha}$	–	Functions with Holder derivative with exponent α.
$\lvert X\rvert$	–	The cardinality of the set X.
$\lvert\alpha\rvert$	–	The number of letters of the word α.
$\lvert I\rvert$	–	The length of the interval I.
$\mathcal{O}(x)$	–	The φ-orbit of the point x.
$\nabla f(x)$	–	The gradient of the real map f in the point x.
$\alpha(x)$	–	the α-limit set of x.
$\omega(x)$	–	the ω-limit set of x.

Chapter 1

Introduction

The classical Lagrange and Markov spectra are closed subsets of the real line related to Diophantine approximations. They arise naturally in the study of rational approximations of irrational numbers and of indefinite binary quadratic forms, respectively. More precisely, given an irrational number α, let

$$\begin{aligned} k(\alpha) &:= \sup\left\{k>0 : \left|\alpha-\frac{p}{q}\right|<\frac{1}{kq^2} \text{ has infinitely many rational solution } \frac{p}{q}\right\} \\ &= \limsup_{\substack{p,q\to\infty \\ p,q\in\mathbb{N}}} (q|q\alpha-p|)^{-1} \end{aligned}$$

be its best constant of Diophantine approximation. The set

$$L := \{k(\alpha) : \alpha\in\mathbb{R}-\mathbb{Q}, k(\alpha)<\infty\}$$

consisting of all finite best constants of Diophantine approximations is the so-called *Lagrange spectrum*.

Similarly, the *Markov spectrum*

$$M := \left\{\left(\inf_{(x,y)\in\mathbb{Z}^2-\{(0,0)\}} |q(x,y)|\right)^{-1}<\infty : q(x,y)=ax^2+bxy+cy^2, b^2-4ac=1\right\}$$

consists of the reciprocal of the minimal values over non-trivial integer vectors $(x,y)\in\mathbb{Z}^2-\{(0,0)\}$ of indefinite binary quadratic forms $q(x,y)$ with unit discriminant.

For our purposes, it is worth to point out here that the Lagrange and Markov spectra have the following *dynamical* interpretation in terms of the continued fraction algorithm, given by Perron (cf. [24]):

Denote by $[a_0, a_1, \dots]$ the continued fraction $a_0 + \frac{1}{a_1 + \frac{1}{\ddots}}$. Let $\Sigma = \mathbb{N}^{\mathbb{Z}}$ the space of bi-infinite sequences of positive integers, $\sigma : \Sigma \to \Sigma$ be the left-shift map $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$, and let $f : \Sigma \to \mathbb{R}$ be the function

$$f((a_n)_{n\in\mathbb{Z}}) = [a_0, a_1, \dots] + [0, a_{-1}, a_{-2}, \dots].$$

Then, the Markov spectrum is the set

$$M = \left\{ \sup_{n\in\mathbb{Z}} f(\sigma^n(\underline{\theta})) < \infty : \underline{\theta} \in \Sigma \right\}$$

and the Lagrange spectrum is the set

$$L = \left\{ \limsup_{n\to\infty} f(\sigma^n(\underline{\theta})) < \infty : \underline{\theta} \in \Sigma \right\}.$$

It follows from these characterizations that M and L are closed subsets of $\mathbb{R}$ and that $L \subset M$.

Markov showed in [15] that

$$L \cap (-\infty, 3) = M \cap (-\infty, 3) = \{k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < ...\},$$

where $k_n^2 \in \mathbb{Q}$ for every $n \in \mathbb{N}$ and $k_n \to 3$ when $n \to \infty$.

M. Hall in [5] proved that

$$C_4 + C_4 = [\sqrt{2} - 1, 4(\sqrt{2} - 1)],$$

where for each positive integer N, C_N is the set of the numbers in $[0, 1]$ in whose continued fractions the coefficients are bounded by N, i.e., $C_N = \{x = [0; a_1, ..., a_n, ...] \in [0, 1] : a_i \le N, \ \forall i \ge 1\}$. Together with Perron characterizations, this implies that L and M contain the whole half-line $[6, +\infty)$.

Freiman in [13] determined the precise beginning of Hall's ray (the biggest half-line contained in L), which is

$$\frac{2221564096 + 283748\sqrt{462}}{491993569} = 4.52782956616\dots$$

Moreira in [16] proved several results on the geometry of the Markov and Lagrange spectra, for example, that the map $d : \mathbb{R} \to [0, 1]$, given by

$$d(t) = HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t))$$

is continuous, surjective and such that $d(3) = 0$ and $d(\sqrt{12}) = 1$. Moreover, that

$$d(t) = \min\{1, 2D(t)\},$$

where $D(t) = HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is also a continuous function from $\mathbb{R}$ to $[0, 1)$. Even more, he proved the limit

$$\lim_{t\to\infty} HD(k^{-1}(t)) = 1.$$

In the sequel, we consider natural generalizations of the classical Lagrange and Markov spectra given above but in the context of horseshoes[1] of smooth diffeomorphisms of compact surfaces. Indeed, if $\varphi : S \to S$ is a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and $f : S \to \mathbb{R}$ is a differentiable function. Following the above characterization of the classical spectra, we define the maps

$$\begin{array}{rcl} m_{\varphi,f} : \Lambda & \to & \mathbb{R} \\ x & \to & m_{\varphi,f}(x) = \sup\limits_{n\in\mathbb{Z}} f(\varphi^n(x)), \\ \ell_{\varphi,f} : \Lambda & \to & \mathbb{R} \\ x & \to & \ell_{\varphi,f}(x) = \limsup\limits_{n\to\infty} f(\varphi^n(x)) \end{array}$$

and call $\ell_{\varphi,f}(x)$ *the Lagrange value* of x associated to f and φ and also $m_{\varphi,f}(x)$ *the Markov value* of x associated to f and φ. The sets

$$L_{\varphi,f}(\Lambda) = \ell_{\varphi,f}(\Lambda) = \{\ell_{\varphi,f}(x) : x \in \Lambda\}$$

and

$$M_{\varphi,f}(\Lambda) = m_{\varphi,f}(\Lambda) = \{m_{\varphi,f}(x) : x \in \Lambda\}$$

are called *Lagrange Spectrum* of (φ, f, Λ) and *Markov Spectrum* of (φ, f, Λ).

It turns out that dynamical Markov and Lagrange spectra associated to hyperbolic dynamics are closely related to the classical Markov and Lagrange spectra. Several results on the Markov and Lagrange dynamical spectra associated to horseshoes in dimension 2 which are analogous to previously known results on the classical spectra were obtained recently: in [18] it is shown that typical dynamical spectra associated to horseshoes with Hausdorff dimensions larger than one have non-empty interior (as

[1] i.e., a non-empty compact invariant hyperbolic set of saddle type which is transitive, locally maximal, and not reduced to a periodic orbit (cf. [25] for more details).

the classical ones). In [17] it is shown that typical Markov and Lagrange dynamical spectra associated to horseshoes have the same minimum, which is an isolated point in both spectra, and is the image by the function of a periodic point of the horseshoe.

In [3], in the context of *conservative* diffeomorphism it is proven that for typical choices of the dynamics and of the real function, the intersections of the corresponding dynamical Markov and Lagrange spectra with half-lines $(-\infty, t)$ have the same Hausdorff dimensions, and this defines a continuous function of t whose image is $[0, D]$, where $D < 1$ is the Hausdorff dimension of the horseshoe.

For more information and results on classical and dynamical Markov and Lagrange spectra, we refer to the books [21] and [8].

If $HD(X)$ denotes the Hausdorff dimension of X, in this work we are interested in the study of the real functions

$$L(t) = L(\varphi, f, \Lambda)(t) = HD(L_{\varphi,f}(\Lambda) \cap (-\infty, t)) \tag{1.0.1}$$

and

$$M(t) = M(\varphi, f, \Lambda)(t) = HD(M_{\varphi,f}(\Lambda) \cap (-\infty, t)).$$

In what follows, the diffeomorphism φ usually determines itself the horseshoe Λ, then we use to write $L_{\varphi,f}$ and $M_{\varphi,f}$ instead $L_{\varphi,f}(\Lambda)$ and $M_{\varphi,f}(\Lambda)$ in those cases.

In order to prove our principal results, it will be useful to study the fractal geometry (Hausdorff dimension) of the set

$$\Lambda_t := \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\{y \in \Lambda : f(y) \leq t\}) = \{x \in \Lambda : m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) \leq t\}$$

for $t \in \mathbb{R}$. Also, we define in the context of mixing horseshoes Λ with $HD(\Lambda) > 1$ the Markov transition parameter as

$$a = a(\varphi, f) = \sup\{t \in \mathbb{R} : HD(\Lambda_t) < 1\}.$$

In [10] is proved that for typical choices of the diffeomorphism φ and the smooth real map f, the Markov parameter is characterized by the conditions

$$\mathrm{Leb}(M_{\varphi,f} \cap (-\infty, a-\delta)) = 0$$

but

$$\mathrm{int}(M_{\varphi,f} \cap (-\infty, a+\delta)) \neq \emptyset,$$

for all $\delta > 0$.[2]

[2] here $\mathrm{Leb}(\cdot)$ denotes the usual Lebesgue measure and $\mathrm{int}(\cdot)$ the interior of the set.

The Lagrange parameter $\tilde{a} = \tilde{a}(\varphi, f)$ is defined in such a way that a similar result is true if we replace $M_{\varphi,f}$ by $L_{\varphi,f}$ and a by $\tilde{a}$ in the last conditions.

In the present thesis, we are going to do first a study of the discontinuities of the map L. By showing geometric consequences of having a discontinuity and introducing the notion of *connection of subhorseshoes* we prove that far away from the first accumulation point of the Lagrange spectra, we have generically only a finite number of discontinuities. That is, given φ_0 a smooth conservative diffeomorphism of a surface S possessing a mixing horseshoe Λ_0 with Hausdorff dimension $HD(\Lambda_0) < 1$, denote by $\mathcal{U}$ a C^2 neighborhood of φ_0 in the space $\text{Diff}^2(S)$ of smooth diffeomorphisms of S such that Λ_0 admits a continuation Λ for every $\varphi \in \mathcal{U}$. Then, we have

Theorem (3.1.1). *If $\mathcal{U} \subset \text{Diff}^2(S)$ is sufficiently small, then there exists a residual subset $\mathcal{U}^{**} \subset \mathcal{U}$ with the following property. For every $\varphi \in \mathcal{U}^{**}$ and $r \geq 2$, there exists a C^r-residual set $\mathcal{R}_{\varphi,\Lambda} \subset C^r(S, \mathbb{R})$ such that given $f \in \mathcal{R}_{\varphi,\Lambda}$ if L is defined by*

$$t \mapsto L(t) = HD(L_{\varphi,f} \cap (-\infty, t))$$

then, the only possible limit of an infinite sequence of discontinuities of L is $c_{\varphi,f} := \min L'_{\varphi,f} = \min\{x : x \text{ is an accumulation point of } L_{\varphi,f}\}$.

On the other hand, we also extend the main results in [3], removing the hypothesis that $HD(\Lambda_0) < 1$. We do this, by replacing the notion of *good-positions* for positions in the alphabet from which is obtained the complete subshift that determines the subhorseshoe with big dimension ([3], proposition 2.9), by the notion of positions that are not contained between any pair of positions that determine the so-called *critical windows*. This notion is more flexible because we suppose only that the gradient of the real map f is different from zero, which is a generic condition without any assumption in the dimension of the horseshoe.

Write $\text{Diff}^2_\omega(S)$ for the set of conservative diffeomorphisms of S with respect to a volume form ω. Using the notations introduced before, we will prove the next theorem

Theorem (4.1.3). *Let $\varphi_0 \in \text{Diff}^2_\omega(S)$ with a mixing horseshoe Λ_0 and $\mathcal{U}$ a C^2-sufficiently small neighbourhood of φ_0 in $\text{Diff}^2_\omega(S)$ such that Λ_0 admits a continuation $\Lambda(= \Lambda(\varphi))$ for every $\varphi \in \mathcal{U}$. There exists a residual set $\tilde{\mathcal{U}} \subset \mathcal{U}$ such that for every $\varphi \in \tilde{\mathcal{U}}$ and $r \geq 2$ there exists a C^r-residual set $\tilde{\mathcal{R}}_{\varphi,\Lambda} \subset C^r(S, \mathbb{R})$ such that for any $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ the function*

$$t \mapsto HD(\Lambda_t)$$

is continuous and

$$\min\{1, HD(\Lambda_t)\} = L(t) = M(t).$$

Remark 1.0.1. In fact, we will prove a continuity result that is valid even in the non-conservative setting (see theorem 4.1.1) and without any generic condition on the diffeomorphism.

Even more, in theorem D of [10] is shown in the conservative case, that generically we have the equality $a = \tilde{a}$ where $a = a(\varphi, f)$ and $\tilde{a} = \tilde{a}(\varphi, f)$ are as before. However, there is a mistake in the proof of the last statement in that theorem. Using the last result, we get a correct proof of the

Corollary (4.1.4). *Let $\varphi_0 \in Diff^2_\omega(S)$ with a mixing horseshoe Λ_0 with $HD(\Lambda_0) > 1$ and $\mathcal{V}$ a C^2-sufficiently small neighbourhood of φ_0 in $Diff^2_\omega(S)$ such that Λ_0 admits a continuation Λ for every $\varphi \in \mathcal{V}$. Then, there exists a residual set $\mathcal{V}^* \subset \mathcal{V}$ such that for every $\varphi \in \mathcal{V}^*$ and $r \geq 2$ there exists a C^r-residual set $\mathcal{P}_{\varphi,\Lambda} \subset C^r(S, \mathbb{R})$ such that for any $f \in \mathcal{P}_{\varphi,\Lambda}$:*

$$Leb(M_{\varphi,f} \cap (-\infty, a-\delta)) = 0 = Leb(L_{\varphi,f} \cap (-\infty, a-\delta))$$

but

$$int(M_{\varphi,f} \cap (-\infty, a+\delta)) \neq \emptyset \neq int(L_{\varphi,f} \cap (-\infty, a+\delta))$$

for all $\delta > 0$. Moreover, one has

$$HD(M_{\varphi,f} \cap (-\infty, a)) = HD(L_{\varphi,f} \cap (-\infty, a)) = 1.$$

Finally, as will be indicated, it is possible to see portions of the classical spectra as dynamical one (associated with some family of horseshoes of diffeomorphisms and real maps defined in $\mathbb{S}^2$). We will use this point of view in order to apply results and notions of the dynamical spectral to the classical setting and show that for t large, in terms of dimension, major part of the set of irrational numbers with best constant of Diophantine approximation bounded by t are concentrated in the set of irrational numbers with best constant being exactly t. That is, we will prove

Theorem (5.1.1). *For $t \geq 6$, the map D is strictly increasing and $D(t) = HD(k^{-1}(t))$ i.e.*

$$HD(k^{-1}((-\infty, t))) = HD(k^{-1}((-\infty, t])) = HD(k^{-1}(t)).$$

1.1 Structure of the work

The present work is divided into four parts

- The first part, chapter 2, contains all the preliminary results and definitions that we will use throughout the text.
- The second one, chapter 3, is dedicated to the study of the discontinuities of the map L, where we prove theorem 3.1.1.
- The third part, chapter 4, is mainly devoted to the proof of the continuity and equality of the maps L and M in the conservative setting. There we prove theorem 4.1.3 and other results related with.
- The fourth part explores the connection between the dynamical spectra with the classical one in order to prove Theorem 5.1.1.

Most of the results of this thesis appear in the papers:

1. C.G. Moreira and C. Villamil. *On the discontinuities of Hausdorff dimension in generic dynamical Lagrange spectrum.*
2. C.G. Moreira, C. Villamil and D. Lima. *Continuity of fractal dimensions in conservative generic Markov and Lagrange dynamical spectra.*
3. C.G. Moreira and C. Villamil. *Concentration of dimension in extremal points of left-half lines in the Lagrange spectrum.*

Chapter 2

Preliminaries

2.1 Preliminaries on hyperbolic dynamics

Let Λ be a closed, φ-invariant set for a C^r-diffeomorphism of a compact manifold S. We say that Λ is a hyperbolic set for φ if there is a continuous splitting of TS_Λ, the tangent bundle of S restricted to Λ, which is $D\varphi$-invariant:

$$TS_\Lambda = E^s \oplus E^u, \quad D\varphi(E^s) = E^s, \quad D\varphi(E^s) = E^s$$

and for which there are real constants c and λ, $c > 0$ and $0 < \lambda < 1$, such that

$$\|D\varphi^n|_{E^s}\| < c\lambda^n \text{ and } \left\|D\varphi^{-n}|_{E^u}\right\| < c\lambda^n, \text{ for } n \geq 0.$$

In the same context, given $x \in S$ and $\epsilon > 0$, we define:

$$\begin{aligned}
W^s_\epsilon(x,\varphi) &= \{y \in S : \lim_{n\to+\infty} d(\varphi^n(x),\varphi^n(y)) = 0 \text{ and } \forall n \geq 0,\ d(\varphi^n(x),\varphi^n(y)) \leq \epsilon\},\\
W^s(x,\varphi) &= \bigcup_{n\geq 0} \varphi^{-n}(W^s_\epsilon(\varphi^n(x),\varphi)),\\
W^u_\epsilon(x,\varphi) &= \{y \in S : \lim_{n\to-\infty} d(\varphi^n(x),\varphi^n(y)) = 0 \text{ and } \forall n \leq 0,\ d(\varphi^n(x),\varphi^n(y)) \leq \epsilon\},
\end{aligned}$$

and

$$W^u(x,\varphi) = \bigcup_{n\geq 0} \varphi^{n}(W^u_\epsilon(\varphi^{-n}(x),\varphi)).$$

The stable manifold theorem states that there is a positive ϵ such that for every point $x \in \Lambda$, $W^s_\epsilon(x,\varphi)$ is an embedded disk of dimension equal to that of E^s_x. Moreover, $T_xW^s_\epsilon(x) = E^s_x$ and also that the manifold $W^s_\epsilon(x,\varphi)$ is as smooth as φ and $W^s(x,\varphi)$ is an immersed submanifold of S. We call this submanifold the global stable manifold of x for φ in contrast to the local stable manifold $W^s_\epsilon(x,\varphi)$. Of course, there are analogous definitions and results for the unstable case.

If Λ is a hyperbolic set for φ, then for $x,\ x' \in \Lambda$ sufficiently close, $W^u_\epsilon(x)$ and $W^s_\epsilon(x')$ have a unique point of intersection. This intersection is transverse and we denote by $[x,x']$. We said that Λ has *local product structure* or that is *locally maximal* if, for $x,\ x' \in \Lambda$ sufficiently close the unique point of intersection $[x,x'] = W^s_\epsilon(x) \cap W^u_\epsilon(x')$ belongs to Λ (cf. [27, pag. 104]) or, equivalently, Λ is the maximal invariant set in some neighborhood of it.

Let $\varphi : S \to S$ a C^r-difeomorphism and Λ a hyperbolic set associated to φ. The *shadowing lemma* says that given $\beta > 0$, there exists $\alpha > 0$ such that every α-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ in Λ is β-shadowed by some orbit. That is, if $\{x_n\}_{n\in\mathbb{Z}} \subset \Lambda$ satisfies $d(\varphi(x_n), x_{n+1}) \le \alpha$ for every $n \in \mathbb{Z}$ then, there exists some $y \in S$ such that $d(\varphi^n(y), x_n) \le \beta$ for every $n \in \mathbb{Z}$. Moreover, if Λ has local product structure $y \in \Lambda$.

As a consequence of the shadowing lemma, we have for Λ hyperbolic and locally maximal

$$W^s(\Lambda) = \bigcup_{y\in\Lambda} W^s(y) \quad \text{and} \quad W^u(\Lambda) = \bigcup_{y\in\Lambda} W^u(y), \tag{2.1.1}$$

where the stable and unstable sets of Λ, are respectively defined by

$$W^s(\Lambda) = \{y \in S : \lim_{n\to\infty} d(\varphi^n(y), \Lambda) = 0\}$$

and

$$W^u(\Lambda) = \{y \in S : \lim_{n\to-\infty} d(\varphi^n(y), \Lambda) = 0\}.$$

Given Λ a hyperbolic set associated to φ with local product structure and dense periodic orbits, we have the so-called *spectral decomposition.* That is, there are sets Λ_i for $i = 1, \dots, m$, which are compact, φ-invariant, pairwise disjoint and transitive. Even more, each Λ_i, $i = 1, \dots, m$ also admits a decomposition in a union of compact sets $\Lambda_i = \Lambda_{i,1} \cup \cdots \cup \Lambda_{i,n_i}$, such that $\varphi(\Lambda_{i,j}) = \Lambda_{i,j+1}$ for $j = 1, \dots, n_i - 1$ and $\varphi(\Lambda_{i,n_i}) = \Lambda_{i,1}$ and $\varphi^{n_i}|_{\Lambda_{i,j}}$ is mixing.

According to [27] (theorems 8.3 and 8.22) we also have that hyperbolicity is persistent under small perturbations. More specifically, let $U \subset S$ be an open set such that $\Lambda = \bigcap_{n\in\mathbb{Z}} \varphi^n(U)$ is a hyperbolic set for φ. Then, there is a neighborhood $\mathcal{U}$ of φ in $\mathrm{Diff}^r(S)$ and a continuous function $\Phi : \mathcal{U} \to C^0(\Lambda, S)$ such that for every $\psi \in \mathcal{U}$, $\Lambda_\psi = \Phi(\psi)(\Lambda) = \bigcap_{n\in\mathbb{Z}} \psi^n(U)$ is a hyperbolic set for ψ which is conjugated to Λ by

$\Phi(\psi)$:

$$\begin{tikzcd} \Lambda \arrow[r, "\Phi(\psi)"] \arrow[d, "\varphi"'] & \Lambda_\psi \arrow[d, "\psi"] \\ \Lambda \arrow[r, "\Phi(\psi)"'] & \Lambda_\psi \end{tikzcd}$$

When Λ is a hyperbolic set associate to C^2-diffeomorphism, there are stable and unstable foliations, $\mathcal{F}^s(\Lambda)$ and $\mathcal{F}^u(\Lambda)$ that are $C^{1+\alpha}$ for some $\alpha > 0$. Moreover, these foliations can be extended to C^1 foliations defined on a full neighborhood of Λ (cf. [28, pag. 604]).

Here, unless explicitly stated otherwise, we will assume that Λ is a horseshoe: non-empty compact, locally maximal, transitive hyperbolic invariant set of saddle type, and so it contains a dense subset of periodic orbits. We suppose also that Λ is not just a periodic orbit.

In the next theorem, we recall a result concerning differentiability of the invariant stable and unstable manifold and foliations themselves of horseshoes in two dimensions with respect to the diffeomorphism. Consider the diffeomorphism $\Psi : \mathcal{U} \times S \to \mathcal{U} \times S$ defined by $\Psi(\psi, x) = (\psi, \psi(x))$ where $\mathcal{U}$ is as before. According to [25] in Appendix 1, one has

Theorem 2.1.1. *If $\Psi : \mathcal{U} \times S \to \mathcal{U} \times S$ is C^2, then there are transverse invariant foliations $\mathcal{F}^s_\psi(x)$, $\mathcal{F}^u_\psi(x)$ defined on U such that the maps $(\psi, x) \to T_x\mathcal{F}^s_\psi(x)$, and $(\psi, x) \to T_x\mathcal{F}^u_\psi(x)$ are $C^{1+\epsilon}$.*

Now we come to the definition of a *Markov partition* for a horseshoe Λ as introduced above. Such a Markov partition consists of a finite set of rectangles, i.e. diffeomorphics images of the square $Q = [-1, +1]^2$, say $B_1 = \psi_1(Q), \ldots, B_\ell = \psi_\ell(Q)$ such that

- $\Lambda \subset \bigcup\limits_{i=1}^{\ell} B_i$,
- $intB_i \cap intB_j = \emptyset$, $i \neq j$ where $intB$ denotes the interior of the set B,
- $\varphi(\partial_s B_i) \subset \bigcup\limits_{i=1}^{\ell} \partial_s B_i$ and $\varphi^{-1}(\partial_u B_i) \subset \bigcup\limits_{i=1}^{\ell} \partial_u B_i$, where $\partial_s B_i = \psi_i(\{(x, y) : -1 \leq x \leq 1, |y| = 1\})$ and $\partial_u B_i = \psi_i(\{(x, y) : |x| = 1, -1 \leq y \leq 1\})$,
- there is a positive integer n such that $\varphi^n(B_i) \cap B_j \neq \emptyset$ for all $1 \leq i, j \leq \ell$

Usually one also requires that $\varphi(B_i) \cap B_j$ is either empty or connected. But we can always satisfy that condition by taking the boxes of the Markov partition sufficiently small:

Theorem 2.1.2. *If Λ is mixing, there is a Markov partition for Λ with arbitrarily small diameter.*

Let Λ be a mixing horseshoe of φ and consider a finite collection $(R_a)_{a\in\mathcal{A}}$ of disjoint rectangles of S, which are a Markov partition of Λ. The set $\mathcal{B} \subset \mathcal{A}^2$ of admissible transitions consist of pairs (a, b) such that $\varphi(R_a) \cap R_b \neq \emptyset$. So, we can define the following transition matrix B which induces the same transitions that $\mathcal{B} \subset \mathcal{A}^2$

$$b_{ab} = 1 \text{ if } \varphi(R_a) \cap R_b \neq \emptyset \text{ and } b_{ab} = 0 \quad \text{otherwise, for } (a, b) \in \mathcal{A}^2.$$

Let $\Sigma_{\mathcal{A}} = \{\underline{a} = (a_n)_{n\in\mathbb{Z}} : a_n \in \mathcal{A} \text{ for all } n \in \mathbb{Z}\}$. Consider the homeomorphism of $\Sigma_{\mathcal{A}}$, the shift, $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ defined by $\sigma(\underline{a})_n = a_{n+1}$. Let $\Sigma_B = \left\{\underline{a} \in \Sigma_{\mathcal{A}} : b_{a_n a_{n+1}} = 1\right\}$, this set is closed and σ-invariant subspace of $\Sigma_{\mathcal{A}}$. Still denote by σ the restriction of σ to Σ_B, the pair (Σ_B, σ) is a subshift of finite type (cf. [27, chap 10]).

Subshifts of finite type have a sort of local product structure. First we define the local stable and unstable sets for $\underline{a} \in \Sigma_{\mathcal{A}}$:

$$\begin{aligned} W^s_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \geq 0,\ a_n = b_n\}, \\ W^u_{1/3}(\underline{a}) &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\ d(\sigma^n(\underline{a}), \sigma^n(\underline{b})) \leq 1/3\} \\ &= \{\underline{b} \in \Sigma_B : \forall n \leq 0,\ a_n = b_n\}, \end{aligned}$$

where $d(\underline{a}, \underline{b}) = \sum_{n=-\infty}^{\infty} 2^{-(2|n|+1)} \delta_n(\underline{a}, \underline{b})$ and $\delta_n(\underline{a}, \underline{b})$ is 0 when $a_n = b_n$ and 1 otherwise. So, if $\underline{a}, \underline{b} \in \Sigma_B$ and $d(\underline{a}, \underline{b}) < 1/2$, then $a_0 = b_0$ and $W^s_{1/3}(\underline{a}) \cap W^u_{1/3}(\underline{b})$ is a unique point, denoted by the bracket $[\underline{a}, \underline{b}] = (\cdots, b_{-n}, \cdots, b_{-1}, b_0, a_1, \cdots, a_n, \cdots)$.

The dynamics of φ on Λ is topologically conjugate to the sub-shift Σ_B defined by B, namely, there is a homeomorphism $\Pi : \Lambda \to \Sigma_B$ such that, the following diagram commutes

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & \Lambda \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \Sigma_B & \xrightarrow[\sigma]{} & \Sigma_B \end{array}$$

Moreover, Π is a morphism of the local structure, that is, $\Pi([x, y]) = [\Pi(x), \Pi(y)]$, (cf. [27, chap 10]).

2.2 Preliminaries on regular Cantor sets and their fractal dimensions

Let $X \subset \mathbb{R}^n$ and $\mathcal{U} = \{U_i\}_{i\in I}$ a countable covering of X by open sets. We define the diameter $diam(\mathcal{U})$ of $\mathcal{U}$ as the supremum of $\ell(U_i)$, $i \in I$, where $\ell(U_i)$ denotes the length of U_i. Define the α-sum of $\mathcal{U}$ as $H_\alpha(\mathcal{U}) = \sum\limits_{i\in I} \ell(U_i)^\alpha$. Then the Hausdorff α-measure of X is

$$m_\alpha(X) = \lim_{\epsilon\to 0}\left(\inf_{\substack{\mathcal{U} \text{ covers } X\\ diam(\mathcal{U})<\epsilon}} H_\alpha(\mathcal{U})\right).$$

It is possible to see that there is a unique number, called the Hausdorff dimension of X, denoted by $HD(X)$, such that for $\alpha < HD(X)$, $m_\alpha(X) = \infty$ and for $\alpha > HD(X)$, $m_\alpha(X) = 0$.

The Hausdorff dimension has the following properties (cf. [29, chap 3]):

- If $X \subset Y$, then $HD(X) \le HD(Y)$;
- $HD(\bigcup\limits_{i\in\mathbb{N}} X_i) = \sup\limits_{i\in\mathbb{N}} HD(X_i)$. In particular $HD(X) = 0$ if X is a countable set;
- If $f : X \to Y$ is α-Holder continuous then $HD(f(X)) \le \frac{1}{\alpha} HD(X)$;
- $HD(\mathbb{R}^n) = n$ and, more generally, $HD(X) = m$ when $X \subset \mathbb{R}^n$ is a m-dimensional submanifold;
- $HD(X \times Y) \ge HD(X) + HD(Y)$.

Another notion of dimension that will be used frequently is the limit capacity or box-counting dimension. In order to define it, let $N_\epsilon(X)$ be the smallest number of boxes of side lengths $\le \epsilon$ needed to cover X. Then the box-counting dimension of X, denoted by $D(X)$, is defined as

$$D(X) = \limsup_{\epsilon\to 0} \frac{\log N_\epsilon(X)}{-\log \epsilon}.$$

The box-counting dimension has the following properties (cf. [29, chap 2])

- If $X \subset Y$, then $D(X) \le D(Y)$;
- $D(\bigcup\limits_{i=1}^{n} X_i) = \max\limits_{i\in\mathbb{N}} D(X_i)$;

- If $f : X \to Y$ is α-Holder continuous then $D(f(X)) \leq \frac{1}{\alpha} D(X)$;
- $D(X \times Y) \leq D(X) + D(Y)$.

A notion that will play an important role in our results is the notion of dynamically defined (or regular) Cantor set

Definition 2.2.1. A set $K \subset \mathbb{R}$ is called a $C^{1+\alpha}$-*regular Cantor set*, $\alpha > 0$, if there exists a collection $\mathcal{P} = \{I_1, I_2, ..., I_r\}$ of compacts intervals and a $C^{1+\alpha}$-expanding map ψ, defined in a neighbourhood of $\cup_{1\leq j\leq r} I_j$ such that

(i) $K \subset \cup_{1\leq j\leq r} I_j$ and $\cup_{1\leq j\leq r} \partial I_j \subset K$,

(ii) For every $1 \leq j \leq r$ we have that $\psi(I_j)$ is the convex hull of a union of I_r's, for l sufficiently large $\psi^l(K \cap I_j) = K$ and

$$K = \bigcap_{n\geq 0} \psi^{-n}\Big(\bigcup_{1\leq j\leq r} I_j\Big).$$

More precisely, we also say that the triple $(K, \mathcal{P}, \psi)$ is a $C^{1+\alpha}$-regular Cantor set.

For regular Cantor sets we have the so-called *bounded distortion property*, (cf. [25, chap 4])

Theorem 2.2.2. *Let $K \subset \mathbb{R}$ a regular Cantor set defined by an expanding map $\psi \in C^{1+\alpha}$ as before. Given $\delta > 0$ there exist $C(\delta) > 0$, decreasing function of δ with $\lim\limits_{\delta\to 0} C(\delta) = 0$ such that for each $x, y \in K$ satisfying*

- $|\psi^n(x) - \psi^n(y)| \leq \delta$
- *For $0 \leq j \leq n$ the interval determined by $\psi^n(x)$ and $\psi^n(y)$ is contained in the domain of ψ.*

one has $\log|(\psi^n)'(x)| - \log|(\psi^n)'(y)| \leq C(\delta)$.

With the same notation as the above theorem. It follows that if $z \in K$ satisfies also for $0 \leq j \leq n$ that the interval determined by $\psi^n(x)$ and $\psi^n(z)$ is contained in the domain of ψ, then

$$e^{-c}\frac{|z-x|}{|y-x|} \leq \frac{|\psi^n(z) - \psi^n(x)|}{|\psi^n(y) - \psi^n(x)|} \leq e^{c}\frac{|z-x|}{|y-x|} \tag{2.2.1}$$

where c is a constant that may be taken small if $\psi^n(x)$, $\psi^n(y)$ and $\psi^n(z)$ are close. So ψ^n essentially preserves ratios of distances between close points: they change but not by more than a uniform, multiplicative constant.

Moreover, if we define inductively $\mathcal{R}_1 = \{I_1, \dots, I_k\}$ and for $n \geq 2$, $\mathcal{R}_n$ as the set of connected components of $\psi^{-1}(J)$, $J \in \mathcal{R}_{n-1}$. And also, for each $R \in \mathcal{R}_n$ we denote by

$$\lambda_{n,R} = \inf|(\psi^n)'|_R| \quad \text{and} \quad \Lambda_{n,R} = \sup|(\psi^n)'|_R|,$$

the bounded distortion property shows the existence of some $a = a(K) \geq 1$, such that $\Lambda_{n,R} \leq a.\lambda_{n,R}$, for all $n \geq 1$.

In the present work, we will deal many times with regular Cantor sets and their fractal dimensions. In this direction, we have the following result, (cf. [25, chap 4])

Theorem 2.2.3. *Let $K \subset \mathbb{R}$ be a dynamically defined Cantor set. Then $D(K) = HD(K)$.*

Indeed, it follows from the proof of the above theorem that for the sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ given by

$$\sum_{R\in\mathcal{R}_n} \left(\frac{1}{\Lambda_{n,R}}\right)^{\alpha_n} = 1 \quad \text{and} \quad \sum_{R\in\mathcal{R}_n} \left(\frac{1}{\lambda_{n,R}}\right)^{\beta_n} = 1, \tag{2.2.2}$$

when ψ is a full Markov map i.e., $\psi(K \cap I_j) = K$ for $1 \leq j \leq k$, one has

$$\alpha_n \leq HD(K) = D(K) \leq \beta_n \tag{2.2.3}$$

and if $n \geq \log a / \log \lambda$, where $\lambda = \lambda(K) = \inf|\psi'| > 1$

$$\beta_n - \alpha_n \leq \frac{HD(K) \log a}{n \log \lambda - \log a} = O(1/n). \tag{2.2.4}$$

2.3 Stable and unstable Cantor sets associated with horseshoes

Let Λ be some mixing horseshoe of some diffeomorphism φ as before. Fix a Markov partition $\{R_a\}_{a\in\mathcal{A}}$ with sufficiently small diameter consisting of rectangles $R_a \sim I_a^s \times I_a^u$ delimited by compact pieces I_a^s, I_a^u, of stable and unstable manifolds of certain points of Λ as before. And recall that the stable and unstable manifolds of Λ can be extended to locally invariant $C^{1+\alpha}$ foliations in a neighborhood of Λ for some $\alpha > 0$.

Therefore, we can use these foliations to define projections $\pi_a^u : R_a \to I_a^s \times \{i_a^u\}$ and $\pi_a^s : R_a \to \{i_a^s\} \times I_a^u$ of the rectangles into the connected components $I_a^s \times \{i_a^u\}$ and $\{i_a^s\} \times I_a^u$ of the stable and unstable boundaries of R_a, where $i_a^u \in \partial I_a^u$ and $i_a^s \in \partial I_a^s$ are fixed arbitrarily. Using these projections, we have the unstable and stable Cantor sets

$$K^u = \bigcup_{a \in \mathcal{A}} \pi_a^s(\Lambda \cap R_a) \text{ and } K^s = \bigcup_{a \in \mathcal{A}} \pi_a^u(\Lambda \cap R_a).$$

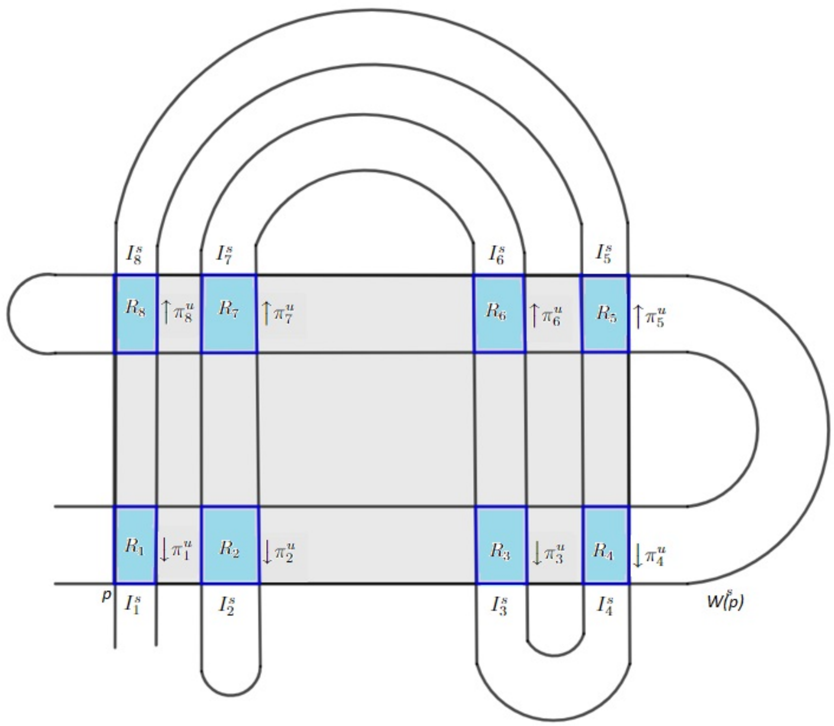

Figure 2.1: Markov partition and projections.

In fact K^u and K^s are $C^{1+\alpha}$ dynamically defined Cantor sets. We define g_s and g_u in the following way: If $y \in R_{a_1} \cap \varphi(R_{a_0})$ we put

$$g_s(\pi_{a_1}^u(y)) = \pi_{a_0}^u(\varphi^{-1}(y))$$

and if $z \in R_{a_0} \cap \varphi^{-1}(R_{a_1})$ we put

$$g_u(\pi_{a_0}^s(z)) = \pi_{a_1}^s(\varphi(z)).$$

We have that g_s and g_u are $C^{1+\alpha}$ expanding maps of type $\Sigma_{\mathcal{B}}$ defining K^s and K^u in the sense that

(i) The domains of g_s and g_u are disjoint unions
$$\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^s(a_1,a_0) \text{ and } \bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^u(a_0,a_1),$$
where $I^s(a_1,a_0)$, resp. $I^u(a_0,a_1)$, are compact subintervals of $I^s_{a_1}$, resp. $I^u_{a_0}$;

(ii) For each $(a_0,a_1)\in\mathcal{B}$, the restrictions $g_s|_{I^s(a_1,a_0)}$ and $g_u|_{|I^u(a_0,a_1)}$ are $C^{1+\alpha}$ diffeomorphisms onto $I^s_{a_0}$ and $I^u_{a_1}$ with $|Dg_s(t)|,|Dg_u(t)|>1$, for all $t\in I^s(a_1,a_0)$, $t\in I^u(a_0,a_1)$ (for appropriate choices of the parametrization of I^s_a and I^u_a);

(iii) K^s and K^u satisfies
$$K^s=\bigcap_{n\geq 0} g_s^{-n}\left(\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^s(a_1,a_0)\right) \quad K^u=\bigcap_{n\geq 0} g_u^{-n}\left(\bigsqcup_{(a_0,a_1)\in\mathcal{B}} I^u(a_0,a_1)\right).$$

The stable and unstable Cantor sets, K^s and K^u, respectively, are closely related to the fractal geometry of the horseshoe Λ; for instance, it is well-known that
$$HD(\Lambda)=HD(K^s)+HD(K^u)=D(K^s)+D(K^u) \tag{2.3.1}$$
see [12] theorem 2 or [25] proposition 4, pag. 75.

Following the above construction, we will study the subsets Λ_t introduced in the previous chapter through its projections on the stable and unstable Cantor sets of Λ:
$$K^u_t=\bigcup_{a\in\mathcal{A}} \pi^s_a(\Lambda_t\cap R_a) \text{ and } K^s_t=\bigcup_{a\in\mathcal{A}} \pi^u_a(\Lambda_t\cap R_a).$$

2.4 Preliminaries on Differential Topology

Let $f\in C^r(S,\mathbb{R})$ with $r\geq 2$, we say that f is a Morse function, if for all $x\in S$ such that $Df_x=0$ we have that the Hessian
$$D^2f(x)\colon T_xS\times T_xS\to\mathbb{R}$$
is nondegenerate, i.e. if $D^2f(x)(v,w)=0$ for all $w\in T_xS$ implies $v=0$. Denote this set by $\mathcal{M}$ and note that in this case, the set $\operatorname{Crit}(f)=\{x\in S: Df_x=0\}$ is discrete. A known result says that for $r\geq 2$, the set of Morse functions is open and dense in $C^r(S,\mathbb{R})$ with the Whitney topology.

Also $C^r(S,\mathbb{R})$, $\operatorname{Diff}^2(S)$ and $\operatorname{Diff}^2_\omega(S)$ are Baire spaces, that is, in these spaces, every countable intersection of open and dense sets is dense.

2.5 Preliminaries on continued fractions

The continued fraction expansion of an irrational number α is denoted by

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

so that the Gauss map $G : (0,1) \to [0,1)$, $G(x) = \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor$ acts on continued fraction expansions by

$$G([0; a_1, a_2, \dots]) = [0; a_2, \dots].$$

For an irrational number $\alpha = \alpha_0 \in (0,1)$, the continued fraction expansion $\alpha = [0; a_1, \dots]$ is recursively obtained by setting $a_n = \lfloor \alpha_n \rfloor$ and $\alpha_{n+1} = \frac{1}{\alpha_n - a_n} = \frac{1}{G^n(\alpha_0)}$. The rational approximations

$$\frac{p_n}{q_n} := [0; a_1, \dots, a_n] \in \mathbb{Q}$$

of α satisfy the recurrence relations

$$p_n = a_n p_{n-1} + p_{n-2} \text{ and } q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 0 \tag{2.5.1}$$

with the convention that $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$. If $0 < a_j \leq N$ for all j, it follows that

$$\frac{p_n}{N+1} \leq p_{n-1} \leq p_n \text{ and } \frac{q_n}{N+1} \leq q_{n-1} \leq q_n, \quad n \geq 1.$$

Given a finite sequence $(a_1, a_2, \dots, a_n) \in (\mathbb{N}^*)^n$, we define

$$I(a_1, a_2, \dots, a_n) = \{x \in [0,1] : x = [0; a_1, a_2, \dots, a_n, \alpha_{n+1}],\ \alpha_{n+1} \geq 1\}$$

then by 2.5.1, $I(a_1, a_2, \dots, a_n)$ is the interval with extremities $[0; a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}$ and $[0; a_1, a_2, \dots, a_n + 1] = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}$ and so

$$|I(a_1, a_2, \dots, a_n)| = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})},$$

because $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

Also, for $(a_0, a_1, \dots, a_n) \in (\mathbb{N}^*)^{n+1}$ we set

$$I(a_0; a_1, \dots, a_n) = \{x \in [0,1] : x = [a_0; a_1, a_2, \dots, a_n, \alpha_{n+1}],\ \alpha_{n+1} \geq 1\},$$

clearly as $I(a_0; a_1, \dots, a_n) = a_0 + I(a_1, a_2, \dots, a_n)$, we have

$$|I(a_0; a_1, \dots, a_n)| = |I(a_1, a_2, \dots, a_n)|. \tag{2.5.2}$$

For example, in our context of sets of continued fractions. Let, as before, G be the Gauss map and $C_N = \{x = [0; a_1, a_2, ...] : a_i \leq N,\ \forall i \geq 1\}$. Then,

$$C_N = \bigcap_{n \geq 0} G^{-n}(I_N \cup ... \cup I_1),$$

where $I_j = [a_j, b_j]$ and $a_j = [0; j, \overline{1, N}]$ and $b_j = [0; j, \overline{N, 1}]$. That is, C_N is a regular Cantor set.

An elementary result for comparing continued fractions is the following lemma

Lemma 2.5.1. *Let $\alpha = [a_0; a_1, \dots, a_n, a_{n+1}, \dots]$ and $\tilde{\alpha} = [a_0; a_1, \dots, a_n, b_{n+1}, \dots]$, then:*

- $|\alpha - \tilde{\alpha}| < 1/2^{n-1}$,
- *If $a_{n+1} \neq b_{n+1}$, $\alpha > \tilde{\alpha}$ if and only if $(-1)^{n+1}(a_{n+1} - b_{n+1}) > 0$.*

Finally, the next two lemmas are from [16] (see lemmas A.1 and A.2)

Lemma 2.5.2. *If $a_0, a_1, a_2 \dots, a_n, a_{n+1}, \dots$ and $b_{n+1}, b_{n+2}, \dots$ are positive integers bounded by $N \in \mathbb{N}$ and $a_{n+1} \neq b_{n+1}$ then*

$$\begin{aligned} |[a_0; a_1, a_2 \dots, a_n, a_{n+1}, \dots] - [a_0; a_1, a_2 \dots, a_n, b_{n+1}, \dots]| &> c(N)/q_{n-1}^2 \\ &> c(N)|I(a_1, a_2, \dots, a_n)| \end{aligned}$$

for some positive constant $c(N)$.

Lemma 2.5.3. *For finite words α and β*

$$\frac{1}{2}|I(\alpha)||I(\beta)| < |I(\alpha\beta)| < 2|I(\alpha)||I(\beta)|.$$

Chapter 3

On the discontinuities of Hausdorff dimension in generic dynamical Lagrange spectrum

3.1 Introduction

Let $\varphi : S \to S$ be a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and let $f : S \to \mathbb{R}$ be a differentiable function. For $x \in S$, following the characterization of the classical spectra, we defined *the Lagrange value* of x associated to f and φ as being the number $\ell_{\varphi,f}(x) = \limsup_{n\to\infty} f(\varphi^n(x))$ and also the *Markov value* of x associated to f and φ as the number $m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x))$.

The sets

$$L_{\varphi,f}(\Lambda) = \{\ell_{\varphi,f}(x) : x \in \Lambda\}$$

and

$$M_{\varphi,f}(\Lambda) = \{m_{\varphi,f}(x) : x \in \Lambda\}$$

were called *Lagrange Spectrum* of (φ, f, Λ) and *Markov Spectrum* of (φ, f, Λ).

In this chapter, we are interested in the study of the real functions

$$L(t) = L(\varphi, f, \Lambda)(t) = HD(L_{\varphi,f}(\Lambda) \cap (-\infty, t)) \tag{3.1.1}$$

and

$$M(t) = M(\varphi, f, \Lambda)(t) = HD(M_{\varphi,f}(\Lambda) \cap (-\infty, t)).$$

Firstly note that L (and also M) is left-continuous because

$$\begin{aligned} L(t) &= HD(L_{\varphi,f}(\Lambda) \cap \bigcup_{n\in\mathbb{N}}(-\infty, t-1/n)) = HD(\bigcup_{n\in\mathbb{N}} L_{\varphi,f}(\Lambda) \cap (-\infty, t-1/n)) \\ &= \sup_{n\in\mathbb{N}} HD(L_{\varphi,f}(\Lambda) \cap (-\infty, t-1/n)) = \sup_{s<t} HD(L_{\varphi,f}(\Lambda) \cap (-\infty, s)) \\ &= \lim_{s\to\ t^-} HD(L_{\varphi,f}(\Lambda) \cap (-\infty, s)) = \lim_{s\to\ t^-} L(s). \end{aligned}$$

In order to prove our principal result, we will first study the Hausdorff dimension of the set

$$\Lambda_t = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\{y \in \Lambda : f(y) \le t\}) = \{x \in \Lambda : m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) \le t\}$$

for $t \in \mathbb{R}$. We do that seeing Λ_t through its projections on the stable and unstable Cantor sets of Λ

$$K_t^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(\Lambda_t \cap R_a) \text{ and } K_t^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(\Lambda_t \cap R_a),$$

where the projections π_a, for $a \in \mathcal{A}$, were defined in chapter 2.

In this setting, our theorem (cf. Theorem 3.1.2 below) will be a kind of generalization of the result of [3] on the continuity of Hausdorff dimension across Lagrange dynamical spectra but away from the first accumulation point of that spectra. Here, we will drop the hypothesis of the neighborhood of the initial conservative diffeomorphism be in the space of conservative diffeomorphisms. However, we can only conclude finiteness of the number of discontinuities but not continuity else.

3.1.1 Statement of the result

Let φ_0 be a smooth conservative diffeomorphism of a surface S possessing a mixing horseshoe Λ_0 with Hausdorff dimension $HD(\Lambda_0) < 1$. Denote by $\mathcal{U}$ a C^2 neighborhood of φ_0 in the space $\mathrm{Diff}^2(S)$ of smooth diffeomorphisms of S such that Λ_0 admits a continuation Λ for every $\varphi \in \mathcal{U}$ with $HD(\Lambda) < 1$. Using the notations of the previous subsection, our main result is the following

Theorem 3.1.1. *If $\mathcal{U} \subset \mathrm{Diff}^2(S)$ is sufficiently small, then there exists a residual subset $\mathcal{U}^{**} \subset \mathcal{U}$ with the following property. For every $\varphi \in \mathcal{U}^{**}$ and $r \ge 2$, there exists a C^r-residual set $\mathcal{R}_{\varphi,\Lambda} \subset C^r(S,\mathbb{R})$ such that given $f \in \mathcal{R}_{\varphi,\Lambda}$ if L is defined by*

$$t \mapsto L(t) = HD(L_{\varphi,f} \cap (-\infty, t))$$

then, the only possible limit of an infinite sequence of discontinuities of L is $c_{\varphi,f} := \min L'_{\varphi,f} = \min\{x : x$ *is an accumulation point of* $L_{\varphi,f}\}$.

We will prove the next result equivalent to theorem 3.1.1

Theorem 3.1.2. *If $\mathcal{U} \subset Diff^2(S)$ is sufficiently small, then there exists a Baire residual subset $\mathcal{U}^{**} \subset \mathcal{U}$ with the following property. For every $\varphi \in \mathcal{U}^{**}$ and $r \geq 2$, there exists a C^r-residual set $\mathcal{R}_{\varphi,\Lambda} \subset C^r(S,\mathbb{R})$ such that given $f \in \mathcal{R}_{\varphi,\Lambda}$ and $\epsilon > 0$ the function*

$$t \mapsto L(t) = HD(L_{\varphi,f} \cap (-\infty, t))$$

has finitely many discontinuities in the interval $[c_{\varphi,f} + \epsilon, \infty)$ where $c_{\varphi,f} = \min L'_{\varphi,f}$.

Remark 3.1.3. The proof of theorem 3.1.2 also shows the existence of the number $c_{\varphi,f}$ and that it is the least point with the property that $L(c_{\varphi,f} + \epsilon) > 0$ for each $\epsilon > 0$.

3.2 Preliminary results

Given a Markov partition $\mathcal{P} = \{R_a\}_{a\in\mathcal{A}}$; recall that the geometrical description of Λ in terms of the Markov partition $\mathcal{P}$ has a combinatorial counterpart in terms of the Markov shift $\Sigma_{\mathcal{B}} \subset \mathcal{A}^{\mathbb{Z}}$. And we can use Π (see section 2.1) to transfer the function f from Λ to a function (still denoted f) on $\Sigma_{\mathcal{B}}$. In this setting, $\Pi(\Lambda_t) = \Sigma_t$ where

$$\Sigma_t = \{\theta \in \Sigma_{\mathcal{B}} : \sup_{n\in\mathbb{Z}} f(\sigma^n(\theta)) \leq t\}.$$

Given an admissible finite sequence $\theta = (a_1, ..., a_n) \in \mathcal{A}^n$ (i.e., $(a_i, a_{i+1}) \in \mathcal{B}$) for all $1 \leq i < n$, we define

$$I^u(\theta) = \{x \in K^u : g_u^i(x) \in I^u(a_i, a_{i+1}), i = 1, 2, ..., n-1\}$$

and

$$I^s(\theta^t) = \{y \in K^s : g_s^i(y) \in I^s(a_i, a_{i-1}), i = 2, ..., n\}$$

where $\theta^t = (a_n, a_{n-1}, ..., a_2, a_1)$. In a similar way, let $\alpha = (a_{s_1}, a_{s_1+1}, ..., a_{s_2}) \in \mathcal{A}^{s_2-s_1+1}$ an admissible word where $s_1, s_2 \in \mathbb{Z}, s_1 < s_2$ and fix $s_1 \leq s \leq s_2$. Define

$$R(\alpha; s) := \bigcap_{m=s_1-s}^{s_2-s} \varphi^{-m}(R_{a_{m+s}}).$$

Note that if $x \in R(\alpha; s) \cap \Lambda$ then the symbolic representation of x is in the way $x = ...a_{s_1}...a_{s-1}; a_s, a_{s+1}...a_{s_2}...$ where on the right of the ; is the 0-th position.

We write $s^{(u)}(\alpha)$ for the unstable size of α, that is, the length of the interval $I^u(\alpha)$ and the unstable scale of α is $r^{(u)}(\alpha) = \lfloor \log(1/s^{(u)}(\alpha)) \rfloor$. Similarly, we write $s^{(s)}(\alpha)$ the stable size of α as being the length of $I^s(\alpha^t)$ and the stable scale of α is $r^{(s)}(\alpha) = \lfloor \log(1/s^{(s)}(\alpha)) \rfloor$.

In our context of $C^{1+\varepsilon}$-dynamically defined Cantor sets, we can relate the unstable and stable sizes of α to its length as a word in the alphabet $\mathcal{A}$ via the *bounded distortion property* (see theorem 2.2.2) saying that there exists a constant $c_1 = c_1(\varphi, \Lambda) > 0$ such that

$$e^{-c_1} \leq \frac{|I^u(\alpha\beta)|}{|I^u(\alpha)||I^u(\beta)|} \leq e^{c_1} \text{ and } e^{-c_1} \leq \frac{|I^s((\alpha\beta)^t)|}{|I^s(\alpha^t)||I^s(\beta^t)|} \leq e^{c_1}. \tag{3.2.1}$$

Write $\alpha^* = (a_1, a_2, ..., a_{n-1})$ if $\alpha = (a_1, a_2, ..., a_n)$ and for $r \in \mathbb{N}$ define the sets

$$P_r^{(u)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(u)}(\alpha) \geq r \text{ and } r^{(u)}(\alpha^*) < r\}$$

and

$$P_r^{(s)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(s)}(\alpha) \geq r \text{ and } r^{(s)}(\alpha^*) < r\}.$$

Now, given any $X \subset \Lambda$ compact φ-invariant we define its projections

$$\pi^u(X) = \bigcup_{a \in \mathcal{A}} \pi_a^s(X \cap R_a) \text{ and } \pi^s(X) = \bigcup_{a \in \mathcal{A}} \pi_a^u(X \cap R_a).$$

We also set

$$\mathcal{C}_u(X, r) = \{\alpha \in P_r^{(u)} : I^u(\alpha) \cap \pi^u(X) \neq \emptyset\}$$

and

$$\mathcal{C}_s(X, r) = \{\alpha \in P_r^{(s)} : I^s(\alpha^t) \cap \pi^s(X) \neq \emptyset\}$$

whose cardinalities are denoted $N_u(X, r) = |\mathcal{C}_u(X, r)|$ and $N_s(X, r) = |\mathcal{C}_s(X, r)|$.

In the article [3] the authors proved the following lemma in the case of $X = \Lambda_t$ with $t \in \mathbb{R}$, for completeness we reproduce the proof here:

Lemma 3.2.1. *If X is a compact φ-invariant subset of Λ, then the sequences $\{N_u(X, r)\}_{r \in \mathbb{N}}$, and $\{N_s(X, r)\}_{r \in \mathbb{N}}$ are essentially submultiplicative, in the sense that, there exists a constant $c = c(\varphi, \Lambda) \in \mathbb{N}$ such that*

$$N_u(X, m + n) \leq |\mathcal{A}|^c \cdot N_u(X, m) \cdot N_u(X, n)$$

and

$$N_s(X, m + n) \leq |\mathcal{A}|^c \cdot N_s(X, m) \cdot N_s(X, n).$$

Proof. By symmetry (i.e., exchanging the roles of φ and φ^{-1}), it suffices to show that the sequence $N_u(X, r)$, $r \in \mathbb{N}$, is essentially submultiplicative.

By 3.2.1 we have for all α, β, γ finite words such that the concatenation $\alpha\beta\gamma$ is admissible

$$|I^u(\alpha\beta\gamma)| \leq e^{2c_1}|I^u(\alpha)| \cdot |I^u(\beta)| \cdot |I^u(\gamma)|$$

Next, we observe that, if $\gamma = \gamma_1 \dots \gamma_c$ is a finite word in the letters $\gamma_i \in \mathcal{A}$, $1 \leq i \leq c$, then

$$|I^u(\gamma)| \leq \frac{1}{\mu^c} \max_{a \in \mathcal{A}} |I^u_a|$$

where $\mu = \min |Dg_u| > 1$,

Now, we note that, for each $c \in \mathbb{N}$, one can cover $\pi^u(X)$ with $\leq \#\mathcal{A}^c \cdot N_u(X, n) \cdot N_u(X, m)$ intervals $I^u(\alpha\beta\gamma)$ with $\alpha \in \mathcal{C}_u(X, n)$, $\beta \in \mathcal{C}_u(X, m)$, $\gamma \in \mathcal{A}^c$ and $\alpha\beta\gamma$ admissible.

Therefore, by taking

$$c_3 = c_3(\varphi, \Lambda) = \lceil \frac{\log(e^{2c_1} \max_{a \in \mathcal{A}} |I^u_a|)}{\log \mu} \rceil \in \mathbb{N},$$

it follows that we can cover $\pi^u(X)$ with $\leq |\mathcal{A}|^{c_3} \cdot N_u(X, n) \cdot N_u(X, m)$ intervals $I^u(\alpha\beta\gamma)$ whose scales satisfy

$$r^{(u)}(\alpha\beta\gamma) \geq r^{(u)}(\alpha) + r^{(u)}(\beta) \geq n + m$$

whenever $\alpha \in \mathcal{C}_u(X, n)$, $\beta \in \mathcal{C}_u(X, m)$, $\gamma \in \mathcal{A}^{c_3}$ and $\alpha\beta\gamma$ is admissible. Hence, we conclude that

$$N_{\mathbf{u}}(X, n + m) \leq |\mathcal{A}|^{c_3} \cdot N_u(X, n) \cdot N_u(X, m)$$

for all $n, m \in \mathbb{N}$. □

From this Lemma we get that for each $X \subset \Lambda$ compact φ-invariant there exist the limits

$$D_u(X) = \lim_{r \to \infty} \frac{\log N_u(X, r)}{r} = \inf_{r \in \mathbb{N}} \frac{\log(|\mathcal{A}|^c N_u(X, r))}{r} \in (0, 1)$$

and

$$D_s(X) = \lim_{r \to \infty} \frac{\log N_s(X, r)}{r} = \inf_{r \in \mathbb{N}} \frac{\log(|\mathcal{A}|^c N_s(X, r))}{r} \in (0, 1).$$

And that the numbers $D_u(X)$ and $D_s(X)$ are the box-counting dimension of $\pi^u(X)$ and $\pi^s(X)$ respectively.

By hyperbolicity we have for some constants $C > 1$ and $\beta \geq 1$ that depends only on Λ and any α admissible that

$$C^{-1}|I^u(\alpha)|^\beta \leq |I^s(\alpha^t)| \leq C|I^u(\alpha)|^{1/\beta}$$

and for this, we conclude that for every $X \subset \Lambda$, compact and φ-invariant, $D_s(X)$ and $D_u(X)$ are comparable i.e. there exist some constant $\tilde{C} > 1$ that only depends on Λ such that

$$\tilde{C}^{-1}D_u(X) \leq D_s(X) \leq \tilde{C}D_u(X) \tag{3.2.2}$$

and so,

$$HD(X) \leq D_s(X) + D_u(X) \leq (\tilde{C}+1)D_s(X) \tag{3.2.3}$$

and

$$HD(X) \leq D_s(X) + D_u(X) \leq (\tilde{C}+1)D_u(X). \tag{3.2.4}$$

Now, fix $r \geq 1$ and for $x \in \Lambda$, let e^s_x and e^u_x unit vectors in the stable and unstable directions of T_xS. We set

$$\mathcal{R}^1_{\varphi,\Lambda} = \{f \in C^r(S,\mathbb{R}) : \nabla f(x) \text{ is not perpendicular neither to } e^s_x \text{ nor } e^u_x \text{ for all } x \in \Lambda\}.$$

In other terms, $\mathcal{R}^1_{\varphi,\Lambda}$ is the class of C^r-functions $f : S \to \mathbb{R}$ that are locally monotone along stable and unstable directions. The next proposition follows from the results proved in [3] (see remark 1.4 in that paper)

Proposition 3.2.2. *Fix $r \geq 2$. If the mixing horseshoe Λ has Hausdorff dimension smaller than 1, then $\mathcal{R}^1_{\varphi,\Lambda}$ is C^r-open and dense in $C^r(S,\mathbb{R})$ and $t \mapsto D_u(\Lambda_t)$ and $t \mapsto D_s(\Lambda_t)$ are continuous functions.*

3.3 Proof of Theorem 3.1.2

The proof is by contradiction [1]: We suppose the existence of an infinite sequence of discontinuities of the map L after the first accumulation point of the Lagrange spectrum and associate to every term of that sequence a pair of subhorseshoes (see A.0.3) that *connect* in specific times. Then, using the constructed sequence of pair of subhorsehoes we obtain arbitrarily big finite sequences of subhorseshoes that don't connect two by two. Choosing correct scales (at the level of sequences) we show that for every term of such a sequence, we can associate a periodic orbit (with bounded period that doesn't depend on the sequence) in such a way that it is possible to connect two subhorseshoes with the same associated periodic orbit, letting us obtain the desired contradiction.

[1] The precise definitions and statements will be present in the sequel.

3.3.1 Geometric consequences of a discontinuity

In this subsection, we show how to associate to each discontinuity the pair o sub-horseshoes described in the last paragraph.

Let us consider for $X \subset \Lambda$ and $h > 0$ the set $C(X, h)$ of admissible finite words p of the form $p = (a_{-m}, \dots, a_0, \dots, a_n)$, $m, n \in \mathbb{N}$, such that the rectangle $R(a_{-m}, \dots, a_0, \dots, a_n; 0) = \bigcap_{j=-m}^{n} \varphi^{-j}(R_{a_j})$ satisfies that $X \cap R(a_{-m}, \dots, a_0, \dots, a_n; 0) \neq \emptyset$ and has diameter $\leq h$ but one of the rectangles $R(a_{-m}, \dots, a_0, \dots, a_{n-1}; 0)$ or $R(a_{-m+1}, \dots, a_0, \dots, a_n; 0)$ has diameter $> h$.

Also set

$$l(h) = \max\{m \in \mathbb{N} : \exists\ p = (a_{-m}, \dots, a_0, \dots, a_n) \in C(\Lambda, h)\}$$

and

$$r(h) = \max\{n \in \mathbb{N} : \exists\ p = (a_{-m}, \dots, a_0, \dots, a_n) \in C(\Lambda, h)\}.$$

We have the following result

Proposition 3.3.1. *Given $\epsilon > 0$ and $c_0 > 0$ there exists a constant $\delta = \delta(\epsilon, c_0) > 0$ such that for every $t \in \mathbb{R}$, if X is a compact φ-invariant subset of Λ_t such that the limit capacities $D_u(X)$ and $D_s(X)$ satisfy both $D_u(X), D_s(X) \geq c_0$. Then there are subhorseshoes $\Lambda^s(X)$ and $\Lambda^u(X)$ of Λ such that*

$$D_u(\Lambda^u(X)) > (1-\epsilon)D_u(X), \quad D_s(\Lambda^s(X)) > (1-\epsilon)D_s(X)$$

and

$$\Lambda^u(X) \cup \Lambda^s(X) \subset \Lambda_{t-\delta}.$$

Furthermore, for every $x \in \Lambda^u(X) \cup \Lambda^s(X)$ the sets

$$X_\epsilon^+(x) := \{n \in \mathbb{N} : \exists\ \alpha = (a_{-l(\epsilon)}, \dots, a_0, \dots a_{r(\epsilon)}) \text{ admissible and } y \in X \text{ with } \varphi^n(x), y \in R(\alpha; 0)\}$$

and

$$X_\epsilon^-(x) := \{n \in \mathbb{Z}^- : \exists\ \beta = (b_{-l(\epsilon)}, \dots, b_0, \dots b_{r(\epsilon)}) \text{ admissible and } y \in X \text{ with } \varphi^n(x), y \in R(\beta; 0)\}$$

are both infinite.

Proof. We will follow closely the proof of proposition 2.9 of [3]. Take $f \in \mathcal{R}^1_{\varphi,\Lambda}$, $t \in \mathbb{R}$ and $X \subset \Lambda_t$, where X is compact and φ-invariant as in the statement of the proposition. We observe that the same proof of that proposition let us conclude that for each $0 < \eta < 1$, there exists $\delta_1 > 0$ and a complete subshift $\Sigma(\mathcal{B}_u) \subset \Sigma \subset \mathcal{A}^{\mathbb{Z}}$ associated to a finite set $\mathcal{B}_u$, of finite sequences such that

$$\Sigma(\mathcal{B}_u) \subset \Sigma_{t-\delta_1} \quad \text{and} \quad D_u(\Lambda(\Sigma(\mathcal{B}_u))) > (1-\eta)D_u(X),$$

where $\Lambda(\Sigma(\mathcal{B}_u))$ denotes the subhorseshoe of Λ associated to $\mathcal{B}_u$. We point here that $\Lambda(\Sigma(\mathcal{B}_u))$ needs not to be contained in X.

For fixing ideas and for future use we will remember some facts about that proof: The construction of $\mathcal{B}_u$ depends on three combinatorial lemmas (2.13-2.15). In our case, to prove that lemmas, we take r_0 large so that

$$\left|\frac{\log N_u(X,r)}{r} - D_u(X)\right| < \frac{\tau}{2}D_u(X) \tag{3.3.1}$$

for all $r \in \mathbb{N}$, $r \geq r_0$ where $\tau = \eta/100$.

The alphabet $\mathcal{B}_u$ is obtained from the set

$$\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_u = \{\beta = \beta_1 \dots \beta_k : \beta_j \in \ \mathcal{C}_u(X,r_0) \ \ \forall\, 1 \leq j \leq k \ \text{ and } \ \pi^u(X) \cap I^u(\beta) \neq \emptyset\}$$

where $k = 8N_u(X,r_0)^2\lceil 2/\tau \rceil$.

Defining the notion of *good position* for positions $j \in \{1,...,k\}$ (see definition 3.3.14 below) is showed that most positions of most words of $\widetilde{\mathcal{B}}$ are good and for that set of words, say $\mathcal{E}$, we can find natural numbers $1 \leq s_1 \leq \cdots \leq s_{3N_0^2} \leq k$, ($N_0 = N_u(X,r_0)$) with

$$s_{m+1} - s_m \geq 2\lceil 2/\tau \rceil \quad \text{for} \quad 1 \leq m < 3N_0^2$$

and words $\widehat{\beta}_{s_1}, \widehat{\beta}_{s_1+1}, \dots, \widehat{\beta}_{s_{3N_0^2}}, \widehat{\beta}_{s_{3N_0^2}+1} \in \ \mathcal{C}_u(X,r_0)$ such that the set $\mathcal{P}$ of words in $\mathcal{E}$ with s_m, s_m+1 good positions and $\beta_{s_m} = \widehat{\beta}_{s_m}, \beta_{s_m+1} = \widehat{\beta}_{s_m+1}$ for $1 \leq m < 3N_0^2$ has cardinality

$$|\mathcal{P}| > N_0^{(1-2\tau)k}.$$

Then is proved that there are $1 \leq p_0 < q_0 \leq 3N_0^2$ such that $\widehat{\beta}_{s_{p_0}} = \widehat{\beta}_{s_{q_0}}$, $\widehat{\beta}_{s_{p_0+1}} = \widehat{\beta}_{s_{q_0+1}}$ and the cardinality of $\mathcal{B}_u = \pi_{p_0,q_0}(\mathcal{P})$ is

$$|\mathcal{B}_u| > N_0^{(1-10\tau)(s_{q_0}-s_{p_0})},$$

where

$$\pi_{p_0,q_0} : \mathcal{P} \to \mathcal{C}_u(X, r_0)^{s_{q_0}-s_{p_0}} \quad \text{is the projection} \quad \pi_{p_0,q_0}(\beta_1 \dots \beta_k) = (\beta_{s_{p_0+1}}, \dots, \beta_{s_{q_0}})$$

obtained by cutting a word $\beta_1 \dots \beta_k \in \mathcal{P}$ at the positions s_{p_0} and s_{q_0} and discarding the words β_j with $j \le s_{p_0}$ and $j > s_{q_0}$.

The conclusion on the cardinality of $\mathcal{B}_u$ allows us to show that $D_u(\Lambda(\Sigma(\mathcal{B}_u))) > (1-\eta)D_u(X)$ and that s_{p_0}, $s_{p_0}+1$, s_{q_0} and $s_{q_0}+1$ are good positions for words in $\mathcal{P}$ that $\Sigma(\mathcal{B}_u) \subset \Sigma_{t-\delta_1}$.

Even more, the proof of that proposition gives us the next formula: $\delta_1 = \min\{\delta^1, \delta^2, \delta^3, \delta^4\}$ where if $\gamma_1 = \widehat{\beta}_{s_{p_0+1}} = a_1 \dots a_{\widehat{m}_1}$, $\beta_{s_{p_0}+2} \dots \beta_{s_{q_0}-1} = b_1 \dots b_{\widehat{m}}$ and $\gamma_2 = \widehat{\beta}_{s_{q_0}} = d_1 \dots d_{\widehat{m}_2}$ then

- $\delta^1 = p \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \ \min\limits_{1 \le j \le \widehat{m}-1} |I^u(b_j \dots b_{\widehat{m}} \gamma_2)|$
- $\delta^2 = p \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \ \min\limits_{1 \le j \le \widehat{m}-1} |I^s((\gamma_1 b_1 \dots b_{j-1})^T)|$
- $\delta^3 = p \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \ \min\limits_{1 \le \ell \le \widehat{m}_1-1} |I^s((\gamma_2 a_1 \dots a_\ell)^T)|$
- $\delta^4 = p \cdot \min\limits_{\gamma_1 b_1 \dots b_{\widehat{m}} \gamma_2 \in \mathcal{B}_u} \ \min\limits_{1 \le \ell \le \widehat{m}_1-1} |I^u(d_{\ell-\widehat{m_1}-\widehat{m}+1} \dots d_{\widehat{m}_2} \gamma_1)|$

and p is a positive constant that only depends on the function f and φ. Now, using the above facts, we want to give more precise estimates of the value of δ_1. The crucial observation here is that in the proof sketched above (without the dimension estimate) we can replace the conditions on r_0 (and k) given by the equation 3.3.1 by the assumption that $r_0 > \lceil (c_1+1)/\tau^2 \rceil$ and $k = 8N_u(X, r_0)^2 \lceil 2/\tau \rceil$ satisfy the inequality

$$\frac{\log N_u(X, r_0)}{r_0} < (1 + \frac{\tau}{2}) \frac{\log N_u(X, k.(r_0 - c_1))}{k.(r_0 - c_1)},$$

where c_1 comes from the bounded distortion property as in equation 3.2.1, because in that case multiplying that inequality by $(1-\tau)r_0 k$ we have

$$\begin{aligned}
\log N_u(X, r_0)^{(1-\tau)k} &< (1-\tau)(1+\tau/2)\frac{r_0}{r_0 - c_1} \log N_u(X, k(r_0 - c_1)) \\
&\le (1 - \frac{\tau}{2})(1 + \frac{c_1}{r_0 - c_1}) \log N_u(X, k(r_0 - c_1)) \\
&< (1 - \frac{\tau}{2})(1 + \frac{\tau^2}{1 - \tau^2}) \log N_u(X, k(r_0 - c_1)) \\
&< (1 - \frac{\tau}{2})(1 + \frac{\tau}{2}) \log N_u(X, k(r_0 - c_1)) \\
&= \log N_u(X, k(r_0 - c_1))^{1 - \frac{\tau^2}{4}}
\end{aligned}$$

and then $2N_u(X, r_0)^{(1-\tau)k} < N_u(X, k(r_0 - c_1))$ that is just the necessary condition to obtain the equation 2.4 in the proof of lemma 2.13 of that paper and the claims in other parts of the proof of the lemmas that use the assumptions that r_0 and k are large are satisfied provided $r_0 > \lceil (c_1 + 1)/\tau^2 \rceil$.

Now, if $z(\tau, \Lambda) \in \mathbb{N}$ is such that given $r_0 \geq z(\tau, \Lambda)$, for any complete subshift associated to a finite alphabet $\mathcal{B}_u = \mathcal{B}_u(r_0)$ of finite words as before, the Cantor set $K^u(\Sigma(\mathcal{B}_u))$ consisting of points of K^u whose trajectory under g_u follows an itinerary obtained from the concatenation of words in the alphabet $\mathcal{B}_u$[2], satisfies that $\lambda = \lambda(K^u(\Sigma(\mathcal{B}_u)))$ is big (we can take $a = a(K^u(\Sigma(\mathcal{B}_u))) = a(K^u(\Lambda))$ where λ and a are as in section 2.2), then by 2.2.3 and 2.2.4

$$\beta_1 - \alpha_1 \leq \frac{\tau}{2} HD(K^u(\Sigma(\mathcal{B}_u))) \leq \frac{\tau}{2}\beta_1.$$

Using this, 2.2.2 and 2.2.3 we obtain

$$HD(K^u(\Sigma(\mathcal{B}_u))) \geq \alpha_1 \geq \left(1 - \frac{\tau}{2}\right)\beta_1 \geq \left(1 - \frac{\tau}{2}\right)\frac{\log|\mathcal{B}_u|}{-\log(\min_{\alpha \in \mathcal{B}_u}|I^u(\alpha)|)}$$

that is the equation used to obtain the dimension estimate.

In order to continue, observe first that for $m \in \mathbb{N}$ and $\beta \in \mathcal{C}_u(X, m)$, $|I^u(\beta^*)| > e^{-m}$ and then for some $c > 1$, $c.(\lambda_{1,u}^{-1})^{|\beta|-1} > e^{-m}$ where $\lambda_{1,u}$ is the smallest modulus of eigenvalues in Λ at the unstable direction. From this follows that $|\beta| < \tilde{\alpha}_1 m + \tilde{\alpha}_2$ where $\tilde{\alpha}_1 = \log(\lambda_{1,u})^{-1}$ and $\tilde{\alpha}_2 = \log(c.\lambda_{1,u})/\log(\lambda_{1,u})$ and then

$$N_u(X, m) = |\mathcal{C}_u(X, m)| \leq |\mathcal{A}|^{\tilde{\alpha}_1 m + \tilde{\alpha}_2} = e^{\alpha_1 m + \alpha_2}, \tag{3.3.2}$$

where $\alpha_1 = \tilde{\alpha}_1.\log|\mathcal{A}| > 0$ and $\alpha_2 = \tilde{\alpha}_2.\log|\mathcal{A}| > 0$ depends only on Λ.

Suppose then $c_0 \leq D_u(X)$ and without loss of generality also that

$$\eta < \min\{c_0, 5000/(c_3|\log\mathcal{A}|), 3\lambda_{1,s}, 3\lambda_{2,u}^{-1}, \kappa\},$$

where $\kappa > 0$ is such that the maps $x \to e^{e^x} - 8e^{2\alpha_1 x + 2\alpha_2}x^2$ and $x \to e^{e^x} - 8\log x.e^{2\alpha_1 x + 2\alpha_2}.x.(\alpha_1 x + \alpha_2)$ are positive if $x > 1/\kappa^2$. Following the observations described above we define the sequence (p_n) as follows: $p_0 = \max\{\lceil (c_1+1)/\tau^2 \rceil, z(\tau, \Lambda)\}$ and for $n \geq 0$ put

$$p_{n+1} = 8N_u(X, p_n)^2 \lceil 2/\tau \rceil (p_n - c_1).$$

[2] which is $C^{1+\alpha}$-dynamically defined associated to certain iterates of g_u on the intervals $I^u(\beta)$. with $\beta \in \mathcal{B}$

We claim that, for some integer $0 \leq s_0 < (1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$, we have

$$\frac{\log N_u(X, p_{s_0})}{p_{s_0}} < (1+\frac{\tau}{2})\frac{\log N_u(X, p_{s_0+1})}{p_{s_0+1}} = (1+\frac{\tau}{2})\frac{\log N_u(X, k(p_{s_0}-c_1))}{k(p_{s_0}-c_1)},$$

with $k = 8N_u(X, p_{s_0})^2\lceil 2/\tau\rceil$.

Indeed, if it is not the case, then for $0 \leq n < (1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$, we have

$$\frac{\log N_u(X, p_{n+1})}{p_{n+1}} < (1+\frac{\tau}{2})^{-1}\frac{N_u(X, p_n)}{p_n}$$

and then, for $M = \lceil(1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}\rceil$ we would have

$$\frac{\log N_u(X, p_M)}{p_M} \leq (1+\frac{\tau}{2})^{-M}.\frac{\log N_u(X, p_0)}{p_0} < \frac{\eta}{4(\alpha_1+\alpha_2+1)}.\frac{\log N_u(X, p_0)}{p_0}$$

because

$$(1+\frac{\tau}{2})^{-M} \leq ((1+\frac{\tau}{2})^{-(1+\frac{2}{\tau})})^{\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}} < e^{-\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}} = \frac{\eta}{4(\alpha_1+\alpha_2+1)}.$$

And so, by 3.3.2

$$\frac{\log N_u(X, p_M)}{p_M} \leq \frac{\eta}{4(\alpha_1+\alpha_2)}.\frac{\log N_u(X, p_0)}{p_0} \leq \frac{\eta}{4(\alpha_1+\alpha_2)}.\frac{\alpha_1.p_0+\alpha_2}{p_0} < \frac{\eta}{2}.$$

But this is a contradiction because

$$\eta < c_0 \leq D_u(X) \leq \frac{\log(|\mathcal{A}|^{c_3}.N_u(X, p_M))}{p_M} \leq \frac{c_3.\log|\mathcal{A}|}{p_M} + \frac{\log N_u(X, p_M)}{p_M}$$

and then $\frac{\log N_u(X,p_M)}{p_M} > \eta - \eta/2 = \eta/2$.

Now if $L = \min\{\lambda_{1,s}, \lambda_{2,u}^{-1},\}^{-1} > 1$ where $\lambda_{1,s}$ is the smallest modulus of eigenvalues in Λ at the stable direction and $\lambda_{2,u}$ is the greatest modulus of eigenvalues in Λ at the unstable direction then for some constant $\tilde{c} = \tilde{c}(\varphi, f) > 0$, by taking $r_0 = p_{s_0}$ and $k = 8N_u(X, r_0)^2\lceil 2/\tau\rceil$ the argument for the construction of $\mathcal{B}_u$ works and then we have:

$$\delta_1 \geq \tilde{c}.\frac{1}{L^{\widehat{m}_1+\widehat{m}_2+\widehat{m}}} \geq \tilde{c}.\frac{1}{L^{k.\max\{|\beta|:\beta\in\mathcal{C}_u(X,r_0)\}}} \geq \tilde{c}.\frac{1}{L^{k.(\alpha_1 r_0+\alpha_2)}}. \tag{3.3.3}$$

We will now give an explicit positive lower bound for δ_1 in terms of η. In order to do this, we define recursively, for each integer $n \geq 0$ and $x \in \mathbb{R}$, the functions $\mathcal{T}(n)$

and $\mathcal{T}(n,x)$ by $\mathcal{T}(x,0)=x$, $\mathcal{T}(x,n+1)=e^{\mathcal{T}(x,n)}$ and $\mathcal{T}(n)=\mathcal{T}(1,n)$. We have for $n\geq 0$,

$$p_{n+1}=8N_u(X,p_n)^2\lceil 2/\tau\rceil(p_n-c_1)\leq 8e^{2\alpha_1 p_n+2\alpha_2}.\frac{3}{\tau}.p_n<8e^{2\alpha_1 p_n+2\alpha_2}p_n^2<e^{e^{p_n}}.$$

Since, for every $n\geq 0$, $p_n\geq p_0>\lceil 1/\tau^2\rceil>\frac{3}{\tau}$, therefore $r_0=p_{s_0}<\mathcal{T}(p_0,2s_0)$ and

$$\begin{aligned}\log L.k.(\alpha_1 r_0+\alpha_2) &= 8\log L.N_u(X,r_0)^2\lceil 2/\tau\rceil.(\alpha_1 r_0+\alpha_2)\\ &< 8\log\frac{3}{\eta}.e^{2\alpha_1 r_0+2\alpha_2}.3/\tau.(\alpha_1 r_0+\alpha_2)\\ &< 8\log r_0.e^{2\alpha_1 r_0+2\alpha_2}.r_0.(\alpha_1 r_o+\alpha_2)<e^{e^{r_0}}\end{aligned}$$

so, by 3.3.3

$$\delta_1>\tilde{c}.\frac{1}{L^{k.(\alpha_1 r_0+\alpha_2)}}=\tilde{c}.e^{-\log L.k.(\alpha_1 r_0+\alpha_2)}>\tilde{c}.e^{-e^{e^{r_0}}}>\frac{\tilde{c}}{\mathcal{T}(p_0,2s_0+3)}. \tag{3.3.4}$$

Finally, since $2^r\geq r^2$ for every $r\geq 4$, it follows by induction that, for every $n\geq 4$, $\mathcal{T}(n)\geq(n+1)^6$. Indeed, $\mathcal{T}(4)>2^{16}>5^6$ and for $n\geq 4$, $\mathcal{T}(n+1)>2^{\mathcal{T}(n)}\geq\mathcal{T}(n)^2\geq(n+1)^{12}\geq(n+2)^6$. This implies that

$$\mathcal{T}(\lfloor(c_1+1)/\eta\rfloor)\geq((c_1+1)/\eta)^6>10001(c_1+1)/\eta^2>\lceil 10000(c_1+1)/\eta^2\rceil=p_0$$

and as $s_0<(1+\frac{2}{\tau})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}=(1+\frac{200}{\eta})\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$, we have

$$\lfloor(c_1+1)/\eta\rfloor+2s_0+3<\frac{202+c_1}{\eta}\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}$$

and then

$$\begin{aligned}\mathcal{T}(p_0,2s_0+3)<\mathcal{T}(\mathcal{T}(\lfloor(c_1+1)/\eta\rfloor),2s_0+3) &= \mathcal{T}(\lfloor(c_1+1)/\eta\rfloor+2s_0+3)\\ &< \mathcal{T}(\lfloor\frac{202+c_1}{\eta}\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}\rfloor)\end{aligned}$$

and by 3.3.4, $\delta_1>\frac{\tilde{c}}{\mathcal{T}(c_0,2s_0+3)}>\frac{\tilde{c}}{\mathcal{T}(\lfloor\frac{202+c_1}{\eta}\log\frac{4(\alpha_1+\alpha_2+1)}{\eta}\rfloor)}$.

Now, it is clear from the construction of $\mathcal{B}_u$ and from the fact that

$$s_{q_0}-s_{p_0}\geq 2\lceil 2/\tau\rceil(q_0-p_0)\geq 2\lceil 2/\tau\rceil$$

that for $\eta<\epsilon$ small enough and $x\in\Lambda(\Sigma(\mathcal{B}_u))$, the sets $X_\epsilon^+(x)$ and $X_\epsilon^-(x)$ are infinite.

Finally, the same construction can be repeated to obtain a $\delta_2 = \delta_2(\epsilon, c_0) > 0$ and a complete subshift $\Sigma(\mathcal{B}_s) \subset \Sigma \subset \mathcal{A}^{\mathbb{Z}}$ associated to a finite set $\mathcal{B}_s$ such that $\Sigma(\mathcal{B}_s) \subset \Sigma_{t-\delta_2}$, $D_s(\Lambda(\Sigma(\mathcal{B}_s))) > (1-\epsilon)D_s(X)$ and for $x \in \Lambda(\Sigma(\mathcal{B}_s))$ the sets $X_\epsilon^+(x)$ and $X_\epsilon^-(x)$ are infinite where $\Lambda(\Sigma(\mathcal{B}_s))$ denotes the subhorseshoe of Λ associated to $\mathcal{B}_s$. Taking $\delta = \min\{\delta_1, \delta_2\}$, $\Lambda^s(X) = \Lambda(\Sigma(\mathcal{B}_s))$ and $\Lambda^u(X) = \Lambda(\Sigma(\mathcal{B}_u))$, we have proved the result.

$\square$

Next, we return to the map $t \to L(t) = L_{\varphi,f}(t) = HD(L_{\varphi,f} \cap (-\infty, t))$ and try to describe its discontinuities. In this direction, we have the following result

Lemma 3.3.2. *For every $t \in \mathbb{R}$ we have*

$$L(t) = \sup_{s<t} HD(\ell_{\varphi,f}(\Lambda_s)) = \lim_{s\to\ t^-} HD(\ell_{\varphi,f}(\Lambda_s))$$

and

$$M(t) = \sup_{s<t} HD(m_{\varphi,f}(\Lambda_s)) = \lim_{s\to\ t^-} HD(m_{\varphi,f}(\Lambda_s)).$$

Proof. Let $x \in \Lambda$ with $\ell_{\varphi,f}(x) = \eta < t$, then there exist a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} f(\varphi^{n_k}(x)) = \eta$. By compactness, without loss of generality, we can suppose also that $\lim\limits_{k\to\infty} \varphi^{n_k}(x) = y$ for some $y \in \Lambda$ and so that $f(y) = \eta$.

We affirm that $m_{\varphi,f}(y) = \eta$: in other case we would have for some $\tilde{k} \in \mathbb{Z}$ and $r \in \mathbb{R}$, $f(\varphi^{\tilde{k}}(y)) > r > \eta$ and then for k big enough by continuity $f(\varphi^{\tilde{k}+n_k}(x)) > \eta$ that contradicts the definition of η. Then, we conclude that

$$\ell_{\varphi,f}(\ell_{\varphi,f}^{-1}(-\infty,t)) = \ell_{\varphi,f}(\{x \in \Lambda : \ell_{\varphi,f}(x) < t\}) \subset \bigcup_{s<t} \ell_{\varphi,f}(\Lambda_s)$$

and as for $s < t$, $\Lambda_s \subset \ell_{\varphi,f}^{-1}(-\infty,t)$, the other inclusion also holds and we have the equality

$$\ell_{\varphi,f}(\ell_{\varphi,f}^{-1}(-\infty,t)) = \bigcup_{s<t} \ell_{\varphi,f}(\Lambda_s).$$

From this follows the result

$$\begin{aligned} L(t) &= HD(\ell_{\varphi,f}(\ell_{\varphi,f}^{-1}(-\infty,t))) = HD(\bigcup_{s<t} \ell_{\varphi,f}(\Lambda_s)) = HD(\bigcup_{n\in\mathbb{N}} \ell_{\varphi,f}(\Lambda_{t-1/n})) \\ &= \sup_{n\in\mathbb{N}} HD(\ell_{\varphi,f}(\Lambda_{t-1/n})) = \sup_{s<t} HD(\ell_{\varphi,f}(\Lambda_s)). \end{aligned}$$

For the second identity, as before

$$M(t) = HD(M_{\varphi,f}(\Lambda)\cap(-\infty,t)) = \sup_{s<t} HD(M_{\varphi,f}(\Lambda)\cap(-\infty,s]) = \sup_{s<t} HD(m_{\varphi,f}(\Lambda_s)).$$

□

Now, using the spectral decomposition theorem, it follows the next result from [14]:

Proposition 3.3.3. *There exists a residual subset $\mathcal{U}^{**} \subset \mathcal{U}$ with the property that for every subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and any $f \in C^1(S,\mathbb{R})$ such that there exists some point in $\tilde{\Lambda}$ with its gradient not parallel neither the stable direction nor the unstable direction, one has*

$$HD(f(\widetilde{\Lambda})) = HD(\widetilde{\Lambda}).$$

that we use to prove the next proposition

Proposition 3.3.4. *If $\mathcal{U}^{**}$ is as in the proposition 3.3.3 and $r \geq 2$, then for any $\varphi \in \mathcal{U}^{**}$, there exists a C^r-residual subset $\mathcal{R}_{\varphi,\Lambda} \subset \mathcal{R}^1_{\varphi,\Lambda}$ such that for every subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and any $f \in \mathcal{R}_{\varphi,\Lambda}$ one has*

$$HD(\widetilde{\Lambda}) = HD(\ell_{\varphi,f}(\widetilde{\Lambda})) = HD(m_{\varphi,f}(\widetilde{\Lambda})).$$

Proof. Following the ideas of the proof of the theorem 1 of [20] we see that for every subhorsehoe $\widetilde{\Lambda} \subset \Lambda$, there exist a C^r- open and dense set $\mathcal{R}_{\widetilde{\Lambda}} \subset C^r(S,\mathbb{R})$ such that for $f \in \mathcal{R}_{\widetilde{\Lambda}}$, $M_{\widetilde{\Lambda},f} = \{x \in \widetilde{\Lambda} : \forall y \in \widetilde{\Lambda}\ ,\ f(x) \geq f(y)\}$ is a unitary set. Take then

$$\mathcal{R}_{\varphi,\Lambda} := \bigcap_{\substack{\widetilde{\Lambda}\subset\Lambda \\ subhorseshoe}} \mathcal{R}_{\widetilde{\Lambda}} \cap \mathcal{R}^1_{\varphi,\Lambda}.$$

In the mentioned paper is also proved that for any such subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and $f \in R_{\widetilde{\Lambda}}$ if x_M is the unique element where $f|_{\widetilde{\Lambda}}$ take its maximum value, then for any $\epsilon > 0$ there exists some subhorseshoe $\widetilde{\Lambda}^\epsilon \subset \widetilde{\Lambda} \setminus \{x_M\}$ with

$$HD(\widetilde{\Lambda}^\epsilon) \geq HD(\widetilde{\Lambda})(1-\epsilon)$$

and such that for some point $d \in \widetilde{\Lambda}^\epsilon$ there exists a local C^1-diffeomorphism $\tilde{A}$ defined in a neighborhood U_d of d such that

$$f(\varphi^{j_0}(\tilde{A}(\tilde{\Lambda}_{j_0}))) \subset \ell_{\varphi,f}(\widetilde{\Lambda}),$$

where j_0 is an integer and $\tilde{\Lambda}_{j_0} \subset \widetilde{\Lambda}^\epsilon$ has nonempty interior in $\widetilde{\Lambda}^\epsilon$ and then is such that $HD(\tilde{\Lambda}_{j_0}) = HD(\widetilde{\Lambda}^\epsilon)$. Moreover, it is proved also that $\dfrac{\partial \tilde{A}}{\partial e_x^{s,u}} \parallel e^{s,u}_{\tilde{A}(x)}$, for $x \in U_d \cap \widetilde{\Lambda}^\epsilon$ and then, $\nabla(f \circ \varphi^{j_0} \circ \tilde{A})(x) \nparallel e_x^{s,u}$ for every $x \in \tilde{\Lambda}_{j_0}$.

Extending properly $f \circ \varphi^{j_0} \circ \tilde{A}$, and letting ϵ tends to 0; it follows from this and proposition 3.3.3 that

$$HD(\widetilde{\Lambda}) \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})).$$

An elementary compactness argument (similar with the proof of lemma 3.3.2) shows that $\{\ell_{\varphi,f}(x) : x \in X\} \subset \{m_{\varphi,f}(x) : x \in X\} \subset f(X)$ whenever $X \subset M$ is a compact φ-invariant subset. It follows that

$$HD(\widetilde{\Lambda}) \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})) \leq HD(m_{\varphi,f}(\widetilde{\Lambda})) \leq HD(f(\widetilde{\Lambda})) \leq HD(\widetilde{\Lambda}).$$

As we wanted to see. $\square$

Take $\varphi \in \mathcal{U}^{**}$, $f \in \mathcal{R}_{\varphi,\Lambda}$ and $t_0 \in \mathbb{R}$ with $L(t_0) \neq 0$. For lemma 3.3.2 we have

$$0 < L(t_0) = \sup_{s<t_0} HD(\ell_{\varphi,f}(\Lambda_s)) \leq HD(\ell_{\varphi,f}(\Lambda_{t_0})) \leq HD(f(\Lambda_{t_0})) \leq HD(\Lambda_{t_0}),$$

then $D_u(\Lambda_{t_0}) > 0$ (also $D_s(\Lambda_{t_0}) > 0$), and by proposition 2.9 in [3] we can find some horseshoe $\Lambda^0 \subset \Lambda_{t_0}$.

Now, suppose that t_0 is a discontinuity for L. So, there exist an $a > 0$ such that

$$L(q) + a < L(s) \text{ for } q \leq t_0 < s. \tag{3.3.5}$$

For $0 < \epsilon < a/2$ and $c_0 = HD(\Lambda^0)/(\tilde{C} + 1) > 0$ take $\delta = \delta(\epsilon/2k, c_o) < \epsilon$ as in the proposition 3.3.1 where $k > 1$ is a Lipschitz's constant for f, and let us consider for $t \in \mathbb{R}$ and $h > 0$ the set $C(\Lambda_t, h)$. Then by compactness, for each $h > 0$, one has

$$C(\Lambda_{t_0}, h) = \bigcap_{t>t_0} C(\Lambda_t, h).$$

In particular, for each h, there exists $t(h) > t_0$ such that for $t_0 < t < t(h)$

$$C(\Lambda_t, h) = C(\Lambda_{t(h)}, h) = C(\Lambda_{t_0}, h).$$

Take then $0 < h < \delta/2k$ and consider the maximal invariant set

$$P = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\bigcup_{p\in C(\Lambda_{t_0},h)} R(p;0)) = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\bigcup_{p\in C(\Lambda_t,h)} R(p;0))$$

for $t_0 < t < t(h)$.

Observe that for $x \in P$ and $n \in \mathbb{Z}$ if $y \in \Lambda_{t_0}$ belongs to the same rectangle $R(p;0)$ as $\varphi^n(x)$ for some $p \in C(\Lambda_{t_0}, h)$ then

$$\begin{aligned} f(\varphi^n(x)) &= f(\varphi^n(x)) - f(y) + f(y) \le |f(\varphi^n(x)) - f(y)| + t_0 \le k.d(\varphi^n(x), y) + t_0 \\ &\le k.h + t_0 < \delta/2 + t_0 \end{aligned}$$

and so $P \subset \Lambda_{t_0+\delta/2}$.

Now, by proposition A.0.3, the set P admits a decomposition $P = \bigcup\limits_{i\in\mathcal{I}} \tilde{\Lambda}_i$ where $\mathcal{I}$ is a finite index set and for $i \in \mathcal{I}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set i.e a set of the form $\tau = \{x \in P : \alpha(x) \subset \tilde{\Lambda}_{i_1} \text{ and } \omega(x) \subset \tilde{\Lambda}_{i_2}\}$ where $\tilde{\Lambda}_{i_1}$ and $\tilde{\Lambda}_{i_2}$ with $i_1, i_2 \in \mathcal{I}$ are subhorseshoes.

Remember that for any subhorseshoe $\tilde{\Lambda} \subset \Lambda$ being locally maximal we have

$$W^s(\tilde{\Lambda}) = \bigcup_{y\in\tilde{\Lambda}} W^s(y) \quad \text{and} \quad W^u(\tilde{\Lambda}) = \bigcup_{y\in\tilde{\Lambda}} W^u(y).$$

Then, there exists an $y \in \tilde{\Lambda}$ with $\lim\limits_{n\to\infty} d(f(\varphi^n(x)), f(\varphi^n(y))) = 0$ for every $x \in P$, such that $\omega(x) \subset \tilde{\Lambda}$, and so $\ell_{\varphi,f}(x) = \ell_{\varphi,f}(y)$. Using this, one has

$$\ell_{\varphi,f}(P) = \bigcup_{i\in\mathcal{I}} \ell_{\varphi,f}(\tilde{\Lambda}_i) = \bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i \text{ is} \\ \text{horseshoe}}} \ell_{\varphi,f}(\tilde{\Lambda}_i) \cup \bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i \\ \text{is orbit}}} \ell_{\varphi,f}(\tilde{\Lambda}_i)$$

and then, by proposition 3.3.4

$$\begin{aligned} HD(\ell_{\varphi,f}(P)) &= HD(\bigcup_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i \text{ is} \\ \text{horseshoe}}} \ell_{\varphi,f}(\tilde{\Lambda}_i)) = \max_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i \text{ is} \\ \text{horseshoe}}} HD(\ell_{\varphi,f}(\tilde{\Lambda}_i)) \\ &= \max_{\substack{i\in\mathcal{I}:\ \tilde{\Lambda}_i \text{ is} \\ \text{horseshoe}}} HD(\tilde{\Lambda}_i). \end{aligned}$$

Let $\tilde{\Lambda}_{i_0}$ with $HD(\ell_{\varphi,f}(P)) = HD(\tilde{\Lambda}_{i_0})$. As $\Lambda^0 \subset P$, by 3.2.3 and 3.2.4 one has

$$c_0 \le HD(\tilde{\Lambda}_{i_0})/(\tilde{C}+1) \le D_s(\tilde{\Lambda}_{i_0}) \text{ and also } c_0 \le HD(\tilde{\Lambda}_{i_0})/(\tilde{C}+1) \le D_u(\tilde{\Lambda}_{i_0})$$

then, proposition 3.3.1 applied to $\tilde{\Lambda}_{i_0}$ let us show the existence of two horseshoes $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ of Λ such that

$$D_u(\Lambda^u(t_0)) > D_u(\tilde{\Lambda}_{i_0}) - \epsilon/2k, \quad D_s(\Lambda^s(t_0)) > D_s(\tilde{\Lambda}_{i_0}) - \epsilon/2k,$$

$$\Lambda^u(t_0) \cup \Lambda^s(t_0) \subset \Lambda_{(t_0+\delta/2)-\delta} = \Lambda_{t_0-\delta/2},$$

and for every $x \in \Lambda^u(t_0) \cup \Lambda^s(t_0)$ the sets $(\tilde{\Lambda}_{i_0})^+_{\epsilon/2k}(x)$ and $(\tilde{\Lambda}_{i_0})^-_{\epsilon/2k}(x)$ are both infinite.

Now, suppose there exists a subhorseshoe $\widetilde{\Lambda} \subset \Lambda_q$ for some $q < t_0$ with $\Lambda^u(t_0) \cup \Lambda^s(t_0) \subset \widetilde{\Lambda}$, then as $\Lambda_t \subset P$ for $t_0 < t < t(h)$ we have by 3.3.5 and lemma 3.3.2

$$\begin{aligned} L(t_0) + a/2 &< L(t_0) + a - \epsilon/k < HD(\ell_{\varphi,f}(P)) - \epsilon/k = HD(\tilde{\Lambda}_{i_0}) - \epsilon/k \\ &< D_u(\Lambda^u(t_0)) + D_s(\Lambda^s(t_0)) \leq HD(\widetilde{\Lambda}) = HD(\ell_{\varphi,f}(\widetilde{\Lambda})) \leq HD(\ell_{\varphi,f}(\Lambda_q)) \\ &\leq \sup_{s<t_0} HD(\ell_{\varphi,f}(\Lambda_s)) = L(t_0) \end{aligned}$$

but this is a contradiction.

On the other hand, take $x \in \Lambda^s(t_0)$, $y \in \Lambda^u(t_0)$ and any $\rho_1, \rho_2 > 0$. If x and y have kneading sequences $(x_n)_{n\in\mathbb{Z}}$, respectively $(y_n)_{n\in\mathbb{Z}}$, choose N_{ρ_1} and N_{ρ_2} big enough such that

$$R(x_{-N_{\rho_1}}, \dots, x_0, \dots, x_{N_{\rho_1}}; 0) \subset B(x, \rho_1) \text{ and } R(y_{-N_{\rho_2}}, \dots, y_0, \dots, y_{N_{\rho_2}}; 0) \subset B(y, \rho_2).$$

Then as the sets $(\tilde{\Lambda}_{i_0})^+_{\epsilon/2k}(x)$ and $(\tilde{\Lambda}_{i_0})^-_{\epsilon/2k}(y)$ are infinite, we can find two words $\alpha = (a_{-l(\epsilon/2k)}, \dots, a_0, \dots a_{r(\epsilon/2k)})$ and $\beta = (b_{-l(\epsilon/2k)}, \dots, b_0, \dots b_{r(\epsilon/2k)})$ that appear far away in the right and in the left respectively of the sequences $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ and also appear in the kneading sequence of two points $\tilde{x}_1, \tilde{y}_1 \in \tilde{\Lambda}_{i_0}$, i.e. $\tilde{x}_1 \in R(\alpha; 0)$, $\tilde{y}_1 \in R(\beta; 0)$ and $(x_{N_1}, \dots x_{N_1-|\alpha|-1}) = \alpha$ for some $N_1 > N_{\rho_1} + 1$ and also $(y_{-N_2-|\beta|+1}, \dots y_{-N_2}) = \beta$ for some $N_2 > N_{\rho_2} + 1$.

As $\tilde{\Lambda}_{i_0}$ is horseshoe we can find a point $z_1 \in \tilde{\Lambda}_{i_0}$ with kneading sequence of the form

$$\begin{aligned} \Pi^{-1}(z_1) = (\dots, z_{-2}, z_{-1}; z_0, z_1, z_2 \dots) = (&\dots, z_{-2}, z_{-1}; \alpha, z_{|\alpha|}, \dots, \\ &z_{|\alpha|+r_1}, \beta, z_{|\alpha|+r_1+|\beta|+1}, \dots) \end{aligned}$$

for some $r_1 > 0$. And then consider the point z with the kneading sequence

$$\begin{aligned} \Pi^{-1}(z) = (\dots, x_{-N_{\rho_1}-1}, x_{-N_{\rho_1}}, \dots; x_0, \dots, x_{N_{\rho_1}}, \dots, x_{N_1-1}, \alpha, z_{|\alpha|}, \dots, z_{|\alpha|+r_1}, \\ \beta, y_{-N_2+1}, \dots, y_{-N_{\rho_2}}, \dots, y_0, ..., y_{N_{\rho_2}}, y_{N_{\rho_2}+1}, \dots) \end{aligned}$$

then by construction if

$$\tilde{P} = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}\Big(\bigcup_{p \in C(\Lambda^u(t_0)\cup\Lambda^s(t_0)\cup\tilde{\Lambda}_{i_0}, \epsilon/2k)} R(p; 0)\Big)$$

we have $z \in \tilde{P} \cap B(x, \rho_1)$, and $\varphi^{N_1-1+2|\alpha|+r_1+N_2}(z) \in B(y, \rho_2)$.

Analogously we can find a $\tilde{z} \in \tilde{P} \cap B(y, \rho_2)$ and an $R \in \mathbb{N}$ such that $\varphi^R(\tilde{z}) \in B(x, \rho_1)$. So x and y belong to the same transitive component of $\tilde{P}$ and then, there exists some subhorseshoe $\tilde{\Lambda} \subset \tilde{P}$ with $\Lambda^u(t_0) \cup \Lambda^s(t_0) \subset \tilde{\Lambda}$. Moreover as $\Lambda^u(t_0) \cup \Lambda^s(t_0) \cup \tilde{\Lambda}_{i_0} \subset \Lambda_{t_0+\delta/2}$, reasoning as we did for P, we have

$$\tilde{\Lambda} \subset \tilde{P} \subset \Lambda_{k.\epsilon/2k+t_0+\delta/2} \subset \Lambda_{t_0+\epsilon}.$$

We summarize our conclusions in the following proposition

Proposition 3.3.5. *Take $\varphi \in \mathcal{U}^{**}$, $f \in \mathcal{R}_{\varphi,\Lambda}$ and some discontinuity t_0 of the map*

$$t \to L(t) = HD(L_{\varphi,f} \cap (-\infty, t))$$

such that $L(t_0) > 0$. Then, given $\epsilon > 0$ there are two subhorseshoes $\Lambda^s(t_0)$ and $\Lambda^u(t_0)$ and some $0 < \eta < \epsilon$ such that

- $\Lambda^s(t_0) \cup \Lambda^u(t_0) \subset \Lambda_{t_0-\eta}$,
- *there is no subhorseshoe $\tilde{\Lambda} \subset \Lambda_q$ with $\Lambda^s(t_0) \cup \Lambda^u(t_0) \subset \tilde{\Lambda}$ for any $q < t_0$,*
- *there exist some subhorseshoe $\tilde{\Lambda}^0 \subset \Lambda_s$ for some $s < t_0 + \epsilon$ with $\Lambda^s(t_0) \cup \Lambda^u(t_0) \subset \tilde{\Lambda}^0$.*

3.3.2 First accumulation point of Lagrange spectrum

In this short subsection, we show the existence of the first accumulation point of the Lagrange spectrum and show that it is exactly at that point where the map L begins to be positive.

Proposition 3.3.6. *Take $\varphi \in \mathcal{U}^{**}$ and $f \in \mathcal{R}_{\varphi,f}$. Then*

$$L'_{\varphi,f} = \{x : x \text{ is an accumulation point of } L_{\varphi,f}\} \neq \emptyset$$

and if $c_{\varphi,f} = \min L'_{\varphi,f}$, we have for $\epsilon > 0$

$$L(c_{\varphi,f} - \epsilon) = 0 \text{ and } L(c_{\varphi,f} + \epsilon) > 0.$$

Proof. First, by proposition 3.3.4

$$HD(L_{\varphi,f}) = HD(\ell_{\varphi,f}(\Lambda)) = HD(\Lambda) > 0,$$

then, $L_{\varphi,f}$ cannot be finite and as $L_{\varphi,f} \subset f(\Lambda)$, it must be true that $L'_{\varphi,f} \neq \emptyset$.

The first affirmation about $c_{\varphi,f}$ is clearly true because for $\epsilon > 0$, $L_{\varphi,f} \cap (-\infty, c_{\varphi,f} - \epsilon)$ is finite. On the other hand, take an injective sequence $(y_n)_{n\in\mathbb{N}} = (\ell_{\varphi,f}(x_n))_{n\in\mathbb{N}} \subset L_{\varphi,f}$ such that $\lim\limits_{n\to\infty} y_n = c_{\varphi,f}$ and consider $N \in \mathbb{N}$ big enough such that for two elements $x, y \in \Lambda$ if their kneading sequences coincide in the central block (centered at the zero position) of size $2N+1$ then $|f(x) - f(y)| < \epsilon/6$.

Take first $n_0 \in \mathbb{N}$ large so that $|\ell_{\varphi,f}(x_n) - c_{\varphi,f}| < \epsilon/6$ for $n \geq n_0$ and there are infinitely many $j \in \mathbb{N}$ such that $|f(\varphi^j(x_n)) - c_{\varphi,f}| < \epsilon/6$. Given such a pair (j, n), consider the finite sequence with $2N+1$ terms $S(j,n) = (b^{(n)}_{j-N}, b^{(n)}_{j-N+1}, \cdots, b^{(n)}_{j}, \cdots, b^{(n)}_{j+N})$ where $\Pi((b^{(n)}_j)_{j\in\mathbb{Z}}) = x_n$. There is a sequence S such that for infinitely many values of n, S appears infinitely many times as $S(j,n)$; i.e., there are $j_1(n) < j_2(n) < \cdots$ with $\lim\limits_{i\to\infty}(j_{i+1}(n) - j_i(n)) = \infty$ and $S(j_i(n), n) = S$ for all $i \geq 1$ and for all n in some infinite set $A \subset \mathbb{N}$.

Consider the sequences $\beta(i,n)$ for $i \geq 1$, $n \in A$ given by

$$\beta(i,n) = (b^{(n)}_{j_i(n)+N+1}, b^{(n)}_{j_i(n)+N+2}, \cdots, b^{(n)}_{j_{i+1}(n)+N}).$$

Taking $n_1,\ n_2 \in A$ distinct and $r = r(n_1, n_2)$ large enough such that for $j \geq r$, $f(\varphi^j(x_{n_1})) < \ell_{\varphi,f}(x_{n_1}) + \epsilon/6$ and $f(\varphi^j(x_{n_2})) < \ell_{\varphi,f}(x_{n_2}) + \epsilon/6$. There are $i_1 \geq r$ and $i_2 \geq r$ for which there is no a sequence γ such that $\beta(i_1, n_1)$ and $\beta(i_2, n_2)$ are concatenations of copies of γ, otherwise $y_{n_1} = y_{n_2}$ because for $n \in A$

$$\Pi^{-1}(x_n) = (b^{(n)}_1, \cdots b^{(n)}_{j_1(n)+N}, \beta(1,n), \beta(2,n), \cdots, \beta(m,n), \cdots).$$

This implies that, taking

$$C = \{\beta(i_1,n_1)\beta(i_2,n_2), \beta(i_2,n_2)\beta(i_1,n_1)\},$$

we have $\Sigma(C)$ is a complete subshift and for $x \in \Lambda(\Sigma(C)) = \Lambda_C$ (the subhorseshoe associated to $\Sigma(C)$) we have $\ell_{\varphi,f}(x) < c_{\varphi,f} + \epsilon$. Indeed, for every $k \in \mathbb{N}$ the kneading sequence of $\varphi^k(x)$ coincides in the central block of size $2N+1$ with the kneading sequence of $\varphi^l(x_\theta)$ where θ is either n_1 or n_2 and $l \geq r$. So

$$f(\varphi^k(x)) < f(\varphi^l(x_\theta)) + \epsilon/6 < \ell_{\varphi,f}(x_\theta) + \epsilon/3 < c_{\varphi,f} + \epsilon/2.$$

Therefore, $\ell_{\varphi,f}(\Lambda_C) \subset L_{\varphi,f} \cap (-\infty, c_{\varphi,f} + \epsilon)$ and using one more time proposition 3.3.4 we conclude

$$0 < HD(\Lambda_C) = HD(\ell_{\varphi,f}(\Lambda_C)) \leq HD(L_{\varphi,f} \cap (-\infty, c_{\varphi,f} + \epsilon)) = L(c_{\varphi,f} + \epsilon).$$

That ends the proof of the proposition. $\square$

3.3.3 Sequences of subhorseshoes

In this subsection, we suppose the existence of an infinity sequence of discontinuities of L before $c_{\varphi,f}$ and then construct arbitrary large finite sequences of subhorseshoes with some specific properties.

First, choose the neighborhood $\mathcal{U}$ of φ_0 small enough such that for some constants r_1, r_2 with $r_1/r_2 > 999/1000$ and any φ-invariant compact subset X of $\Lambda(\varphi) = \Lambda$ we have

$$r_1 D_s(X) \leq D_u(X) \leq r_2 D_s(X). \tag{3.3.6}$$

Fix $\varphi \in \mathcal{U}^{**}$, $f \in \mathcal{R}_{\varphi,\Lambda}$, $\epsilon > 0$ and suppose we have a infinite sequence of discontinuities for L with $s \geq c_{\varphi,f} + \epsilon$ for every s in the sequence. Then, as

$$L(c_{\varphi,f} + \epsilon) \leq L(s) = HD(L_{\varphi,f} \cap (-\infty, s)) \leq HD(f(\Lambda_s)) \leq HD(\Lambda_s)$$

by 3.2.3 and 3.2.4

$$c \leq D_s(\Lambda_s) \text{ and } c \leq D_u(\Lambda_s), \tag{3.3.7}$$

where $c = L(c_{\varphi,f} + \epsilon)/(\tilde{C} + 1)$.

Now, as the maps $t \to HD(K_t^u) = D_u(\Lambda_t)$ and $t \to HD(K_t^s) = D_s(\Lambda_t)$ are continuous (by proposition 3.2.2) and $D_u(\Lambda_t) = D_s(\Lambda_t) = 0$ for $t < \min(f)$ and $D_u(\Lambda_t) = D_u(\Lambda)$, $D_s(\Lambda_t) = D_s(\Lambda)$ for $t > \max(f)$. Then, they are uniformly continuous and so we can find some $\delta > 0$ such that

$$|t - \bar{t}| < \delta \text{ implies } |D_u(\Lambda_t) - D_u(\Lambda_{\bar{t}})| < 0.001c \text{ and } |D_s(\Lambda_t) - D_s(\Lambda_{\bar{t}})| < 0.001c$$

and for the sequence of discontinuities we have some accumulation point and unless pass to a sub-sequence, change the index set and discard some terms, we can suppose that $\{t_n\}$ is of one of the next two types:

- The sequence is strictly increasing $\{t_n\}_{n\geq 1}$ with $\lim\limits_{n\to\infty} t_n := t_0$ and $t_0 - t_1 < \delta$,
- The sequence is strictly increasing $\{t_n\}_{n\leq 0}$ with $\lim\limits_{n\to-\infty} t_n := t^*$ and $t_0 - t^* < \delta$.

In particular for each n

$$0.999 D_u(\Lambda_{t_0}) = D_u(\Lambda_{t_0}) - 0.001 D_u(\Lambda_{t_0}) \leq D_u(\Lambda_{t_0}) - 0.001c < D_u(\Lambda_{t_n}). \tag{3.3.8}$$

Now, in order to get the sequences of subhorseshoes, we will associate to every n a pair of subhorseshoes of Λ. In fact, the two subhorseshoes $\Lambda^s(t_n)$ and $\Lambda^u(t_n)$ are given by proposition 3.3.5 considering some $0 < \epsilon_n < \min\{0.001, (t_{n+1} - t_n)/2\}$ and they satisfy

- $\Lambda^s(t_n) \cup \Lambda^u(t_n) \subset \Lambda_{t_n-\eta_n}$ for some $0 < \eta_n < \epsilon_n$,
- there is no subhorseshoe $\tilde{\Lambda} \subset \Lambda_q$ with $\Lambda^s(t_n) \cup \Lambda^u(t_n) \subset \tilde{\Lambda}$ for any $q < t_n$,
- there exist some subhorseshoe $\tilde{\Lambda}_n \subset \Lambda_{t_n+\epsilon_n} \subset \Lambda_{(t_n+t_{n+1})/2}$ with $\Lambda^s(t_n) \cup \Lambda^u(t_n) \subset \tilde{\Lambda}_n$.

For the next, it will be useful to give the following definition

Definition 3.3.7. Given $\Lambda(1)$ and $\Lambda(2)$ subhorseshoes of Λ and $t \in \mathbb{R}$, we said that $\Lambda(1)$ *connects* with $\Lambda(2)$ or that $\Lambda(1)$ and $\Lambda(2)$ *connect* before t if there exist a subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and some $q < t$ with $\Lambda(1) \cup \Lambda(2) \subset \tilde{\Lambda} \subset \Lambda_q$.

Remark 3.3.8. With the definition given above, we have for each n that $\Lambda^s(t_n)$ doesn't connect with $\Lambda^u(t_n)$ before t_n. But they connect before t_{n+1}.

Lemma 3.3.9. *Suppose $\Lambda(1)$ and $\Lambda(2)$ are subhorseshoes of Λ and for some $x, y \in \Lambda$ we have $x \in W^u(\Lambda(1)) \cap W^s(\Lambda(2))$ and $y \in W^u(\Lambda(2)) \cap W^s(\Lambda(1))$. If for some $t \in \mathbb{R}$, it is true that*

$$\Lambda(1) \cup \Lambda(2) \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_t,$$

then for every $\epsilon > 0$, $\Lambda(1)$ and $\Lambda(2)$ connect before $t + \epsilon$.

Proof. Take a Markov partition $\mathcal{P}$ for Λ with diameter small enough such that $\max f \restriction_{\bigcup\limits_{P \in \mathcal{R}} P} < t + \epsilon$, where $\mathcal{R} = \{P \in \mathcal{P} : P \cap (\Lambda(1) \cup \Lambda(2) \cup \mathcal{O}(x) \cup \mathcal{O}(y)) \neq \emptyset\}$ and consider

$$\Lambda_{\mathcal{R}} = \bigcap_{n \in \mathbb{Z}} \varphi^{-n}(\bigcup_{P \in \mathcal{R}} P).$$

Evidently $\Lambda(1) \cup \Lambda(2) \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_{\mathcal{R}} \subset \Lambda_{t+\epsilon}$, then the lemma will be proved if we show that $\Lambda(1)$ and $\Lambda(2)$ form part of the same transitive component of $\Lambda_{\mathcal{R}}$.

Let $x_1 \in \Lambda(1)$, $x_2 \in \Lambda(2)$ and $\rho_1, \rho_2 > 0$. Take

$$\eta = \frac{1}{2} \min\{\rho_1, \rho_2, \min\{d(P, Q) : P, Q \in \mathcal{R} \text{ and } P \neq Q\}\}.$$

By the shadowing lemma there exist $0 < \delta \leq \eta$ such that every δ-pseudo orbit of Λ is η-shadowed by the orbit of some element of Λ.

On the other hand, as $\varphi \restriction_{\Lambda(1)}$ is transitive and $x \in W^u(\Lambda(1))$ there exist $y_1 \in \Lambda(1) \cap B(x_1, \delta)$ and $N_1, M_1 \in \mathbb{N}$ such that $d(\varphi^{M_1}(y_1), \varphi^{-N_1}(x)) < \delta$ and analogously as $\varphi\restriction_{\Lambda(2)}$ is transitive and $x \in W^s(\Lambda(2))$ there exist $y_2 \in \Lambda(2)$ and $N_2, M_2 \in \mathbb{N}$ such that $d(\varphi^{N_2}(x), y_2) < \delta$ and $d(x_2, \varphi^{M_2}(y_2)) < \delta$; define then the δ-pseudo orbit:

$$\dots, \varphi^{-1}(y_1); y_1, \varphi(y_1), \dots, \varphi^{M_1-1}(y_1), \varphi^{-N_1}(x), \dots, \varphi^{N_2-1}(x), y_2, \varphi(y_2), \dots$$

Then there exists $w \in \Lambda$ that η-shadowed that orbit. Moreover as the δ-pseudo orbit have all its terms in $\bigcup_{P\in\mathcal{R}} P$ and $\eta \leq \frac{1}{2}\min\{d(P,Q) : P, Q \in \mathcal{R} \text{ and } P \neq Q\}$ we have also $\mathcal{O}(w) \subset \bigcup_{P\in\mathcal{R}} P$; that is, $w \in \Lambda_{\mathcal{R}}$ and furthermore

$$w \in B(x_1, \rho_1) \quad \text{and} \quad \varphi^{M_1+N_1-1+N_2+M_2}(w) \in B(x_2, \rho_2).$$

The proof that there exists $w \in B(x_2, \rho_2)$ and $M \in \mathbb{N}$ such that $\varphi^M(w) \in B(x_1, \rho_1)$ is analog. □

Corollary 3.3.10. *Suppose $\Lambda(1)$ and $\Lambda(2)$ are subhorseshoes of Λ with $\Lambda(1)\cup\Lambda(2) \subset \Lambda_t$ for some $t \in \mathbb{R}$. If $\Lambda(1)\cap\Lambda(2) \neq \emptyset$, then for every $\epsilon > 0$, $\Lambda(1)$ and $\Lambda(2)$ connects before $t+\epsilon$.*

Proof. If $\Lambda(1) \cap \Lambda(2) \neq \emptyset$, then every $w \in \Lambda(1) \cap \Lambda(2)$ satisfies $w \in W^u(\Lambda(1)) \cap W^s(\Lambda(2))$ and $w \in W^u(\Lambda(2)) \cap W^s(\Lambda(1))$ and then we have the conclusion. □

Corollary 3.3.11. *Let $\Lambda(1)$, $\Lambda(2)$ and $\Lambda(3)$ subhorseshoes of Λ and $t \in \mathbb{R}$. If $\Lambda(1)$ connects with $\Lambda(2)$ before t and $\Lambda(2)$ connects with $\Lambda(3)$ before t. Then also $\Lambda(1)$ connects with $\Lambda(3)$ before t.*

Proof. By hypothesis we have two subhorseshoes $\Lambda^{1,2}$ and $\Lambda^{2,3}$ and $q_1, q_2 < t$ with

$$\Lambda(1) \cup \Lambda(2) \subset \Lambda^{1,2} \subset \Lambda_{q_1} \text{ and } \Lambda(2) \cup \Lambda(3) \subset \Lambda^{2,3} \subset \Lambda_{q_2}.$$

Applying corollary 3.3.10 to $\Lambda^{1,2}$ and $\Lambda^{2,3}$, with $\tilde{t} = \max\{q_1, q_2\}$ and $\epsilon = (t-\tilde{t})/2$ we have the result. □

We are ready to prove the next proposition

Proposition 3.3.12. *We can take $\theta \in \{s, u\}$ such that given $N \in \mathbb{N}$ arbitrary, there exists a sequence $n_1 < n_2 < \dots < n_N$ of elements of $\mathcal{I}$ (where $\mathcal{I}$ is the index set of the sequence $\{t_n\}$) such that for $i, j \in \{1, \dots, N\}$ with $i \neq j$, $\Lambda^\theta(t_{n_i})$ and $\Lambda^\theta(t_{n_j})$ doesn't connect before* $\max\{t_{n_i}, t_{n_j}\}$.

Proof. We said that a sequence $n_1 < n_2 < \dots < n_r$ of elements of $\mathcal{I}$ is a *r-chain* if $\Lambda^s(t_{n_i})$ connects with $\Lambda^s(t_{n_{i+1}})$ before $t_{n_{i+1}}$ for $i = 1, \dots r-1$. Then we have two cases:

- There exists some $R \in \mathbb{N}$ such that there is no r-chain for $r > R$.
- There are r-chains with r arbitrarily big.

We do the proof when the index set of the sequence is $\mathcal{I} = \{n \in \mathbb{Z} : n \geq 1\}$, and the other case follows similarly.

In the first case take a maximal r_1-chain beginning with 1; that is, a r_1-chain $1 = n_1 < n_2 < ... < n_{r_1}$ such that for any $n > n_{r_1}$, $1 = n_1 < n_2 < ... < n_{r_1} < n$ is not a $(r_1 + 1)$-chain and then $\Lambda^s(t_{n_{r_1}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n. Next take a maximal r_2-chain beginning with $n_{r_1} + 1$: $n_{r_1} + 1 = n_1^{(r_1)} < n_2^{(r_1)} < \cdots < n_{r_2}^{(r_1)}$ then as before for $n_{r_2}^{(r_1)} < n$, $\Lambda^s(t_{n_{r_2}^{(r_1)}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n. Now consider a maximal r_3-chain beginning with $n_{r_2}^{(r_1)} + 1$: $n_{r_2}^{(r_1)} + 1 = n_1^{(r_1,r_2)} < n_2^{(r_1,r_2)} < \cdots < n_{r_3}^{(r_1,r_2)}$ then for $n_{r_3}^{(r_1,r_2)} < n$, $\Lambda^s(t_{n_{r_3}^{(r_1,r_2)}})$ doesn't connect with $\Lambda^s(t_n)$ before t_n.

Continuing in this way we can construct inductively an increasing sequence

$$\{\tilde{n}_k\}_{k \geq 2} = \{n_{r_k}^{(r_1,r_2,\ldots,r_{k-1})}\}_{k \geq 2}$$

such that for $k_1, k_2 \geq 2$ with $k_1 \neq k_2$, $\Lambda^s(t_{\tilde{n}_{k_1}})$ and $\Lambda^s(t_{\tilde{n}_{k_2}})$ doesn't connect before $\max\{t_{\tilde{n}_{k_1}}, t_{\tilde{n}_{k_2}}\}$.

On the other hand, in the second case take $r \in \mathbb{N}$ arbitrarily big and $n_1 < n_2 < ... < n_r$ some r-chain, then we affirm that for $i, j \in \{1, ..., r\}$ with $i \neq j$ $\Lambda^u(t_{n_i})$ and $\Lambda^u(t_{n_j})$ doesn't connect before $\max\{t_{n_i}, t_{n_j}\}$. In other case if for some $i_0, j_0 \in \{1, ..., r\}$ with $i_0 < j_0$, $\Lambda^u(t_{n_{i_0}})$ and $\Lambda^u(t_{n_{j_0}})$ connect before $t_{n_{j_0}}$ then as by corollary 3.3.11 $\Lambda^s(t_{n_{j_0}})$ connect with $\Lambda^s(t_{n_{i_0}})$ before $t_{n_{j_0}}$ and as also $\Lambda^s(t_{n_{i_0}})$ connects with $\Lambda^u(t_{n_{i_0}})$ before $t_{n_{i_0}+1}$ (and then before $t_{n_{j_0}}$). Applying two times more that corollary we have that $\Lambda^s(t_{n_{j_0}})$ connect with $\Lambda^u(t_{n_{j_0}})$ before $t_{n_{j_0}}$ that is a contradiction.

From this follows the result. □

Without loss of generality, we will suppose that in the last proposition $\theta = u$ (for $\theta = s$ the argument is similar) and call $\Lambda^u(t_n) = \Lambda^n$.

3.3.4 Connection of subhorseshoes

In this subsection, we associate to every term of the sequence $\{\Lambda^n\}_{n \in \mathcal{I}}$ a periodic orbit with the property that if Λ^n and Λ^m are associated with the same periodic orbit then they connect before $\max\{t_n, t_m\}$.

In order to do that, given some n, remember the construction of Λ^n given by proposition 3.3.5. A close inspection of the proof of that proposition shows that for some maximal invariant set, said M^n, that contains Λ_{t_n} we took the subhorseshoe with maximal Hausdorff dimension $\Lambda_0^n \subset M^n$ and then applied proposition 3.3.1 in order to obtain the subhorseshoe Λ^n with

$$D_u(\Lambda^n) > (1-\epsilon_n/2k)D_u(\Lambda_0^n) > (1-\epsilon_n)D_u(\Lambda_0^n) > 0.999D_u(\Lambda_0^n). \tag{3.3.9}$$

Next, if $D_u(M^n) = D_u(\Lambda_2^n)$ where $\Lambda_2^n \subset M^n$ is a subhorseshoe of Λ, then as Λ_0^n has maximal dimension, it follows that either $D_u(\Lambda_2^n) \leq D_u(\Lambda_0^n)$ or $D_s(\Lambda_2^n) \leq D_s(\Lambda_0^n)$. In the first case

$$D_u(\Lambda_{t_n}) \leq D_u(M^n) = D_u(\Lambda_2^n) \leq D_u(\Lambda_0^n) \leq \frac{r_2}{r_1} D_u(\Lambda_0^n)$$

and in the second, 3.3.6 let us conclude that

$$D_u(\Lambda_{t_n}) \leq D_u(M^n) = D_u(\Lambda_2^n) \leq r_2 D_s(\Lambda_2^n) \leq r_2 D_s(\Lambda_0^n) \leq \frac{r_2}{r_1} D_u(\Lambda_0^n)$$

that is,

$$D_u(\Lambda_{t_n}) \leq \frac{r_2}{r_1} D_u(\Lambda_0^n). \tag{3.3.10}$$

Now, take r_0 big enough such that $2^{2020} < N_u(\Lambda_{t_0}, r_0)$ and

$$\frac{\log N_u(\Lambda_{t_0}, r_0)}{r_0 - c_1} < 1.001 D_u(\Lambda_{t_0}). \tag{3.3.11}$$

We set $\mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$, $N_0 = N_u(\Lambda_{t_0}, r_0)$ and for $n, M \in \mathbb{N}$ define the set

$$\mathcal{B}_M(\Lambda^n) := \{\beta = \beta_1 \dots \beta_M : \beta_j \in \mathcal{B}_0 \ \ \forall\, 1 \leq j \leq M \ \text{ and } \ \Pi^u(\Lambda^n) \cap I^u(\beta) \neq \emptyset\}.$$

Before continuing, we introduce some notation. Consider $\beta = \beta_{k_1}\beta_{k_2}...\beta_{k_\ell} = a_1...a_p \in \mathcal{A}^p$, $\beta_{k_i} \in \mathcal{B}_0$, $1 \leq i \leq \ell$. We say that $n \in \{1, ..., p\}$ is the n-th position of β. If $\beta_{k_i} \in \mathcal{A}^{n_{k_i}}$ we write $|\beta_{k_i}| = n_{k_i}$ for its length and $P(\beta_{k_i}) = \{1, 2, ..., n_{k_i}\}$ for its set of positions as a word in the alphabet $\mathcal{A}$ and given $s \in P(\beta_{k_i})$ we call $P(\beta, k_i; s) = n_{k_1} + ... + n_{k_{i-1}} + s$ the position in β of the position s of β_{k_i}.

Recall that the sizes of the intervals $I^u(\alpha)$ behave essentially submultiplicatively due the bounded distortion property of g_u (see equation 3.2.1) so that, one has

$$|I^u(\beta)| \leq \exp(-M(r_0 - c_1))$$

for any $\beta \in \mathcal{B}_M(\Lambda^n)$, and thus, $\{I^u(\beta) : \beta \in \mathcal{B}_M(\Lambda^n)\}$ is a covering of $\Pi^u(\Lambda^n)$ by intervals of sizes $\leq \exp(-M(r_0 - c_1))$. In particular for $M(\Lambda^n) = M_n$ big enough

$$\begin{aligned}
\frac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{\log N_0^{M_n}} &= \frac{\dfrac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{-\log\exp(-M_n(r_0-c_1))}}{\dfrac{M_n\log N_0}{M_n(r_0-c_1)}} \\
&\geq \frac{\dfrac{\log|\mathcal{B}_{M_n}(\Lambda^n)|}{-\log\exp(-M_n(r_0-c_1))}}{1.001 D_u(\Lambda_{t_0})} \quad \text{(by equation 3.3.11)} \\
&\geq \frac{0.999 D_u(\Lambda^n)}{1.001 D_u(\Lambda_{t_0})} \quad (M_n \text{ is big}) \\
&\geq \frac{0.999\cdot 0.999 D_u(\Lambda_0^n)}{1.001 D_u(\Lambda_{t_0})} \quad \text{(by equation 3.3.9)} \\
&\geq \frac{r_1}{r_2}\frac{0.999\cdot 0.999 D_u(\Lambda_{t_n})}{1.001 D_u(\Lambda_{t_0})} \quad \text{(by equation 3.3.10)} \\
&\geq \frac{r_1}{r_2}\frac{0.999\cdot 0.999\cdot 0.999}{1.001} \quad \text{(by equation 3.3.8)} \\
&> 0.999^4/1.001 \\
&> 991/1000.
\end{aligned}$$

Then we have proved the next result

Lemma 3.3.13. *Given $n\in\mathbb{N}$ and M_n big enough*

$$|\mathcal{B}_{M_n}(\Lambda^n)| \geq N_0^{991/1000 M_n}.$$

Remember $f\in\mathcal{R}^1_{\varphi,\Lambda}$ where $\mathcal{R}^1_{\varphi,\Lambda}$ was defined in Section 3.2 above. Then, by definition, we can refine the initial Markov partition $\{R_a\}_{a\in\mathcal{A}}$ (if necessary) so that the restriction of f to each of the intervals $\{i^s_a\}\times I^u_a$, $a\in\mathcal{A}$, is monotone (i.e., strictly increasing or decreasing), and, furthermore, for some constant $c_6=c_6(\varphi,f)>0$, the following estimates hold:

$$\begin{aligned}
|f(\underline{\theta}^{(1)}a_1\dots a_n a_{n+1}\underline{\theta}^{(3)}) - f(\underline{\theta}^{(1)}a_1\dots a_n a'_{n+1}\underline{\theta}^{(4)})| &> c_6|I^u(a_1\dots a_n)|, \qquad (3.3.12)\\
|f(\underline{\theta}^{(1)}a_{m+1}a_m\dots a_1\underline{\theta}^{(3)}) - f(\underline{\theta}^{(2)}a'_{m+1}a_m\dots a_1\underline{\theta}^{(3)})| &> c_6|I^s(a_m\dots a_1)|
\end{aligned}$$

whenever $a_{n+1}\neq a'_{n+1}$, $a_{m+1}\neq a'_{m+1}$ and $\underline{\theta}^{(1)},\underline{\theta}^{(2)}\in\mathcal{A}^{\mathbb{Z}^-}$, $\underline{\theta}^{(3)},\underline{\theta}^{(4)}\in\mathcal{A}^{\mathbb{N}}$ are admissible.

Moreover, we observe that, since f is Lipschitz (actually $f \in C^2$), there exists $c_7 = c_7(\varphi, f) > 0$ such that one also has the following estimates:

$$|f(\underline{\theta}^{(1)}a_1 \dots a_n a_{n+1}\underline{\theta}^{(3)}) - f(\underline{\theta}^{(1)}a_1 \dots a_n a'_{n+1}\underline{\theta}^{(4)})| < c_7|I^u(a_1 \dots a_n)|, \quad (3.3.13)$$
$$|f(\underline{\theta}^{(1)}a_{m+1}a_m \dots a_1\underline{\theta}^{(3)}) - f(\underline{\theta}^{(2)}a'_{m+1}a_m \dots a_1\underline{\theta}^{(3)})| < c_7|I^s(a_1 \dots a_m)|$$

whenever $a_{n+1} \neq a'_{n+1}$, $a_{m+1} \neq a'_{m+1}$ and $\underline{\theta}^{(1)}, \underline{\theta}^{(2)} \in \mathcal{A}^{\mathbb{Z}^-}$, $\underline{\theta}^{(3)}, \underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$ are admissible.

Next, we give a definition

Definition 3.3.14. Given $n \in \mathcal{I}$, $M \in \mathbb{N}$ and $\beta = \beta_1 \dots \beta_M \in \mathcal{B}_M(\Lambda^n)$ with $\beta_i \in \mathcal{B}_0$ for all $1 \leq i \leq M$, we say that $j \in \{1, \dots, M\}$ is a *M-right-good position* of β if there are two elements of $\mathcal{B}_M(\Lambda^n)$

$$\beta^{(p)} = \beta_1 \dots \beta_{j-1}\beta_j^{(p)} \dots \beta_M^{(p)}, \quad p = 1, 2$$

with $\beta_i^{(p)} \in \mathcal{B}_0$ for all $j \leq i \leq M$, $p = 1, 2$ and such that $\sup I^u(\beta_j^{(1)}) < \inf I^u(\beta_j) \leq \sup I^u(\beta_j) < \inf I^u(\beta_j^{(2)})$, i.e., the interval $I^u(\beta_j)$ is located between $I^u(\beta_j^{(1)})$ and $I^u(\beta_j^{(2)})$.

Similarly, we say that $j \in \{1, \dots, M\}$ is a *M-left-good position* of β if there are two elements of $\mathcal{B}_M(\Lambda^n)$

$$\beta^{(p)} = \beta_1^{(p)} \dots \beta_j^{(p)}\beta_{j+1} \dots \beta_M, \quad p = 3, 4$$

with $\beta_i^{(p)} \in \mathcal{B}_0$ for all $1 \leq i \leq j$, $p = 3, 4$ such that $\sup I^s((\beta_j^{(3)})^T) < \inf I^s(\beta_j^T) \leq \sup I^s(\beta_j^T) < \inf I^s((\beta_j^{(4)})^T)$, i.e., the interval $I^s(\beta_j^T)$ is located between $I^s((\beta_j^{(3)})^T)$ and $I^s((\beta_j^{(4)})^T)$.

Finally, we say that $j \in \{1, \dots, M\}$ is a *M-good position* of β if it is both a M-right-good and a M-left-good position of β.

The bounded distortion property (equation 3.2.1) let us fix $L \in \mathbb{N}$ big enough such that for $\beta_1\beta_2 \dots \beta_L$ and $\beta_{L+1}\beta_{L+2}$ admissible with $\beta_1, \beta_2, \dots, \beta_L, \beta_{L+1}, \beta_{L+2} \in \mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$

$$|I^u(\beta_1\beta_2 \dots \beta_L)| \leq |I^s((\beta_{L+1}\beta_{L+2})^T)|$$

and

$$|I^s((\beta_1\beta_2 \dots \beta_L)^T)| \leq |I^u(\beta_{L+1}\beta_{L+2})|.$$

Set $k := 8LN_0^2$ (observe that k does not depend on n). The next lemma says that most positions of some word of $\mathcal{B}_{5N_nk}(\Lambda^n)$ are $5N_nk$-good.

Lemma 3.3.15. *For N_n big enough, there exists $\beta_n \in \mathcal{B}_{5N_nk}(\Lambda^n)$ such that the number of $5N_nk$-good positions of β_n is greater or equal than $49N_nk/10$.*

Proof. Let us first estimate the cardinality of the subset of $\mathcal{B}_{5N_nk}(\Lambda^n)$ consisting of words β such that at least $N_nk/20$ positions are not $5N_nk$-right-good. First, we notice that there are at most 2^{5N_nk} choices for the set of $m \geq N_nk/20$, $5N_nk$-right-bad (i.e., not $5N_nk$-right-good) positions. Secondly, once this set of $5N_nk$-right-bad positions is fixed:

- if j is a $5N_nk$-right-bad position and $\beta_1, \ldots, \beta_{j-1} \in \mathcal{B}_0$ were already chosen, then we see that there are at most two possibilities for $\beta_j \in \mathcal{B}_0$ (namely, the choices leading to the leftmost and rightmost subintervals of $I^u(\beta_1 \ldots \beta_{j-1})$ of the form $I^u(\beta_1 \ldots \beta_{5N_nk})$ intersecting $\pi^u(\Lambda^n)$),

- if j is not a $5N_nk$-right-bad position, then there are at most N_0 choices of β_j.

In particular, once a set of $m \geq N_nk/20$ $5N_nk$-right-bad positions is fixed, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda_n)$ with this set of m, $5N_nk$-right-bad positions is at most

$$2^m \cdot N_0^{5N_nk-m} \leq 2^{N_nk/20} \cdot N_0^{99N_nk/20}.$$

Therefore, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda^n)$ with at least $N_nk/20$, $5N_nk$-right-bad positions is

$$\leq 2^{5N_nk} \cdot 2^{N_nk/20} \cdot N_0^{99N_nk/20} = 2^{101N_nk/20} \cdot N_0^{99N_nk/20}.$$

Analogously, the quantity of words in $\mathcal{B}_{5N_nk}(\Lambda^n)$ with at least $N_nk/20$, $5N_nk$-left-bad positions is also $\leq 2^{101N_nk/20} \cdot N_0^{99N_nk/20}$.

It follows that the set of words $\beta \in \mathcal{B}_{5N_nk}(\Lambda_n)$ with at least $N_nk/10$, $5N_nk$-bad (i.e., not $5N_nk$-good) positions is $\leq 2.2^{101N_nk/20} \cdot N_0^{99N_nk/20}$.

Since $|\mathcal{B}_{5N_nk}(\Lambda^n)| > N_0^{991N_nk/200}$ (by lemma 3.3.13) and $2^{1+101N_nk/20} \cdot N_0^{99N_nk/20} < N_0^{991N_nk/200}$ (from our choices of r_0, N_0 large), we have that there exists some $\beta_n \in \mathcal{B}_{5N_nk}(\Lambda^n)$ with less than $N_nk/10$, $5N_nk$-bad positions. That is, with at least $5N_nk - N_nk/10 = 49N_nk/10$ good positions. □

Given $n \in \mathcal{I}$ take N_n big enough as in the lemma 3.3.15 and such that for two elements $x, y \in \Lambda$ if their kneading sequences coincide in the central block (centered at the zero position) of size $2N_n + 1$ then $|f(x) - f(y)| < \eta_n/2$.

The next proposition shows that the notion of good positions allows us to have some control over the values that f takes in some rectangles.

Proposition 3.3.16. *If $\beta_n = \beta_1^n\beta_2^n\dots\beta_{5N_nk}^n$ with $\beta_r^n \in \mathcal{B}_0$ for $i = 1,\dots,5N_nk$ is as in the latest lemma and for some $1 < i < j < 5N_nk$, the positions $i-1, i, j, j+1$ are $5N_nk$-good positions of β_n and $j - i \geq L$. Then for each $i \leq s \leq j$ and $\bar{n} \in P(\beta_s^n)$ if $\eta = \beta_{i-1}^n\beta_i^n\dots\beta_j^n\beta_{j+1}^n$ and $x \in R(\eta; P(\eta, s; \bar{n})) \cap \Lambda$ we have $f(x) < t_n$.*

Proof. By hypothesis, we have

$$\sup I^s((\beta_i')^T) < \inf I^s((\beta_i^n)^T) \leq \sup I^s((\beta_i^n)^T) < \inf I^s((\beta_i'')^T)$$

and

$$\sup I^u(\beta_j') < \inf I^u(\beta_j^n) \leq \sup I^u(\beta_j^n) < \inf I^u(\beta_j''),$$

for some words $\beta_i', \beta_i'', \beta_j', \beta_j'' \in \mathcal{B}_0$ verifying

$$I^u(\beta_i'\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n) \cap \pi^u(\Lambda^n) \neq \emptyset, \quad I^u(\beta_i''\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n) \cap \pi^u(\Lambda^n) \neq \emptyset,$$

$$I^u(\beta_{i-1}^n\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j') \cap \pi^u(\Lambda^n) \neq \emptyset, \quad I^u(\beta_{i-1}^n\beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j'') \cap \pi^u(\Lambda^n) \neq \emptyset$$

In order to prove the result, we consider sequences of the form

$$\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)},$$

where $\underline{\theta}^{(2)} \in \mathcal{A}^{\mathbb{N}}$ and $\underline{\theta}^{(1)} \in \mathcal{A}^{\mathbb{Z}^-}$ and the symbol ; serves to mark the location of the entry of index 0 of the bi-infinite sequence $\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}$.

In this notation, our task is equivalent to show that

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})) < t_n \tag{3.3.14}$$

for all $0 \leq \ell \leq m_1 + m + m_2 - 1$ where $\beta_i^n := a_1\dots a_{m_1}$, $\beta_{i+1}^n\dots\beta_{j-1}^n := b_1\dots b_m$ and $\beta_j^n := d_1\dots d_{m_2}$.

We consider two regimes for $0 \leq \ell \leq m_1 + m + m_2 - 1$:

I) $m_1 \leq \ell \leq m_1 + m - 1$,

II) $0 \leq \ell \leq m_1 - 1$ or $m_1 + m \leq \ell \leq m_1 + m + m_2 - 1$.

In case I), we write $\ell = m_1 - 1 + r$ so that

$$\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n\beta_{i+1}^n\dots\beta_{j-1}^n\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}) = \underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1}; b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)} \tag{3.3.15}$$

We have two possibilities:

I.a) $|I^s((\beta_i^n b_1\dots b_{r-1})^T)| \leq |I^u(b_r\dots b_m\beta_j^n)|$

I.b) $|I^u(b_r\dots b_m\beta_j^n)|\leq |I^s((\beta_i^n b_1\dots b_{r-1})^T)|$

In case I.a), we choose $\beta_j^*\in\{\beta_j',\beta_j''\}$ such that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)}) < f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})$$

for any $\underline{\theta}^{(4)}\in\mathcal{A}^{\mathbb{N}}$ (because of the local monotonicity of f along stable and unstable manifolds). By (3.3.12), it follows that

$$\begin{aligned}&f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})+c_6|I^u(b_r\dots b_m\beta_j^n)|\\&< f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})\end{aligned}$$

for some $c_6>0$. On the other hand, by (3.3.13), we also know that, for some $c_7>0$, the function f obeys the Lipschitz estimate

$$\begin{aligned}&|f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})-f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})|\\&< c_7|I^s((\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1})^T)|\end{aligned}$$

for any $\underline{\theta}^{(3)}\in\mathcal{A}^{\mathbb{Z}^-}$. From these estimates, we obtain that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})+c_6|I^u(b_r\dots b_m\beta_j^n)|<$$

$$f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})+c_7|I^s((\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1})^T)|$$

for any $\underline{\theta}^{(3)}\in\mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}^{(4)}\in\mathcal{A}^{\mathbb{N}}$. Now, we observe that the usual bounded distortion property implies that

$$|I^s((\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1})^T)|\leq e^{c_1}|I^s((\beta_{i-1}^n)^T)|\cdot|I^s((\beta_i^n b_1\dots b_{r-1})^T)|$$

By plugging this information into the previous estimate, we have

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})+c_6|I^u(b_r\dots b_m\beta_j^n)|<$$

$$f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})+c_7e^{c_1}|I^s((\beta_{i-1}^n)^T)|\cdot|I^s((\beta_i^n b_1\dots b_{r-1})^T)|.$$

Since we are dealing with case I.a), i.e., $|I^s((\beta_i^n b_1\dots b_{r-1})^T)|\leq|I^u(b_r\dots b_m\beta_j^n)|$, we deduce that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^n\beta_{j+1}^n\underline{\theta}^{(2)})<$$

$$f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1\dots b_{r-1};b_r\dots b_m\beta_j^*\underline{\theta}^{(4)})-(c_6-c_7e^{c_1}|I^s((\beta_{i-1}^n)^T)|)\cdot|I^u(b_r\dots b_m\beta_j^n)|.$$

Next, we note that the usual bounded distortion property ensures that $c_7 e^{c_1} \cdot |I^s((\beta_{i-1}^n)^T)| < c_6/2$ if $r_0 \in \mathbb{N}$ is sufficiently large. In particular, we have that

$$\begin{aligned} f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) < \\ f(\underline{\theta}^{(3)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}^{(4)}) - (c_6/2) \cdot |I^u(b_r \dots b_m \beta_j^n)| \end{aligned} \tag{3.3.16}$$

for any $\underline{\theta}^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}^{(4)} \in \mathcal{A}^{\mathbb{N}}$. Now, we recall that $\beta_j^* \in \{\beta_j', \beta_j''\}$, so that

$$I^u(\beta_{i-1}^n \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^*) \cap \pi^u(\Lambda^n) \neq \emptyset.$$

By definition, this implies that there are $\underline{\theta}_*^{(3)} \in \mathcal{A}^{\mathbb{Z}^-}$ and $\underline{\theta}_*^{(4)} \in \mathcal{A}^{\mathbb{N}}$ with

$$\underline{\theta}_*^{(3)}; \beta_{i-1}^n \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^* \underline{\theta}_*^{(4)} \in \Sigma_{t_n},$$

and, *a fortiori*,

$$f(\sigma^{m_2+\ell}(\underline{\theta}_*^{(3)}; \beta_{i-1}^n \beta_i^n b_1 \dots b_m \beta_j^* \underline{\theta}_*^{(4)})) = f(\underline{\theta}_*^{(3)} \beta_{i-1}^n \beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^* \underline{\theta}_*^{(4)})) \leq t_n.$$

Here, we used (3.3.15) for the first equality. Combining this with (3.3.16), we see that

$$f(\underline{\theta}^{(1)}\beta_{i-1}^n\beta_i^n b_1 \dots b_{r-1}; b_r \dots b_m \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) < t_n - (c_6/2) \cdot |I^u(b_r \dots b_m \beta_j^n)|.$$

Therefore, in case I.a), we conclude that

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)})) < t_n \tag{3.3.17}$$

The case I.b) is dealt with in a symmetric manner: in fact, by mimicking the argument above for case I.a), one gets that in case I.b)

$$f(\sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)})) < t_n - (c_6/2) \cdot |I^s((\beta_i^n b_1 \dots b_{r-1})^T)| < t_n. \tag{3.3.18}$$

Finally, the case II) is also similar with the case I.a). We write

$$\begin{aligned} \sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) = \\ \underline{\theta}^{(1)}\beta_{i-1}^n a_1 \dots a_\ell; a_{\ell+1} \dots a_{m_1} \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)} \end{aligned} \tag{3.3.19}$$

for $0 \leq \ell \leq m_1 - 1$, and

$$\begin{aligned} \sigma^\ell(\underline{\theta}^{(1)}\beta_{i-1}^n; \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n \beta_j^n \beta_{j+1}^n \underline{\theta}^{(2)}) = \\ \underline{\theta}^{(1)}\beta_{i-1}^n \beta_i^n \beta_{i+1}^n \dots \beta_{j-1}^n d_1 \dots d_{\ell-m_1-m}; d_{\ell-m_1-m+1} \dots d_{m_2} \beta_{j+1}^n \underline{\theta}^{(2)} \end{aligned} \tag{3.3.20}$$

for $m_1 + m \leq \ell \leq m_1 + m + m_2 - 1$.

Since $j - i \geq L$ and $\beta^n_{i-1}, \beta^n_i, \dots, \beta^n_{j-1}, \beta^n_j \in \mathcal{B}_0 = \mathcal{C}_u(\Lambda_{t_0}, r_0)$, it follows from our choice of L that

$$|I^u(a_{\ell+1} \dots a_{m_1} \beta^n_{i+1} \dots \beta^n_{j-1} \beta^n_j)| \leq |I^s((\beta^n_{i-1} a_1 \dots a_\ell)^T)|$$

for $0 \leq \ell \leq m_1 - 1$, and

$$|I^s((\beta^n_i \beta^n_{i+1} \dots \beta^n_{j-1} d_1 \dots d_{\ell-m_1-m})^T)| \leq |I^u(d_{\ell-m_1-m+1} \dots d_{m_2} \beta^n_{j+1})|$$

for $m_1 + m \leq \ell \leq m_1 + m + m_2 - 1$. By plugging this into the argument for case I.a), one deduces that

$$f(\sigma^\ell(\underline{\theta}^{(1)} \beta^n_{i-1}; \beta^n_i \beta^n_{i+1} \dots \beta^n_{j-1} \beta^n_j \beta^n_{j+1} \underline{\theta}^{(2)})) < t_n - (c_6/2) \cdot |I^s((\beta^n_{i-1} a_1 \dots a_\ell)^T)| < t_n \tag{3.3.21}$$

for $0 \leq \ell \leq m_1 - 1$, and

$$f(\sigma^\ell(\underline{\theta}^{(1)} \beta^n_{i-1}; \beta^n_i \beta^n_{i+1} \dots \beta^n_{j-1} \beta^n_j \beta^n_{j+1} \underline{\theta}^{(2)})) < t_n - (c_6/2) \cdot |I^u(d_{\ell-m_1-m+1} \dots d_{m_2} \beta^n_{j+1})| < t_n \tag{3.3.22}$$

for $m_1 + m \leq \ell \leq m_1 + m + m_2 - 1$

In summary, from (3.3.17), (3.3.18), (3.3.21), and (3.3.22) we deduce that (3.3.14) holds, i.e.,

$$f(\sigma^\ell(\underline{\theta}^{(1)} \beta^n_{i-1}; \beta^n_i \beta^n_{i+1} \dots \beta^n_{j-1} \beta^n_j \beta^n_{j+1} \underline{\theta}^{(2)})) < t_n$$

for $0 \leq \ell \leq m_1 + m + m_2 - 1$. As we wanted to see. $\square$

Consider $\beta_n = \beta^n_1 \beta^n_2 \dots \beta^n_{5N_n k}$ and divide its position set $I = \{1, 2, \dots, 5N_n k\}$ in positions packages of size $N_n k$. In the central package $I^* = \{2N_n k + 1, 2N_n k + 2, \dots, 3N_n k\}$, the number of $5N_n k$-bad positions is less than $5N_n k - 49N_n k/10 = N_n k/10$ and then subdividing that package now in N_n package of positions of size k we can find some package of size k with less than $k/10$, $5N_n k$-bad positions, said

$$I^{**} = \{2N_n k + sk + 1, 2N_n k + sk + 2, \dots, 2N_n k + (s+1)k\} \text{ for some } 0 \leq s < N_n.$$

Then we can find $\lceil 2k/5 \rceil$ positions

$$2N_n k + sk + 1 \leq i_1 \leq \dots \leq i_{\lceil 2k/5 \rceil} \leq 2N_n k + (s+1)k$$

such that $i_{r+1} \geq i_r + 2$ for all $1 \leq r < \lceil 2k/5 \rceil$ and the positions

$$i_1, i_1 + 1, \dots, i_{\lceil 2k/5 \rceil}, i_{\lceil 2k/5 \rceil} + 1$$

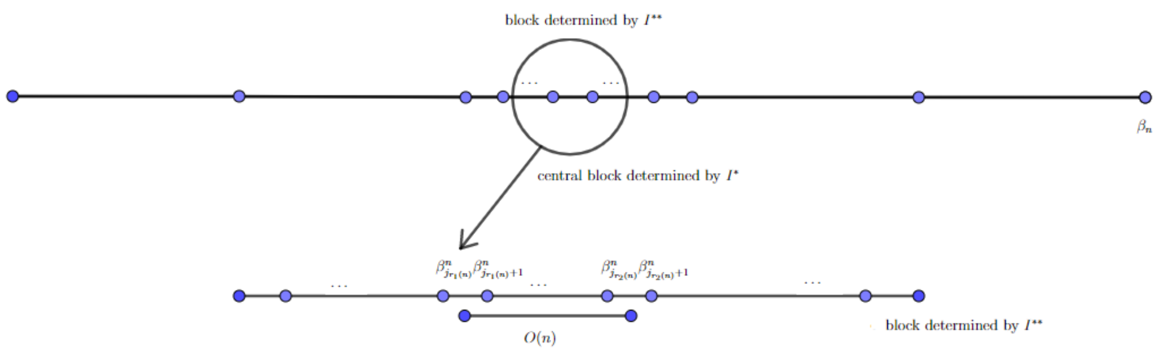

Figure 3.1: Construction of $O(n)$.

are $5N_nk$-good.

Since we took $k = 8LN_0^2$, it makes sense to set

$$j_r := i_{rL} \quad \text{for } 1 \leq r \leq 3N_0^2$$

because $3LN_0^2 < (16/5)LN_0^2 = 2k/5$. In this way, we obtain positions such that

$$j_{r+1} - j_r \geq 2L \quad \text{for } 1 \leq r \leq 3N_0^2$$

and $j_1, j_1+1, \ldots, j_{3N_0^2}, j_{3N_0^2}+1$ are $5N_nk$-good positions.

Since for $1 \leq r \leq 3N_0^2$ the number of possibilities for $(\beta^n_{j_r}, \beta^n_{j_r+1})$ is at most N_0^2, we conclude that for some different $1 \leq r_1(n), r_2(n) \leq 3N_0^2$ we have

$$(\beta^n_{j_{r_1(n)}}, \beta^n_{j_{r_1(n)}+1}) = (\beta^n_{j_{r_2(n)}}, \beta^n_{j_{r_2(n)}+1})$$

then, we can define the following map:

$$\begin{aligned} O : \mathcal{I} &\to \bigcup_{j=2}^{k-1} \mathcal{B}_0^j \\ n &\to \beta^n_{j_{r_1(n)}+1}\beta^n_{j_{r_1(n)}+2}\ldots\beta^n_{j_{r_2(n)}} \end{aligned}$$

Next, we see that if for some $m, n \in \mathcal{I}$ we have $O(m) = O(n)$ then it is possible to go from Λ^m to Λ^n without leaving $\Lambda_{\max\{t_n,t_m\}}$ and staying arbitrarily close of the orbit of the periodic point $p := \Pi^{-1}(\overline{O(m)})$ for times arbitrarily big.

Proposition 3.3.17. *Take $m, n \in \mathcal{I}$ such that $O(m) = O(n)$. Then given $N \in \mathbb{N}$ and $\epsilon > 0$ there exist some $x = x(N, \epsilon) \in W^u(\Lambda^m) \cap W^s(\Lambda^n)$ and $\overline{m} = \overline{m}(N, \epsilon) \in \mathbb{N}$ such that for $\overline{m} \leq i \leq \overline{m} + N$, $d(\mathcal{O}(p), \phi^i(x)) < \epsilon$. Even more, we have $m_{\phi,f}(x) < \max\{t_n, t_m\}$.*

Remark 3.3.18. By symmetry, we have also the existence of some $y \in W^u(\Lambda^n) \cap W^s(\Lambda^m)$ and $\overline{n} \in \mathbb{N}$ with similar properties as x and $\overline{m}$.

Proof. As $\beta_m \in \mathcal{B}_{5N_m k}(\Lambda^m)$ and $\beta_n \in \mathcal{B}_{5N_n k}(\Lambda^n)$ we can find $\theta^1_m, \theta^1_n \in \mathcal{A}^{\mathbb{Z}^-}$ and $\theta^2_m, \theta^2_n \in \mathcal{A}^{\mathbb{N}}$ such that

$$\theta^1_m; \beta_m \theta^2_m \in \Pi(\Lambda^m) \quad \text{and} \quad \theta^1_n; \beta_n \theta^2_n \in \Pi(\Lambda^n).$$

By lemma 3.3.15, arguing as before; we can find positions $1 \leq j_{r_0(m)} < N_m k$ and $1 \leq j_{r_0(n)} < N_n k$ such that $j_{r_0(m)}$, $j_{r_0(m)}+1$ are $5N_m k$-good positions for β_m and $j_{r_0(n)}$, $j_{r_0(n)}+1$ are $5N_n k$-good positions for β_n; and also positions $4N_m k+1 \leq j_{r_3(m)} < 5N_m k$ and $4N_n k + 1 \leq j_{r_3(n)} < 5N_n k$ such that $j_{r_3(m)}$, $j_{r_3(m)} + 1$ are $5N_m k$-good positions for β_m and $j_{r_3(n)}$, $j_{r_3(n)} + 1$ are $5N_n k$-good positions for β_n.

Define then for $R \in \mathbb{N}$

$$x_R = \theta^1_m; \beta^m_1 \beta^m_2 \dots \beta^m_{j_{r_1(m)}} O(n)^R \beta^n_{j_{r_2(n)}+1} \beta^n_{j_{r_2(n)}+2} \dots \beta^n_{5N_n k} \theta^2_n.$$

Clearly the proposition will be proved if we show that for some $t < \max\{t_n, t_m\}$, $x_R \in \Sigma_t$:

Let $l \in \mathbb{Z}$. In any of the next three cases:

- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^n_{j_{r_1(n)}} \beta^n_{j_{r_1(n)}+1} \dots \beta^n_{j_{r_2(n)}} \beta^n_{j_{r_2(n)}+1} (= \beta^m_{j_{r_1(m)}} \beta^m_{j_{r_1(m)}+1} \dots \beta^m_{j_{r_2(m)}} \beta^n_{j_{r_2(m)+1}})$, some $j_{r_1(n)} < s \leq j_{r_2(n)}$ and $\bar{n} \in P(\beta^n_s)$.
- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^m_{j_{r_0(m)}} \beta^m_{j_{r_0(m)}+1} \dots \beta^m_{j_{r_1(m)}} \beta^m_{j_{r_2(m)+1}}$, some $j_{r_0(m)} < s \leq j_{r_1(m)}$ and $\bar{n} \in P(\beta^m_s)$.
- If $\Pi^{-1}(\sigma^l(x_R)) \in R(\eta; P(\eta, s; \bar{n}))$ for $\eta = \beta^2_{j_{r_2(n)}} \beta^2_{j_{r_2(2)}+1} \dots \beta^2_{j_{r_3(n)}} \beta^n_{j_{r_3(n)+1}}$, some $j_{r_2(n)} < s \leq j_{r_3(n)}$ and $\bar{n} \in P(\beta^n_s)$

proposition 3.3.16 let us conclude that $f(\Pi^{-1}(\sigma^l(x_R))) < \max\{t_n, t_m\}$.

Let $r_1 = |\beta^m_1 \beta^m_2 \dots \beta^n_{j_{r_0(m)}}|$ then, for $l \leq r_1 - 1$

$$f(\Pi^{-1}(\sigma^l(x_R))) < f(\Pi^{-1}(\sigma^l(\theta^1_m; \beta_m \theta^2_m))) + \eta_m/2 < t_m - \eta_m/2$$

because $\Lambda^m \subset \Lambda_{t_m-\eta_m}$ and as $j_{r_1(m)} - j_{r_0(m)} > 2N_m k - N_m k = N_m k$ we have that $\sigma^l(x_R)$ coincides with $\sigma^l(\theta_m^1; \beta_m \theta_m^2)$ in the central block of size $2N_m + 1$ centered at the zero position.

Analogously, for $r_2 = |\beta_1^m \beta_2^m \dots \beta_{j_{r_1(m)}}^m O(n)^R \beta_{j_{r_2(n)}+1}^n \beta_{j_{r_2(n)}+2}^n \dots \beta_{j_{r_3(n)}}^m|$, $j = r_2 - |\beta_1^n \beta_2^n \dots \beta_{j_{r_3(n)}}^n|$ and $l \geq r_2$

$$f(\Pi^{-1}(\sigma^l(x_R))) < f(\Pi^{-1}(\sigma^{l-j}(\theta_n^1; \beta_n \theta_n^2))) + \eta_n/2 < t_n - \eta_n/2$$

because $\Lambda^n \subset \Lambda_{t_n-\eta_n}$ and as $j_{r_3(n)} - j_{r_2(n)} > 4N_n k - 3N_n k = N_n k$ we have that $\sigma^l(x_R)$ coincides with $\sigma^{l-j}(\theta_n^1; \beta_n \theta_n^2)$ in the central block of size $2N_n + 1$ centered at the zero position.

As the previous cases describe all the possibilities for $l \in \mathbb{Z}$ and for $l \leq r_1 - 1$ and $l \geq r_2$ we have uniform limitation for the values of $f(\Pi^{-1}(\sigma^l(x_R))) < \max\{t_n, t_m\}$ then we have proved the result. $\square$

Using proposition 3.3.17 we can prove that if for some $m, n \in \mathbb{N}$, $O(m) = O(n)$ then we can connect Λ^m with Λ^n without leaving $\Lambda_{max\{t_n,t_m\}}$ as is expressed in definition 3.3.7

Corollary 3.3.19. *Let $m, n \in \mathcal{I}$ such that $O(m) = O(n)$. Then Λ^m connects with Λ^n before* $\max\{t_n, t_m\}$.

Proof. Proposition 3.3.17 let us find some $x, y \in \Lambda$ with $x \in W^u(\Lambda^m) \cap W^s(\Lambda^n)$, $y \in W^u(\Lambda^n) \cap W^s(\Lambda^m)$ and some $t < \max\{t_n, t_m\}$ such that

$$\Lambda^n \cup \Lambda^m \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_t.$$

Then lemma 3.3.9 let us conclude that Λ^n and Λ^m connects before $\max\{t_n, t_m\}$. $\square$

3.3.5 End of the proof of theorem 3.1.2

We are ready to obtain the desired contradiction. As the map O take only a finite number of different values, said M. Then by corollary 3.3.19 it would be impossible to have a sequence $n_1 < n_2 < ... < n_{M+1}$ of elements of $\mathcal{I}$ such that for $i, j \in \{1, ..., M+1\}$ with $i \neq j$, Λ^{n_i} and Λ^{n_j} doesn't connect before $\max\{t_{n_i}, t_{n_j}\}$ in contradiction with proposition 3.3.12.

Chapter 4

Continuity of fractal dimensions in conservative generic Markov and Lagrange dynamical spectra

4.1 Introduction

Given $\varphi : S \to S$ be a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and $f : S \to \mathbb{R}$ be a differentiable function. Consider the Lagrange Spectrum and Markov Spectrum of (φ, f, Λ)

$$L_{\varphi,f}(\Lambda) = \{\ell_{\varphi,f}(x) : x \in \Lambda\} \text{ and } M_{\varphi,f}(\Lambda) = \{m_{\varphi,f}(x) : x \in \Lambda\}$$

where for $x \in S$, $\ell_{\varphi,f}(x) = \limsup_{n\to\infty} f(\varphi^n(x))$ is the Lagrange value of x associated to f and φ and also $m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x))$ is the Markov value of x associated to f and φ. An elementary compactness argument (cf. Remark in Section 3 of [20]) shows that $\{\ell_{\varphi,f}(x) : x \in X\} \subset \{m_{\varphi,f}(x) : x \in X\} \subset f(X)$ whenever $X \subset M$ is a compact φ-invariant subset.

In this chapter, we are interested in the study of the relation between the real functions

$$L(t) = L(\varphi, f, \Lambda)(t) = HD(L_{\varphi,f}(\Lambda) \cap (-\infty, t)),$$
$$M(t) = M(\varphi, f, \Lambda)(t) = HD(M_{\varphi,f}(\Lambda) \cap (-\infty, t))$$

and

$$t \mapsto HD(\Lambda_t).$$

As in the previous chapter, we do that considering the projections of Λ_t on the stable and unstable Cantor sets of Λ

$$K_t^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(\Lambda_t \cap R_a) \text{ and } K_t^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(\Lambda_t \cap R_a).$$

In this setting, our main result (cf. Theorem 4.1.3 below) will be a generalization of the results of [3] on the continuity of Hausdorff dimension across Lagrange and Markov dynamical spectra.

Also, we define in the context of mixing horseshoes Λ with $HD(\Lambda) > 1$ the Markov transition parameter as

$$a = a(\varphi, f) = \sup\{t \in \mathbb{R} : HD(\Lambda_t) < 1\}.$$

In [10] is proved that for typical choices of the diffeomorphism φ and the smooth real map f, the Markov parameter is characterized by the conditions

$$\text{Leb}(M_{\varphi,f} \cap (-\infty, a-\delta)) = 0$$

but

$$\text{int}(M_{\varphi,f} \cap (-\infty, a+\delta)) \neq \emptyset,$$

for all $\delta > 0$.

The Lagrange parameter $\tilde{a} = \tilde{a}(\varphi, f)$ is defined in such a way that a similar result is true if we replace $M_{\varphi,f}$ by $L_{\varphi,f}$ and a by $\tilde{a}$ in the last conditions. Note, that as $L_{\varphi,f} \subset M_{\varphi,f}$, we always have $a(\varphi, f) \leq \tilde{a}(\varphi, f)$.

4.1.1 Statement of the results

The aim of this work is to extend the main theorem in [3], removing the hypothesis that $HD(\Lambda) < 1$. Using the notations of the previous subsection, our results are the following

Theorem 4.1.1. *Let $\varphi \in Diff^2(S)$ with a mixing horseshoe Λ. For every $r \geq 2$ there exists a C^r-open and dense set $\mathcal{R}_{\varphi,\Lambda}$ such that for any function $f \in \mathcal{R}_{\varphi,\Lambda}$ the functions*

$$t \mapsto d_u(t) := HD(K_t^u) \text{ and } t \mapsto d_s(t) := HD(K_t^s)$$

are continuous.

Remark 4.1.2. Our proof of theorem 4.1.1 shows that d_u and d_s coincide with the box-counting dimensions of K_t^s and K_t^u respectively.

We write $\mathrm{Diff}^2_\omega(S)$ for the set of conservative diffeomorphisms of S with respect to a volume form ω. Then we have the

Theorem 4.1.3. *Let $\varphi_0 \in Diff^2_\omega(S)$ with a mixing horseshoe Λ_0 and $\mathcal{U}$ a C^2-sufficiently small neighbourhood of φ_0 in $Diff^2_\omega(S)$ such that Λ_0 admits a continuation $\Lambda(=\Lambda(\varphi))$ for every $\varphi \in \mathcal{U}$. There exists a residual set $\tilde{\mathcal{U}} \subset \mathcal{U}$ such that for every $\varphi \in \tilde{\mathcal{U}}$ and $r \geq 2$ there exists a C^r-residual set $\tilde{\mathcal{R}}_{\varphi,\Lambda} \subset C^r(S,\mathbb{R})$ such that for any $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ the functions:*

$$t \mapsto d_u(t) = HD(K^u_t) \text{ and } t \mapsto d_s(t) = HD(K^s_t)$$

are continuous and in fact, they are equal with

$$HD(\Lambda_t) = d_u(t) + d_s(t) = 2d_u(t)$$

and

$$\min\{1, HD(\Lambda_t)\} = L(t) = M(t).$$

Finally, in theorem D of [10] is shown in the conservative case, that generically we have the equality $a = \tilde{a}$ where $a = a(\varphi, f)$ and $\tilde{a} = \tilde{a}(\varphi, f)$ are as in the previous section. However, there is a mistake in the proof of that theorem; more specifically, in the proof of the affirmation

$$HD(M_{\varphi,f} \cap (-\infty, a)) = HD(L_{\varphi,f} \cap (-\infty, a)) = 1.$$

Nevertheless, working in the setting of theorem 4.1.3 we have

$$\begin{aligned} L(a) = M(a) = \min\{1, HD(\Lambda_a)\} &= \lim_{t\to a^-} \min\{1, HD(\Lambda_t)\} \\ &= \lim_{t\to a^-} HD(\Lambda_t) \\ &= HD(\Lambda_a) = 1 \end{aligned}$$

then, intersecting the residual sets of the theorem D with the residual sets that we obtained here, we get a correct proof of the

Corollary 4.1.4 (Theorem D of [10])**.** *Let $\varphi_0 \in Diff^2_\omega(S)$ with a mixing horseshoe Λ_0 with $HD(\Lambda_0) > 1$ and $\mathcal{V}$ a C^2-sufficiently small neighbourhood of φ_0 in $Diff^2_\omega(S)$ such that Λ_0 admits a continuation Λ for every $\varphi \in \mathcal{V}$. Then, there exists a residual set $\mathcal{V}^* \subset \mathcal{V}$ such that for every $\varphi \in \mathcal{V}^*$ and $r \geq 2$ there exists a C^r-residual set $\mathcal{P}_{\varphi,\Lambda} \subset C^r(M,\mathbb{R})$ such that for any $f \in \mathcal{P}_{\varphi,\Lambda}$:*

$$Leb(M_{\varphi,f} \cap (-\infty, a-\delta)) = 0 = Leb(L_{\varphi,f} \cap (-\infty, a-\delta))$$

but

$$int(M_{\varphi,f} \cap (-\infty, a+\delta)) \neq \emptyset \neq int(L_{\varphi,f} \cap (-\infty, a+\delta))$$

for all $\delta > 0$. *Moreover, one has*

$$HD(M_{\varphi,f} \cap (-\infty, a)) = HD(L_{\varphi,f} \cap (-\infty, a)) = 1.$$

4.2 Preliminary results

First, we remember some results and notations from the previous chapter. Fix a Markov partition $\mathcal{P} = \{R_a\}_{a\in\mathcal{A}}$ for Λ. Then, there is a homeomorphism $\Pi : \Lambda \to \Sigma$ such that $\Pi(\varphi(x)) = \sigma(\Pi(x))$, where $\Sigma = \Sigma_{\mathcal{B}}$ is the Markov shift of finite type associated to $\mathcal{B}$ and σ is the left-shift map. We can use Π to transfer the function f from Λ to a function (still denoted f) on $\Sigma_{\mathcal{B}}$. In this setting, $\Pi(\Lambda_t) = \Sigma_t$ where

$$\Sigma_t = \{\theta \in \Sigma_{\mathcal{B}} : \sup_{n\in\mathbb{Z}} f(\sigma^n(\theta)) \leq t\}.$$

Given an admissible finite sequence $\theta = (a_1, ..., a_n) \in \mathcal{A}^n$ for all $1 \leq i < n$, we define

$$I^u(\theta) = \{x \in K^u : g_u^i(x) \in I^u(a_i, a_{i+1}), i = 1, 2, ..., n-1\}$$

and

$$I^s(\theta^t) = \{y \in K^s : g_s^i(y) \in I^s(a_i, a_{i-1}), i = 2, ..., n\}$$

where $\theta^t = (a_n, a_{n-1}, ..., a_2, a_1)$. In a similar way, let $\alpha = (a_{s_1}, a_{s_1+1}, ..., a_{s_2}) \in \mathcal{A}^{s_2-s_1+1}$ an admissible word where $s_1, s_2 \in \mathbb{Z}, s_1 < s_2$ and fix $s_1 \leq s \leq s_2$. We define

$$R(\alpha; s) = \bigcap_{m=s_1-s}^{s_2-s} \varphi^{-m}(R_{a_{m+s}}). \tag{4.2.1}$$

We write $s^{(u)}(\alpha)$ for the length of the interval $I^u(\alpha)$ and $r^u(\alpha) = \lfloor \log(1/s^{(u)}(\alpha)) \rfloor$ for the unstable scale of α. Similarly, we write $s^{(s)}(\alpha)$ for the length of $I^s(\alpha^t)$ and the stable scale of α is $r^{(s)}(\alpha) = \lfloor \log(1/s^{(s)}(\alpha)) \rfloor$. The bounded distortion property lets us relate the unstable and stable sizes of α to its length as a word in the alphabet $\mathcal{A}$. That is, there exists a constant $c_1 = c_1(\varphi, \Lambda) > 0$ such that

$$e^{-c_1} \leq \frac{|I^u(\alpha\beta)|}{|I^u(\alpha)||I^u(\beta)|} \leq e^{c_1} \text{ and } e^{-c_1} \leq \frac{|I^s((\alpha\beta)^t)|}{|I^s(\alpha^t)||I^s(\beta^t)|} \leq e^{c_1}. \tag{4.2.2}$$

Remark 4.2.1. In the context of horseshoes of C^2-conservative diffeomorphisms, there is a constant $c_2 = c_2(\varphi, \Lambda) > 0$ such that the stable and unstable sizes of any word $\alpha = (a_1, ..., a_n)$ in the alphabet $\mathcal{A}$ satisfy

$$e^{-c_2} \leq \frac{|I^u(\alpha)|}{|I^s(\alpha^t)|} \leq e^{c_2}. \tag{4.2.3}$$

Indeed, this happens because φ^n maps the unstable rectangle

$$R^u(\alpha) = \{x \in R_{a_0} : \varphi^i(x) \in R_{a_i}, 1 \leq i \leq n\}$$

diffeomorphically onto the stable rectangle

$$R^s(\alpha^t) = \{x \in R_{a_0} : \varphi^j(x) \in R_{a_{n-j}}, 1 \leq j \leq n\},$$

φ preserves areas, and the areas of $R^u(\alpha)$ and $R^s(\alpha^t)$ are comparable to $|I^u(\alpha)|$ and $|I^s(\alpha^t)|$ up to multiplicative factors.

We define for $r \in \mathbb{N}$

$$P_r^{(u)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(u)}(\alpha) \geq r \text{ and } r^{(u)}(\alpha^*) < r\},$$

where, $\alpha^* = (a_1, a_2, ..., a_{n-1})$ if $\alpha = (a_1, a_2, ..., a_n)$ and similarly,

$$P_r^{(s)} = \{\alpha \in \mathcal{A}^n \text{ admissible} : r^{(s)}(\alpha) \geq r \text{ and } r^{(s)}(\alpha^*) < r\}.$$

We also define

$$\mathcal{C}_u(t, r) = \mathcal{C}_u(\Lambda_t, r) = \{\alpha \in P_r^{(u)} : I^u(\alpha) \cap K_t^u \neq \emptyset\}$$

and

$$\mathcal{C}_s(t, r) = \mathcal{C}_s(\Lambda_t, r) = \{\alpha \in P_r^{(s)} : I^s(\alpha^t) \cap K_t^s \neq \emptyset\}$$

whose cardinalities are denoted $N_u(t, r) = |\mathcal{C}_u(t, r)|$ and $N_s(t, r) = |\mathcal{C}_s(t, r)|$.

In the last chapter, we proved that for each $t \in \mathbb{R}$ there exist the limits

$$D_u(t) = D_u(\Lambda_t) = \lim_{r \to \infty} \frac{\log N_u(t, r)}{r} = \inf_{r \in \mathbb{N}} \frac{\log(|\mathcal{A}|^c N_u(t, r))}{r} \in (0, 1), \tag{4.2.4}$$

$$D_s(t) = D_s(\Lambda_t) = \lim_{r \to \infty} \frac{\log N_s(t, r)}{r} = \inf_{r \in \mathbb{N}} \frac{\log(|\mathcal{A}|^c N_s(t, r))}{r} \in (0, 1). \tag{4.2.5}$$

and that the numbers $D_u(t)$ and $D_s(t)$ are the box counting dimension of K_t^u and K_t^s respectively. By proposition 2.6 in [3] we have that $t \mapsto D_u(t)$ and $t \mapsto D_s(t)$

are upper semicontinuous functions. We proceed to give a proof that $t \mapsto D_u(t)$ and $t \mapsto D_s(t)$ are also lower semicontinuous, however in order to do that we need to work with the correct set of functions:

Let $r \geq 2$. We define

$$\mathcal{R}_{\varphi,\Lambda} := \{f \in C^r(S, \mathbb{R}) : \nabla f(z) \neq 0, \ \forall \ z \in \Lambda\}.$$

In others words, $\mathcal{R}_{\varphi,\Lambda}$ is the class of functions C^r, $f : S \to \mathbb{R}$ such that for every $z \in \Lambda$ either $Df(z)e_z^s \neq 0$ or $Df(z)e_z^u \neq 0$ where e_z^s and e_z^u are unit vectors in the stable and unstable directions of T_zS. We end this section with the following lemma:

Lemma 4.2.2. *Given $r \geq 2$, $\mathcal{R}_{\varphi,\Lambda}$ is an open and dense subset of $C^r(S, \mathbb{R})$.*

Proof. Consider the class $\mathcal{M}$ of the Morse functions, we know by the compacity of S that $\mathcal{M}$ is dense in $C^r(S, \mathbb{R})$ and as a corollary of Morse's lemma that every element of $\mathcal{M}$ has only finitely many critical points. Then since we have $\text{int}(\Lambda) = \emptyset$, given $g \in \mathcal{M}$ we can find $f \in \mathcal{R}_{\varphi,\Lambda}$, C^r-arbitrarily close to g and this implies that $\mathcal{R}_{\varphi,\Lambda}$ is also C^r-dense. As $\mathcal{R}_{\varphi,\Lambda}$ is clearly open we have the result.

□

4.3 Critical windows and combinatorial lemmas

To prove the Theorem 4.1.1 we need the following proposition, whose proof depends on the notion of *critical window* and some combinatorial lemmas related with.

Proposition 4.3.1. *Let $\varphi : S \to S$ be a C^2 diffeomorphism with a mixing horseshoe Λ. Fix $f \in \mathcal{R}_{\varphi,\Lambda}$ and $t \in \mathbb{R}$ such that $D_u(t)$, resp. $D_s(t) > 0$. Then, for every $0 < \eta < 1$ there exists $\delta > 0$ and a complete subshift $\Sigma(\mathcal{B}_u) \subset \Sigma \subset \mathcal{A}^{\mathbb{Z}}$, resp. $\Sigma(\mathcal{B}_s) \subset \Sigma \subset \mathcal{A}^{\mathbb{Z}}$, associated to a finite set $\mathcal{B}_u = \{\beta_1^{(u)}, \beta_2^{(u)}, ..., \beta_m^{(u)}\}$, resp. $\mathcal{B}_s = \{\beta_1^{(s)}, \beta_2^{(s)}, ..., \beta_n^{(s)}\}$, of finite sequences, such that*

$$\Sigma(\mathcal{B}_u) \subset \Sigma_{t-\delta} \text{ , resp. } \Sigma(\mathcal{B}_s) \subset \Sigma_{t-\delta}$$

and

$$HD(K^u(\Sigma(\mathcal{B}_u)) > (1-\eta)D_u(t) \text{ , resp. } HD(K^s(\Sigma(\mathcal{B}_s^t)) > (1-\eta)D_s(t).$$

where $K^u(\Sigma(\mathcal{B}_u))$ and $K^s(\Sigma(\mathcal{B}_s^t))$ are the subsets of K^u and K^s, consisting of points whose trajectory under g_u and g_s, follows an itinerary obtained from the concatenation of words in the alphabets $\mathcal{B}_u$ and $\mathcal{B}_s^t$ respectively, where $\mathcal{B}_s^t$ is the alphabet whose words are the transposes of the words of the alphabet $\mathcal{B}_s$.

Remark 4.3.2. By symmetry it suffices to exhibit $\mathcal{B}_u$ satisfying the conclusion of Proposition 4.3.1.

Let m_1 big enough such that if $\alpha = (a_{-s_1}, ..., a_0, ..., a_{s_2})$ is admissible with $s_1, s_2 > m_1$ then either $R(\alpha; 0) \cap f^{-1}(t) = \emptyset$ or $R(\alpha; 0) \cap f^{-1}(t)$ is the graph of a differentiable map f_s defined in a (closed) sub interval of $I^s(a_0, a_{-1}, ..., a_{-s_1})$ with values in $I^u(a_0, a_1..., a_{s_2})$ (case 1) or $R(\alpha; 0) \cap f^{-1}(t)$ is the graph of a differentiable map f_u defined in a sub interval of $I^u(a_0, a_1, ..., a_{s_2})$ with values in $I^s(a_0, a_{-1}..., a_{-s_1})$ (case 2). Note that here we used the implicit function theorem, that we are working with the coordinates of the stable and unstable foliation, and also that we can suppose that with the choice made for r_0 there exists a $\tilde{\delta} > 0$ such that in case 1: $|Df(z)e^u_z| > \tilde{\delta}$, $\forall z \in R(\alpha; 0)$ and in case 2: $|Df(z)e^s_z| > \tilde{\delta}$, $\forall z \in R(\alpha; 0)$.

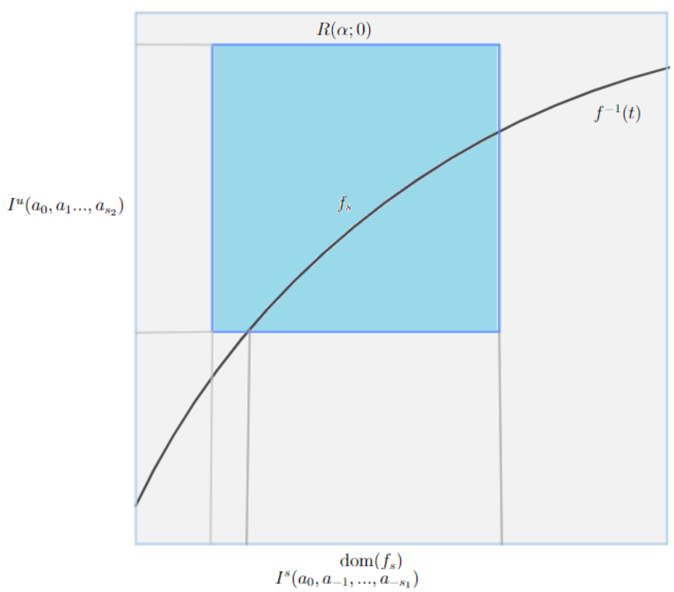

Figure 4.1: Letters on the left of α determine part of the letters on the right.

Suppose that we are in case 1. By the mean value theorem and because $f(\cdot, f_s(\cdot)) = t$, we have for some constant M that depends only on f

$$|Im(f_s)| \leq \frac{M}{\tilde{\delta}} I^s(a_0, a_{-1}..., a_{-s_1}) \leq M_1 {\lambda_{2,s}}^{s_1}, \tag{4.3.1}$$

where $\lambda_{2,s}$ is the greatest modulus of eigenvalues in Λ at the stable direction and M_1

is a constant. Suppose then $s_1 \geq R_1$ where R_1 is big enough such that there is at most only one $q \in \mathcal{A}$ with $I^u(a_0, ..., a_{m_1}, q) \cap Im(f_s) \neq \emptyset$.

Now suppose that there exist $x_1, x_2 \in B(\alpha; 0) \cap \Lambda$ such that $f(x_1) \leq t \leq f(x_2)$, then we have two possibilities: If $a_{m_1+1}, ..., a_{s_2} \in \mathcal{A}$ are unique such that there exist $x_1, x_2 \in B(\alpha; 0) \cap \Lambda = B((a_{-s_1}, ..., a_{m_1}, a_{m_1+1}, ..., a_{s_2}); 0) \cap \Lambda$ with $f(x_1) \leq t \leq f(x_2)$ then by knowing $a_{-s_1}, ..., a_0, ..., a_{m_1}$ we determine all the letters after a_{m_1}, i.e. the letters $a_{m_1+1}, ..., a_{s_2}$.

If for some $j_0 \geq 1$ with $s_2 > j_0 + m_1$ there are $\tilde{a}_{m_1+j_0+1}, ..., \tilde{a}_{s_2}, \tilde{b}_{m_1+j_0+1}, ..., \tilde{b}_{s_2} \in \mathcal{A}$ with $\tilde{a}_{m_1+j_0+1} \neq \tilde{b}_{m_1+j_0+1}$and $\tilde{x}_1, \tilde{x}_2 \in B((a_{-s_1}, ..., a_{m_1+j_0}, \tilde{a}_{m_1+j_0+1}, ..., \tilde{a}_{s_2}); 0) \cap \Lambda$, $\tilde{y}_1, \tilde{y}_2 \in B((a_{-s_1}, ..., a_{m_1+j_0}, \tilde{b}_{m_1+j_0+1}, ..., \tilde{b}_{s_2}); 0) \cap \Lambda$ with $f(\tilde{x}_1) \leq t \leq f(\tilde{x}_2)$ and $f(\tilde{y}_1) \leq t \leq f(\tilde{y}_2)$ then let j_0 minimal that satisfies that condition. So we have that depending on the relative positions of $I^u(a_0, ..., a_{m_1+j_0}, \tilde{a}_{m_1+j_0+1})$, $I^u(a_0, ..., a_{m_1+j_0}, \tilde{b}_{m_1+j_0+1})$ and $Im(f_s)$ that $Im(f_s)$ contains an interval of the form $I^u(a_0, ..., a_{m_1+j_0}, q)$ with $q \in \mathcal{A}$ or contain a gap between two intervals of that form. In any case by 4.3.1 we have for some constant $C > 0$

$$C(\lambda_{2,u}^{-1})^{m_1+j_0+1} \leq M_1 {\lambda_{2,s}}^{s_1},$$

where $\lambda_{2,u}$ is the greatest modulus of eigenvalues in Λ at the unstable direction, and then for some $R > 0$

$$(\lambda_{2,u}^{-1})^{j_0} \leq R {\lambda_{2,s}}^{s_1-m_1}. \tag{4.3.2}$$

So by knowing the letters $a_{-m_1}, ..., a_0, ..., a_{m_1}$ of α, by 4.3.2 the first $s_1 - m_1$ letters determine

$$j_0 \geq \frac{\log(R)}{\log(\lambda_{2,u}^{-1})} + (s_1 - m_1)\frac{\log(\lambda_{2,s})}{\log(\lambda_{2,u}^{-1})} > (s_1 - m_1)\frac{\log(\lambda_{2,s})}{2\log(\lambda_{2,u}^{-1})}$$

letters if $s_1 \geq R_2$ for some R_2.

Then either we determine $s_2 - m_1$ letters or at least

$$\left\lceil (s_1 - m_1)\frac{\log(\lambda_{2,s})}{2\log(\lambda_{2,u}^{-1})} \right\rceil \quad \text{letters if } s_1 \geq \max\{R_1, R_2\}.$$

If we are in case 2, we see analogously that are determine $s_1 - m_1$ letters or at least

$$\left\lceil (s_2 - m_1)\frac{\log(\lambda_{1,u}^{-1})}{2\log(\lambda_{1,s})} \right\rceil \quad \text{letters if } s_2 \geq \max\{\tilde{R}_1, \tilde{R}_2\},$$

where $\lambda_{1,s}$, $\lambda_{1,u}$ are the smallest modulus of eigenvalues in Λ at the stable and unstable direction respectively and $\tilde{R}_1, \tilde{R}_2$ are constants.

Finally, in any case, if $s_1, s_2 \geq \max\{R_1, R_2, \tilde{R}_1, \tilde{R}_2\} := \tilde{R}$, P letters at one side of the central block $(a_{-m_1}, ..., a_0, ..., a_{m_1})$ either determine all the letters of the other side or at least

$$\left\lceil \frac{1}{2} \min\left\{ \frac{\log(\lambda_{1,u}^{-1})}{\log(\lambda_{1,s})}, \frac{\log(\lambda_{2,s})}{\log(\lambda_{2,u}^{-1})} \right\} P \right\rceil \geq \left\lceil \frac{1}{\frac{1}{\theta} + \tilde{R}} P \right\rceil, \tag{4.3.3}$$

where $\theta = \frac{1}{2} \min\left\{ \frac{\log(\lambda_{1,u}^{-1})}{\log(\lambda_{1,s})}, \frac{\log(\lambda_{2,s})}{\log(\lambda_{2,u}^{-1})} \right\} < 1$.

Now, given $r \in \mathbb{N}$ define $\ell_1(r) := min\{|\beta| : \beta \in \mathcal{C}_u(t,r)\}$ and $\ell_2(r) := max\{|\beta| : \beta \in \mathcal{C}_u(t,r)\}$. For any word $\beta \in \mathcal{C}_u(t,r)$, as $r^{(u)}(\beta) \geq r$ and $r^{(u)}(\beta^*) < r$, we have for two constants $C_1, C_2 > 0$, with $\log C_1 \notin \mathbb{Z}$

$$C_1 (\lambda_{2,u}^{-1})^{|\beta|} \leq |I^u(\beta)| \leq e^{-r} < |I^u(\beta^*)| \leq C_2 (\lambda_{1,u}^{-1})^{|\beta|}$$

then,

$$\frac{-(r + \log C_1)}{\log(\lambda_{2,u}^{-1})} \leq |\beta| < \frac{-(r + \log C_2)}{\log(\lambda_{1,u}^{-1})}$$

so, applying this to the words in $\mathcal{C}_u(t,r)$ that realize $\ell_1(r)$ and $\ell_2(r)$, we conclude that

$$\frac{\ell_2(r)}{\ell_1(r)} \leq \frac{\log(\lambda_{2,u}^{-1})}{\log(\lambda_{1,u}^{-1})} \frac{(r + \log C_2)}{(r + \log C_1)}.$$

That is, $\{\frac{\ell_2(r)}{\ell_1(r)}\}_{r \in \mathbb{N}}$ is bounded and then we can define

$$m_0 := 3 \left\lceil \left(\frac{1}{\theta} + \tilde{R} \right) \cdot \left(\sup_{r \in \mathbb{N}} \frac{\ell_2(r)}{\ell_1(r)} \right) \right\rceil.$$

In order to prove the proposition, let us begin by taking $\tau = \eta/(100(2m_0+3)^2)$ and $r_0 = r_0(\varphi, f, \eta, t) \in \mathbb{N}$ large so that $\ell_1(r_0) \geq m_1$ and

$$\left| \frac{\log N_u(t,r)}{r} - D_u(t) \right| < \frac{\tau}{2} D_u(t), \ \forall \ r \geq r_0, \ r \in \mathbb{N} \tag{4.3.4}$$

also call $\mathcal{B}_0 = \mathcal{C}_u(t, r_0)$, $N_0 = N_u(t, r_0)$.

Consider $\beta = \beta_{k_1}\beta_{k_2}...\beta_{k_\ell} = a_1...a_p \in \mathcal{A}^p$, $\beta_{k_i} \in \mathcal{B}_0$, $1 \leq i \leq \ell$. We say that $n \in \{1, ..., p\}$ is the n-th position of β; if $\beta_{k_i} \in \mathcal{A}^{n_{k_i}}$ we write $|\beta_{k_i}| = n_{k_i}$ for its length and $P(\beta_{k_i}) = \{1, 2, ..., n_{k_i}\}$ for its set of positions as a word in the alphabet $\mathcal{A}$ and given $s \in P(\beta_{k_i})$ we call $P(\beta, k_i; s) = n_{k_1} + ... + n_{k_{i-1}} + s$ the position in β of the position s of β_{k_i}.

For the next definition, set

$$\tilde{\mathcal{B}} = \tilde{\mathcal{B}_u} := \{\beta = \beta_1\beta_2...\beta_k : \ \beta_j \in \mathcal{B}_0 \ \forall 1 \le j \le k \text{ and } I^u(\beta) \cap K_t^u \ne \emptyset\}$$

where $k = 4(2m_0 + 3)N_0^{2m_0+3} \cdot \lceil 2/\tau \rceil$.

Definition 4.3.3. Let $\beta = \beta_1\beta_2...\beta_k, \ \beta_r \in \mathcal{B}_0, \ 1 \le r \le k$ an element of $\tilde{\mathcal{B}}$. We say that (i, j) is a critical window for β if $j - i$ is even, $j - i \ge 2m_0 + 2$ and there is $n \in P(\beta_{(j+i)/2})$ such that if $\tilde{\eta} = \beta_i...\beta_j = a_1...a_{|\tilde{\eta}|}$ there are $x_1, x_2 \in R(\tilde{\eta}; P(\eta, (j + i)/2; n)) \cap \Lambda$ with $f(x_1) \le t \le f(x_2)$. We call $r = (j - i)/2$ the radius of the critical window.

Remark 4.3.4. By 4.3.3 we have that in this situation we determine all the $r - 1$ blocks of words of $\mathcal{B}_0$ that are on the left or on the right of $\beta_r, \beta_{r+1}, \beta_{r+2}$ or we determine

$$j_0 \ge \left\lceil \frac{1}{\frac{1}{\theta} + \tilde{R}} \ell_1(r_0)(r-1) \right\rceil \ge \left\lceil \frac{1}{(\frac{1}{\theta} + \tilde{R})\frac{\ell_2(r_0)}{\ell_1(r_0)}} \ell_2(r_0)(r-1) \right\rceil \ge \left\lceil \frac{1}{m_0/3} \ell_2(r_0)(r-1) \right\rceil$$

letters before the position $P(\tilde{\eta}, r+1; n) - m_1$ or after the position $P(\tilde{\eta}, r+1; n) + m_1$ of $\tilde{\eta}$ and then

$$\left\lfloor \frac{r-1}{m_0/3} \right\rfloor - 2 \ge \left\lfloor \frac{r-1}{m_0} \right\rfloor$$

blocks at one side of $\beta_r, \beta_{r+1}, \beta_{r+2}$ are determined in any case.

Given the pair (i, j) we write $\overline{[i, j]}$ for the set $\{i, i + 1, ..., j\}$. Moreover, if $\beta = \beta_1\beta_2...\beta_k, \ \beta_r \in \mathcal{B}_0, \ 1 \le r \le k$ we put

$$C(\beta) = \{1 \le s \le k \ : \ \exists \ (i, j) \text{ critical window of } \beta \text{ and } s \in \overline{[i, j]}\}.$$

In other words, $C(\beta)$ is the set of positions that are "contained" in a critical window.

Now we want to estimate the cardinality of the set

$$\mathcal{E} = \left\{ \beta = \beta_1...\beta_k \in \tilde{\mathcal{B}} : |C(\beta)| < \frac{k}{5(2m_0 + 3)} \right\}.$$

But first, we do that for the set $\tilde{\mathcal{B}}$.

Lemma 4.3.5. *We have* $|\tilde{\mathcal{B}}| > 2N_0^{(1-\tau)k}$.

Proof. Given $\beta = \beta_1\beta_2...\beta_k \in \tilde{\mathcal{B}}$ we have by the inequality 4.2.2 that

$$|I^u(\beta)| \leq \prod_{i=1}^{k} e^{c_1}|I^u(\beta_i)|.$$

By definition, for every $i = 1, 2, ..., k$, as $\beta_i \in \mathcal{C}_u(t, r_0)$

$$r^u(\beta_i) = \left\lfloor \log \frac{1}{|I^u(\beta_i)|} \right\rfloor \geq r_0$$

and then

$$|I^u(\beta)| \leq \prod_{i=1}^{k} e^{c_1}|I^u(\beta_i)| \leq e^{-k(r_0-c_1)}.$$

This implies that $\{I^u(\beta) : \beta \in \tilde{\mathcal{B}}\}$ is a cover of K_t^u by intervals of unstable-size $\geq k(r_0 - c_1)$. In particular, writing $\beta = (b_1b_2...b_{n(k)})$ we have a surjective projection $(b_1b_2...b_{n(k)}) \mapsto (b_1b_2...b_j) \in \mathcal{C}_u(t, k(r_0 - c_1))$ where

$$j = \min\{1 \leq i \leq n(k) : r^u(b_1b_2...b_i)) \geq k(r_0 - c_1)\}.$$

We can take r_0 large enough such that $k(r_0 - c_1) > r_0$ and then, by 4.2.4

$$|\mathcal{C}_u(t, k(r_0 - c_1))| = N_u(t, k(r_0 - c_1)) > \frac{1}{|\mathcal{A}|^c} e^{(k(r_0-c_1)D_u(t))}.$$

In particular,

$$|\tilde{\mathcal{B}}| \geq \frac{1}{|\mathcal{A}|^c} e^{(k(r_0-c_1)D_u(t))} > 2e^{k(r_0-2c_1)D_u(t)},$$

because k is large for r_0 large and $D_u(t) > 0$. Then

$$|\tilde{\mathcal{B}}| > 2e^{(1-\tau/2)kr_0D_u(t)} > 2e^{(1-\tau)(1+\tau/2)kr_0D_u(t)} > 2N_0^{(1-\tau)k},$$

because $N_0 < e^{(1+\tau/2)r_0D_u(t)}$ by 4.3.4. □

Using the above lemma we have the following:

Lemma 4.3.6. *One has* $|\mathcal{E}| > N_0^{(1-\tau)k}$.

Proof. Remember the elementary fact that, given a finite family of intervals, there is a subfamily of disjoint intervals whose sum of lengths is at least half of the measure of the union of the intervals of the original family. If for $\beta \in \tilde{\mathcal{B}}$, $|C(\beta)| \geq \frac{k}{5(2m_0+3)}$ then applying the above fact to the family of intervals $[i, j+1)$ with (i, j) a critical window of β there exist a family $\{(i_x, j_x)\}_{x\in\mathcal{X}}$ of critical windows of β such that $\overline{[i_x, j_x]} \cap \overline{[i_y, j_y]} = \emptyset$ if $x, y \in \mathcal{X}$ with $x \neq y$ and $\frac{k}{10(2m_0+3)} \leq |\bigcup_{x\in\mathcal{X}} \overline{[i_x, j_x]}| := M_{\mathcal{X}}$. Set $r_x = (j_x - i_x)/2$ for $x \in \mathcal{X}$, we observe that if $|\mathcal{X}| \leq \frac{M_{\mathcal{X}}}{2(2m_0+3)}$:

$$\begin{aligned}
\sum_{x\in\mathcal{X}} \left\lfloor \frac{r_x - 1}{m_0} \right\rfloor &= \frac{M_{\mathcal{X}}}{2m_0} - \frac{3|\mathcal{X}|}{2m_0} + \sum_{x\in\mathcal{X}} \left(\left\lfloor \frac{r_x - 1}{m_0} \right\rfloor - \frac{r_x - 1}{m_0} \right) \\
&= \frac{M_{\mathcal{X}}}{2m_0} - \frac{3|\mathcal{X}|}{2m_0} - \sum_{x\in\mathcal{X}} \left(\frac{r_x - 1}{m_0} - \left\lfloor \frac{r_x - 1}{m_0} \right\rfloor \right) \\
&\geq \frac{M_{\mathcal{X}}}{2m_0} - \left(1 + \frac{3}{2m_0}\right) |\mathcal{X}| \quad \left(\text{since } 0 \leq \frac{r_x - 1}{m_0} - \left\lfloor \frac{r_x - 1}{m_0} \right\rfloor < 1\right) \\
&\geq \frac{M_{\mathcal{X}}}{2m_0} - \frac{M_{\mathcal{X}}}{4m_0} = \frac{M_{\mathcal{X}}}{4m_0} \geq \frac{k}{40m_0(2m_0+3)} \geq \frac{k}{20(2m_0+3)^2}
\end{aligned}$$

and if $|\mathcal{X}| > \frac{M_{\mathcal{X}}}{2(2m_0+3)}$:

$$\sum_{x\in\mathcal{X}} \left\lfloor \frac{r_x - 1}{m_0} \right\rfloor \geq \sum_{x\in\mathcal{X}} 1 = |\mathcal{X}| > \frac{M_{\mathcal{X}}}{2(2m_0+3)} \geq \frac{k}{20(2m_0+3)^2}.$$

In any case

$$\begin{aligned}
\prod_{x\in\mathcal{X}} N_0^{2r_x+1-\lfloor (r_x-1)/m_0 \rfloor} \cdot N_0^{k-M_{\mathcal{X}}} &= N_0^{M_{\mathcal{X}} - \sum_{x\in\mathcal{X}} \lfloor (r_x-1)/m_0 \rfloor} \cdot N_0^{k-M_{\mathcal{X}}} \\
&= N_0^{k - \sum_{x\in\mathcal{X}} \lfloor (r_x-1)/m_0 \rfloor} \leq N_0^{(1-1/(20(2m_0+3)^2)k}.
\end{aligned}$$

Then, using that and remark 4.3.4 we have

$$|\tilde{\mathcal{B}} \setminus \mathcal{E}| \leq 2^k \cdot 2^k \cdot N_0^{(1-1/20(2m_0+3)^2)k}. \tag{4.3.5}$$

Since for our choices of r_0, N_0, k large enough and τ sufficiently small we have $2^{2k} \cdot N_0^{(1-1/20(2m_0+3)^2)k} < N_0^{(1-\tau)k}$ it follows from 4.3.5 that:

$$|\mathcal{E}| = |\tilde{\mathcal{B}}| - |\tilde{\mathcal{B}} \setminus \mathcal{E}| > 2N_0^{(1-\tau)k} - 2^{2k} \cdot N_0^{(1-1/20(2m_0+3)^2)k} > N_0^{(1-\tau)k}.$$

This completes the proof of the lemma. $\square$

Our next lemma shows that among the words $\beta \in \mathcal{E}$ we have several words which share the same positions which do not belong to $C(\beta)$ and the same words of $\mathcal{B}_0$ appearing in these positions.

Lemma 4.3.7. *There are $3N_0^{2m_0+3}$ words $(\tilde{\beta}_{s_i-m_0-1}, ..., \tilde{\beta}_{s_i}, ..., \tilde{\beta}_{s_i+m_0+1}) \in \mathcal{B}_0^{2m_0+3}$, with $s_i \in \{m_0+2, m_0+3, ..., k-m_0-1\}$, and $1 \leq i \leq 3N_0^{2m_0+3}$, such that*

$$s_{i+1} - s_i \geq (2m_0+3)\left\lceil \frac{2}{\tau} \right\rceil \text{ for } 1 \leq i < 3N_0^{2m_0+3}$$

and the set

$$X = \{\beta = \beta_1\beta_2...\beta_k \in \mathcal{E} : (\beta_{s_i-m_0-1}, ..., \beta_{s_i}, ..., \beta_{s_i+m_0+1}) = (\tilde{\beta}_{s_i-m_0-1}, ..., \tilde{\beta}_{s_i}, ..., \tilde{\beta}_{s_i+m_0+1}),$$
$$\{s_i - m_0 - 1, ..., s_i, ..., s_i + m_0 + 1\} \cap C(\beta) = \emptyset, 1 \leq i \leq 3N_0^{2m_0+3}\}$$

has cardinality bigger than $N_0^{(1-2\tau)k}$.

Proof. Given $\beta = \beta_1\beta_2...\beta_k \in \mathcal{E}$ we can find $W = \left\lceil \frac{4k}{5(2m_0+3)} \right\rceil$ indices $i_1 < i_2 < ... < i_W$ with $i_p \in \{m_0+2, m_0+3, ...k-m_0-1\}$, $\forall p = 1, 2, ..., W$ such that

- $i_{p+1} - i_p \geq (2m_0+3)$, $p = 1, 2, ..., W-1$
- $\cup_{p=1}^{W}\{i_p - m_0 - 1, ..., i_p, ..., i_p + m_0 + 1\} \cap C(\beta) = \emptyset$.

We remember that $k = 4(2m_0+3)N_0^{2m_0+3} \cdot \lceil 2/\tau \rceil$ and since $3N_0^{2m_0+3}\lceil 2/\tau \rceil < (16/5)N_0^{2m_0+3}$ $\lceil 2/\tau \rceil \leq W$ we can write $j_m = i_{m\lceil 2/\tau \rceil}$ with $1 \leq m \leq 3N_0^{2m_0+3}$. Then for $1 \leq m < 3N_0^{2m_0+3}$, $j_{m+1} - j_m \geq (2m_0+3)\lceil 2/\tau \rceil$ and for $1 \leq m \leq 3N_0^{2m_0+3}$

$$\{j_m - m_0 - 1, ..., j_m, ..., j_m + m_0 + 1\} \cap C(\beta) = \emptyset \ .$$

Note that

- The number of possibilities for $(j_1, ..., j_{3N_0^{2m_0+3}})$ is smaller than 2^k
- For $(j_1, ..., j_{3N_0^{2m_0+3}})$ fixed, the number of possibilities for $(\beta_{j_i-m_0-1}, ..., \beta_{j_i}, ..., \beta_{j_i+m_0+1})$ is at most $N_0^{3(2m_0+3)N_0^{2m_0+3}}$.

Then we can choose $m_0 + 1 < s_1 < s_2 < ... < s_{3N_0^{2m_0+3}} < k - m_0$ with $s_{i+1} - s_i \geq (2m_0+3)\lceil 2/\tau \rceil$ and strings $(\tilde{\beta}_{s_i-m_0-1}, ..., \tilde{\beta}_{s_i}, ..., \tilde{\beta}_{s_i+m_0+1}) \in \mathcal{B}_0^{2m_0+3}$, $1 \leq i \leq 3N_0^{2m_0+3}$ such that the set

$$X = \{\beta = \beta_1\beta_2...\beta_k \in \mathcal{E} : (\beta_{s_i-m_0-1}, ..., \beta_{s_i}, ..., \beta_{s_i+m_0+1}) = (\tilde{\beta}_{s_i-m_0-1}, ..., \tilde{\beta}_{s_i}, ..., \tilde{\beta}_{s_i+m_0+1}),$$
$$\{s_i - m_0 - 1, ..., s_i, ..., s_i + m_0 + 1\} \cap C(\beta) = \emptyset, 1 \leq i \leq 3N_0^{2m_0+3}\}$$

has cardinality

$$|X| \geq \frac{|\mathcal{E}|}{2^k \cdot N_0^{3(2m_0+3)N_0^{2m_0+3}}}.$$

But, $|\mathcal{E}| > N_0^{(1-\tau)k}$ and $2^k \cdot N_0^{3(2m_0+3)N_0^{2m_0+3}} < N_0^{\tau k}$. Therefore,

$$|X| > \frac{|\mathcal{E}|}{2^k \cdot N_0^{3(2m_0+3)N_0^{2m_0+3}}} > N_0^{(1-2\tau)k}.$$

□

Our third combinatorial lemma states that it is possible to cut words in the subset X provided by Lemma 4.3.7 at certain positions in such a way that one obtains a set $\mathcal{B}_u$ with non-neglectible cardinality.

For every $1 \leq p < q \leq 3N_0^{2m_0+3}$ we denote $\pi_{p,q} : X \to \mathcal{B}_0^{s_q - s_p}$ the projection $\pi_{p,q}(\beta) = (\beta_{s_p+1}, \beta_{s_p+2}, ..., \beta_{s_q})$, if $\beta = \beta_1\beta_2...\beta_k$.

Lemma 4.3.8. *There are* $1 \leq p_0 < q_0 \leq 3N_0^{2m_0+3}$ *such that*

i) $(\tilde{\beta}_{s_{p_0}-m_0-1}, ..., \tilde{\beta}_{s_{p_0}}, ..., \tilde{\beta}_{s_{p_0}+m_0+1}) = (\tilde{\beta}_{s_{q_0}-m_0-1}, ..., \tilde{\beta}_{s_{q_0}}, ..., \tilde{\beta}_{s_{q_0}+m_0+1})$

ii) $|\pi_{p_0,q_0}(X)| > N_0^{(1-10\tau)(s_{q_0}-s_{p_0})}$

Proof. Consider $\mathcal{T}$ the set of pairs (p, q) such that $1 \leq p < q \leq 3N_0^{2m_0+3}$ and $\pi_{p,q}(X) \leq N_0^{(1-10\tau)(s_q-s_p)}$. For each pair in $\mathcal{T}$ we exclude from the set $\overline{[1, 3N_0^{2m_0+3}]}$ the indices $j \in \overline{[p, q-1]}$.

Claim: The set $\mathcal{Z} = \bigcup_{(p,q)\in\mathcal{T}} \overline{[p, q-1]}$ has cardinality smaller than $2N_0^{2m_0+3}$. Using the same observation given in Lemma 4.3.6 we can find a subset $\tilde{\mathcal{T}}$ of $\mathcal{T}$ such that $\overline{[p, q-1]} \cap \overline{[\tilde{p}, \tilde{q}-1]} = \emptyset$, for every $(p, q), (\tilde{p}, \tilde{q}) \in \tilde{\mathcal{T}}$ with $(p, q) \neq (\tilde{p}, \tilde{q})$ and

$$\sum_{(p,q)\in\tilde{\mathcal{T}}} (q-p) \geq \frac{1}{2}|\mathcal{Z}|.$$

Suppose that $|\mathcal{Z}| \geq 2N_0^{2m_0+3}$. Since the sequence $s_1 < s_2 < ... < s_{3N_0^{2m_0+3}}$ given by Lemma 4.3.7 is such that $s_{i+1} - s_i \geq (2m_0+3)\lceil 2/\tau \rceil$ we have

$$\sum_{(p,q)\in\tilde{\mathcal{T}}} (s_q - s_p) \geq (2m_0+3)\lceil 2/\tau \rceil \sum_{(p,q)\in\tilde{\mathcal{T}}} (q-p) \geq (2m_0+3)\lceil 2/\tau \rceil N_0^{2m_0+3}. \quad (4.3.6)$$

On the other hand, since $\pi_{p,q}(X) \leq N_0^{(1-10\tau)(s_q-s_p)}$ we have

$$|X| < N_0^{(1-10\tau)\sum_{(p,q)\in\tilde{\mathcal{T}}}(s_q-s_p)} \cdot N_0^{k-\sum_{(p,q)\in\tilde{\mathcal{T}}}(s_q-s_p)}$$

Using 4.3.6 we have

$$|X| < N_0^{(1-10)\tau((2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3})} \cdot N_0^{k-(2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3})} = N_0^{k-10\tau((2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3})}. \tag{4.3.7}$$

By lemma 4.3.7 we know that $|X| > N_0^{(1-2\tau)k}$. Using that and the inequality 4.3.7 we must have

$$(1-2\tau)k < k - 10\tau((2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3})$$

that is,

$$10(2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3} < 2k.$$

Since, $2k = 8(2m_0+3)\lceil 2/\tau\rceil N_0^{2m_0+3}$ we have a contradiction. This implies that $|\mathcal{Z}| < 2N_0^{2m_0+3}$ which proves our claim.

Therefore, we do not exclude at least $N_0^{2m_0+3}+1$ indices. Since for each of that indices we have at most $N_0^{2m_0+3}$ possibilities for choose $(\tilde{\beta}_{s_i-m_0-1}, ..., \tilde{\beta}_{s_i}, ..., \tilde{\beta}_{s_i+m_0+1})$ (see lemma 4.3.7) we conclude that there are two indices $(p_0, q_0) \notin \mathcal{T}$ such that

$$(\tilde{\beta}_{s_{p_0}-m_0-1}, ..., \tilde{\beta}_{s_{p_0}}, ..., \tilde{\beta}_{s_{p_0}+m_0+1}) = (\tilde{\beta}_{s_{q_0}-m_0-1}, ..., \tilde{\beta}_{s_{q_0}}, ..., \tilde{\beta}_{s_{q_0}+m_0+1}).$$

By definition of non-excluded index $|\pi_{p_0,q_0}(X)| > N_0^{(1-10\tau)k}$ as we wanted to see. □

Take $\mathcal{B}_u := \pi_{p_0,q_0}(X)$ were p_0, q_0 are given by the previous lemma. Note that $K^u(\Sigma(\mathcal{B}_u))$ is a $C^{1+\varepsilon}$-dynamically defined Cantor set associated to certain iterates of g_u on the intervals $I^u(\alpha)$ with $\alpha \in \mathcal{B}_u$. In this case, its Hausdorff dimension coincides with its box-counting dimension and as for r_0 sufficiently large, we have $a(K^u(\Sigma(\mathcal{B}_u)))$ is close to 1 and $\lambda(K^u(\Sigma(\mathcal{B}_u)))$ is big (see section 2.2), then $1 \geq \log a/\log\lambda$ and by 2.2.3 and 2.2.4 one has

$$\beta_1 - \alpha_1 \leq \frac{\tau}{2} HD(K^u(\Sigma(\mathcal{B}_u))) \leq \frac{\tau}{2}\beta_1.$$

Using this, 2.2.2 and 2.2.3 we obtain

$$HD(K^u(\Sigma(\mathcal{B}_u))) \geq \alpha_1 \geq \left(1-\frac{\tau}{2}\right)\beta_1 \geq \left(1-\frac{\tau}{2}\right)\frac{\log|\mathcal{B}_u|}{-\log(\min\limits_{\alpha\in\mathcal{B}_u}|I^u(\alpha)|)}. \tag{4.3.8}$$

By the item ii) of the lemma 4.3.8, $|\mathcal{B}_u| > N_0^{(1-10\tau)(s_{q_0}-s_{p_0})}$. On the other hand, by the bounded distortion property (see 4.2.2), we have $|I^u(\alpha)| \geq e^{-(c_1+r_0)(s_{q_0}-s_{p_0})}$, for each $\alpha \in \mathcal{B}_u$. Using this and the inequality 4.3.8 we obtain

$$HD(K^u(\Sigma(\mathcal{B}_u))) \geq \frac{(1-\tau/2)(1-10\tau)\log N_0}{c_1+r_0}. \tag{4.3.9}$$

Since $N_0 = N_u(t, r_0)$ satisfies

$$\left|\frac{\log N_0}{r_0} - D_u(t)\right| < \frac{\tau}{2} D_u(t)$$

we have

$$\log N_0 > (1-\tau/2) r_0 D_u(t).$$

Plugging this in inequality 4.3.9 and using that $\tau = \eta/(100(2m_0+3)^2)$ we have

$$HD(K^u(\Sigma(\mathcal{B}_u))) > \frac{(1-10\tau)(1-\tau/2)^2 r_0}{r_0+c_1} D_u(t) > (1-12\tau)D_u(t) > (1-\eta)D_u(t)$$

for $r_0 = r_0(\eta)$ sufficiently large.

At this point we are ready to end the proof of the Proposition 4.3.1.

Proof. We write by simplicity

$$y_1 = \tilde{\beta}_{s_{p_0}+1}\tilde{\beta}_{s_{p_0}+2}...\tilde{\beta}_{s_{p_0}+m_0+1} = \tilde{\beta}_{s_{q_0}+1}\tilde{\beta}_{s_{q_0}+2}...\tilde{\beta}_{s_{q_0}+m_0+1}$$

and

$$y_2 = \tilde{\beta}_{s_{p_0}-m_0-1}...\tilde{\beta}_{s_{p_0}-1}\tilde{\beta}_{s_{p_0}} = \tilde{\beta}_{s_{q_0}-m_0-1}...\tilde{\beta}_{s_{q_0}-1}\tilde{\beta}_{s_{q_0}}.$$

It follows that any element in $\mathcal{B}_u$ has the form $y_1\beta_{s_{p_0}+m_0+2}...\beta_{s_{q_0}-m_0-2}y_2$, where $\beta_i \in \mathcal{B}_0$ for any $i = s_{p_0}+m_0+2, ..., s_{q_0}-m_0-2$ and

$$I^u(y_2y_1\beta_{s_{p_0}+m_0+2}...\beta_{s_{q_0}-m_0-2}y_2y_1) \cap K_t^u \neq \emptyset.$$

And then the elements of $\Sigma(\mathcal{B}_u)$ have the form

$$x = \theta^{(1)}y_2; y_1\beta_{s_{p_0}+m_0+2}...\beta_{s_{q_0}-m_0-2}y_2y_1\theta^{(2)},$$

where ; indicates that the 0-th position is the first position in y_1 and $y_1\theta^{(2)} \in \mathcal{A}^{\mathbb{N}}$ and $\theta^{(1)}y_2 \in \mathcal{A}^{\mathbb{Z}^-}$.

We will see that $f(\sigma^{\ell}(x)) < t$, $\forall \ell \in \mathbb{Z}$. Repair that in order to prove that, it is sufficient to consider $0 \leq \ell \leq \tilde{m} - 1$ if

$$\alpha = y_1 \beta_{s_{p_0}+m_0+2} \dots \beta_{s_{q_0}-m_0-2} y_2 = a_1 \dots a_{\tilde{m}}.$$

Take then $j \in P(\beta_r)$ for some $r \in \{s_{p_0}+1, ..., s_{q_0}\}$ and suppose that $f(\sigma^{P(\alpha,r;j)}(x)) \geq t$. If $s_{q_0} - r \geq r - s_{p_0} - 1$ let $\tilde{\eta} = \beta_{s_{p_0}-m_0} \dots \beta_r \dots \beta_{2r-s_{p_0}+m_0}$, then $x_1 = \sigma^{P(\alpha,r;j)}(x) \in R(\tilde{\eta}; P(\tilde{\eta}, r; j)) \cap \Lambda$ and since $I^u(y_2 y_1 \beta_{s_{p_0}+m_0+2} \dots \beta_{s_{q_0}-m_0-2} y_2 y_1) \cap K_t^u \neq \emptyset$, by definition there are $\theta^{(3)} \in \mathcal{A}^{\mathbb{Z}_-}$ and $\theta^{(4)} \in \mathcal{A}^{\mathbb{N}}$ such that

$$\theta^{(3)}; y_2 y_1 \beta_{s_{p_0}+m_0+2} \dots \beta_{s_{q_0}-m_0-1} y_2 y_1 \theta^{(4)} \in \Sigma_t,$$

and then, there exists $x_2 \in R(\tilde{\eta}; P(\tilde{\eta}, r; j)) \cap \Lambda$ such that $f(x_2) \leq t$. But this is a contradiction because remembering that $\mathcal{B}_u = \pi_{p_0,q_0}(X)$ then there is some $\beta \in X$ such that $(s_{p_0} - m_0, 2r - s_{p_0} + m_0)$ is a critical window of β, because $2r - s_{q_0} + m_0 - (s_{p_0} - m_0) = 2r - 2s_{p_0} + 2m_0 \geq 2m_0 + 2$, and $s_{p_0} \in \overline{[s_{p_0} - m_0, 2r - s_{p_0} + m_0)]}$.

If $s_{q_0} - r < r - s_{p_0} - 1$ the argument is similar. Therefore, $f(\sigma^{\ell}(x)) < t$, $\forall \ell \in \mathbb{Z}$ and since $\Sigma(\mathcal{B}_u) \subset \bigcup_{i=0}^{k\ell_2} \sigma^i(\Sigma(\mathcal{B}_u))$ where $\bigcup_{i=0}^{k\ell_2} \sigma^i(\Sigma(\mathcal{B}_u)) = \bigcup_{i\in\mathbb{Z}} \sigma^i(\Sigma(\mathcal{B}_u))$ is the compact set, formed by the orbits by σ of elements of $\Sigma(\mathcal{B}_u)$, there exists $\delta > 0$ such that

$$\Sigma(\mathcal{B}_u) \subset \Sigma_{t-\delta}.$$

□

Remark 4.3.9. It is possible to show that if $\Sigma(\mathcal{B}) \subset \Sigma \subset \mathcal{A}^{\mathbb{Z}}$ is a complete subshift associated to a finite alphabet $\mathcal{B}$ of finite words on $\mathcal{A}$ then the set of the previous proof (as a subset of Λ) $\Lambda(\Sigma(\mathcal{B})) = \Pi^{-1}(\bigcup_{i\in\mathbb{Z}} \sigma^i(\Sigma(\mathcal{B})))$ is a subhorseshoe of Λ.

4.4 Proofs of the theorems

We begin proving theorem 4.1.1.

Proof. Proposition 4.3.1 implies that

$$D_u(t) \geq HD(K_t^u) \geq HD(K_{t-\delta}^u) \geq HD(K^u(\Sigma(\mathcal{B}_u))) > (1-\eta) D_u(t).$$

Since $\eta > 0$ is arbitrary we have $D_u(t) = HD(K_t^u) = d_u(t)$. Moreover,

$$(1-\eta) D_u(t) \leq HD(K^u(\Sigma(\mathcal{B}_u))) \leq HD(K_{t-\delta}^u) = D_u(t-\delta)$$

that is, $t \mapsto D_u(t)$ is a lower semicontinuous function. Since by Proposition 2.6 in [3], $t \mapsto D_u(t)$ is also upper semicontinuous, we have that $t \mapsto D_u(t) = d_u(t)$ is continuous.

Similarly, we have the equality $D_s(t) = d_s(t)$ and that $a \mapsto D_s(t) = d_s(t)$ is continuous, so we have proved theorem 4.1.1. $\square$

In the sequel, we will use the following result that follows from the spectral decomposition theorem and from [14]

Proposition 4.4.1. *There exists a residual subset $\tilde{\mathcal{U}} \subset \mathcal{U}$ with the property that for every subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and any $f \in C^1(S, \mathbb{R})$ such that there exists some point in $\tilde{\Lambda}$ with its gradient not parallel neither the stable direction nor the unstable direction, one has*

$$HD(f(\tilde{\Lambda})) = \min\{1, HD(\tilde{\Lambda})\}.$$

to prove the next proposition

Proposition 4.4.2. *If $\tilde{\mathcal{U}}$ is as in the proposition 4.4.1 and $r \geq 2$ then for any $\varphi \in \tilde{\mathcal{U}}$, there exists a C^r-residual subset $\tilde{\mathcal{R}}_{\varphi,\Lambda} \subset \mathcal{R}_{\varphi,\Lambda}$ such that for every subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and any $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ one has*

$$\min\{1, HD(\tilde{\Lambda})\} = HD(\ell_{\varphi,f}(\widetilde{\Lambda})) = HD(m_{\varphi,f}(\widetilde{\Lambda})).$$

Proof. Following the ideas of the proof of the theorem 1 of [20] we see that given a subhorseshoe $\widetilde{\Lambda} \subset \Lambda$, the set

$$H_{\widetilde{\Lambda}} = \{f \in C^r(S, \mathbb{R}) : |M_{\widetilde{\Lambda},f}| = 1 \text{ and if } z \in M_{\widetilde{\Lambda},f},\ Df_z(e_z^{s,u}) \neq 0\}$$

is C^r- open and dense set, where $M_{\widetilde{\Lambda},f} = \{x \in \widetilde{\Lambda} : \forall y \in \widetilde{\Lambda}\ ,\ f(x) \geq f(y)\}$. Take then

$$\tilde{\mathcal{R}}_{\varphi,\Lambda} := \bigcap_{\substack{\widetilde{\Lambda} \subset \Lambda \\ subhorseshoe}} H_{\widetilde{\Lambda}} \cap \mathcal{R}_{\varphi,\Lambda}.$$

In the mentioned paper is also proved that for any such subhorseshoe $\widetilde{\Lambda} \subset \Lambda$ and $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ if x_M is the unique element where $f|_{\widetilde{\Lambda}}$ take its maximum value, then for any $\epsilon > 0$ there exists some subhorseshoe $\widetilde{\Lambda}^\epsilon \subset \widetilde{\Lambda} \setminus \{x_M\}$ with

$$HD(\widetilde{\Lambda}^\epsilon) \geq HD(\widetilde{\Lambda})(1 - \epsilon)$$

and such that for some point $d \in \widetilde{\Lambda}^{\epsilon}$ there exists a local C^1-diffeomorphism $\tilde{A}$ defined in a neighborhood U_d of d such that

$$f(\varphi^{j_0}(\tilde{A}(\tilde{\Lambda}_{j_0}))) \subset \ell_{\varphi,f}(\widetilde{\Lambda}),$$

where j_0 is an integer and $\tilde{\Lambda}_{j_0} \subset \widetilde{\Lambda}^{\epsilon}$ has nonempty interior in $\widetilde{\Lambda}^{\epsilon}$ and then is such that $HD(\tilde{\Lambda}_{j_0}) = HD(\widetilde{\Lambda}^{\epsilon})$. Moreover, it is proved also that $\dfrac{\partial \tilde{A}}{\partial e_x^{s,u}} \parallel e^{s,u}_{\tilde{A}(x)}$, for $x \in U_d \cap \widetilde{\Lambda}^{\epsilon}$ and then, by construction, $\nabla(f \circ \varphi^{j_0} \circ \tilde{A})(x) \nparallel e_x^{s,u}$ for every $x \in \tilde{\Lambda}_{j_0}$.

Extending properly $f \circ \varphi^{j_0} \circ \tilde{A}$, and letting ϵ tends to 0; it follows from this and proposition 4.4.1 that

$$\min\{1, HD(\tilde{\Lambda})\} \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})).$$

And finally

$$\min\{1, HD(\tilde{\Lambda})\} \leq HD(\ell_{\varphi,f}(\widetilde{\Lambda})) \leq HD(m_{\varphi,f}(\widetilde{\Lambda})) \leq HD(f(\widetilde{\Lambda})) \leq \min\{1, HD(\tilde{\Lambda})\}.$$

As we wanted to see. $\square$

Now we proceed with the proof of theorem 4.1.3.

Proof. First, note that as in 4.3.8 we have

$$HD(K^s(\Sigma(\mathcal{B}_u^t))) > \left(1 - \frac{\tau}{2}\right) \frac{\log |\mathcal{B}_u|}{-\log(\min_{\alpha \in \mathcal{B}_u} |I^s(\alpha^t)|)},$$

where $\mathcal{B}_u^t$ is the alphabet whose words are the transposes of the words of the alphabet $\mathcal{B}_u$. Since $|I^s(\alpha^t)|$ is comparable to $|I^u(\alpha)|$, using the notation of the remark 4.2.3 and the calculations after 4.3.8 we have that for r_0 large

$$D_s(t) \geq HD(K^s(\Sigma(\mathcal{B}_u^t))) \geq \frac{(1-10\tau)(1-\tau/2)^2 r_0 D_u(t)}{r_0 + c_1 + c_2} > (1-\eta) D_u(t).$$

Since $\eta > 0$ is arbitrary we have $D_s(t) \geq D_u(t)$ and the other inequality is proved in a similar way. On the other hand, if we take $\varphi \in \tilde{\mathcal{U}} \subset \mathcal{U}$, $t \in \mathbb{R}$ such that $D_u(t) > 0$ and $\eta > 0$ we have

$$2(1-\eta) D_u(t) = (1-\eta)(D_s(t) + D_u(t)) \leq HD(\Lambda(\Sigma(\mathcal{B}_u))), \tag{4.4.1}$$

where $\mathcal{B}_u$ comes from Proposition 4.3.1. By Proposition 4.4.2 it follows that

$$\min\{1, HD(\Lambda(\Sigma(\mathcal{B}_u)))\} = HD(\ell_{\varphi,f}(\Lambda(\Sigma(\mathcal{B}_u))))$$

and then

$$\begin{aligned}\min\{1, 2(1-\eta)D_u(t)\} &\leq \min\{1, HD(\Lambda(\Sigma(\mathcal{B}_u)))\} = HD(\ell_{\varphi,f}(\Lambda(\Sigma(\mathcal{B}_u))))\\ &\leq HD(L_{\varphi,f}\cap(-\infty,t)) \leq HD(M_{\varphi,f}\cap(-\infty,t))\\ &\leq HD(f(\Lambda_t)) \leq \min\{1, HD(\Lambda_t)\}\\ &\leq \min\{1, 2D_u(t)\}.\end{aligned}$$

Since $\eta > 0$ is arbitrary

$$\min\{1, 2D_u(t)\} = L(t) = M(t).$$

Finally, using one more time 4.4.1, we also obtain

$$2(1-\eta)D_u(t) \leq HD(\Lambda(\Sigma(\mathcal{B}_u))) \leq HD(\Lambda_t) \leq 2D_u(t),$$

because $\eta > 0$ is arbitrary, this proves that $HD(\Lambda_t) = 2D_u(t)$. □

Remark 4.4.3. The equality $HD(\Lambda_t) = 2D_u(t)$ in the last proof, in fact, doesn't need any generic condition on φ.

Now we want to prove that the conclusions of proposition 4.4.2 hold not only for subhorseshoes, but also for sets of the form Λ_t for $t \in \mathbb{R}$. In order to do that, we recall the lemma 3.3.2 of chapter 3

Lemma 4.4.4. *For every $t \in \mathbb{R}$ we have*

$$L(t) = \sup_{s<t} HD(\ell_{\varphi,f}(\Lambda_s)) = \lim_{s\to t^-} HD(\ell_{\varphi,f}(\Lambda_s))$$

and

$$M(t) = \sup_{s<t} HD(m_{\varphi,f}(\Lambda_s)) = \lim_{s\to t^-} HD(m_{\varphi,f}(\Lambda_s)).$$

Corollary 4.4.5. *For any $\varphi \in \tilde{\mathcal{U}}$, $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ and $t \in \mathbb{R}$*

$$\min\{1, HD(\Lambda_t)\} = HD(\ell_{\varphi,f}(\Lambda_t)) = HD(m_{\varphi,f}(\Lambda_t)).$$

Proof. This is a direct consequence of lemma 4.4.4 and theorem 4.1.3. Indeed, for $\delta > 0$

$$L(t-\delta) \leq HD(\ell_{\varphi,f}(\Lambda_t)) \leq L(t+\delta)$$

letting δ tends to 0 we have $L(t) = HD(\ell_{\varphi,f}(\Lambda_t))$. Analogously we have $M(t) = HD(m_{\varphi,f}(\Lambda_t))$ and from theorem 4.1.3, $L(t) = M(t) = \min\{1, HD(\Lambda_t)\}$. □

We end this chapter by giving another property of the map $L = M$

Corollary 4.4.6. *For any $\varphi \in \tilde{\mathcal{U}}$ and $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ the map $L = M$ is not a Holder continuous function.*

Proof. First, using proposition 4.4.1, we can argue as in proposition 3.3.6 of the last chapter and show that for $\varphi \in \tilde{\mathcal{U}}$ and $f \in \tilde{\mathcal{R}}_{\varphi,\Lambda}$ it must be true that $HD(L_{\varphi,f}) > 0$, $L^{'}_{\varphi,f} = \{x : x$ is an accumulation point of $L_{\varphi,f}\} \neq \emptyset$ and show that it is exactly at the point $c_{\varphi,f} = \min L^{'}_{\varphi,f}$ where the map L begins to be positive.

Suppose that L is Holder continuous with exponent $\alpha > 0$. Then there is $\epsilon > 0$ such that $0 < L(c_{\varphi,f} + \epsilon) < \alpha$ and being L an α-Holder function, one has

$$\begin{aligned} 1 = HD([0, L(c_{\varphi,f} + \epsilon)]) &= HD(L(L_{\varphi,f} \cap (-\infty, c_{\varphi,f} + \epsilon))) \\ &< \frac{1}{\alpha}.HD(L_{\varphi,f} \cap (-\infty, c_{\varphi,f} + \epsilon)) \\ &= \frac{1}{\alpha} L(c_{\varphi,f} + \epsilon) < 1, \end{aligned}$$

which is absurd. $\square$

Chapter 5

Concentration of dimension in extremal points of left-half lines in the Lagrange spectrum

5.1 Introduction

Remember that given any $\eta \in \mathbb{R} \setminus \mathbb{Q}$, we set

$$\begin{aligned} k(\eta) &= \sup\left\{k > 0 : \left|\eta - \frac{p}{q}\right| < \frac{1}{kq^2} \text{ has infinitely many rational solution } \frac{p}{q}\right\} \\ &= \limsup_{p\in\mathbb{Z}, q\in\mathbb{N}, p,q\to\infty} |q(q\eta - p)|^{-1} \in \mathbb{R} \cup \{\infty\} \end{aligned}$$

for the best constant of Diophantine approximations of η.

The classical Lagrange spectrum is the set

$$L = \{k(\eta) : \eta \in \mathbb{R} \setminus \mathbb{Q}, k(\eta) < \infty\},$$

and the classical Markov spectrum is the set

$$M = \left\{ \left(\inf_{(x,y)\in\mathbb{Z}^2 - \{(0,0)\}} |q(x,y)| \right)^{-1} < \infty : q(x,y) = ax^2 + bxy + cy^2, b^2 - 4ac = 1 \right\}$$

that consists of the reciprocal of the minimal values over non-trivial integer vectors $(x, y) \in \mathbb{Z}^2 - \{(0, 0)\}$ of indefinite binary quadratic forms $q(x, y)$ with unit discriminant.

Given a bi-infinite sequence $\theta = (\theta_n)_{n\in\mathbb{Z}} \in (\mathbb{N}^*)^{\mathbb{Z}}$, let

$$\lambda_i(\theta) := [0; a_{i+1}, a_{i+2}, \dots] + a_i + [0; a_{i-1}, a_{i-2}, \dots].$$

The Markov value $m(\theta)$ of θ is $m(\theta) = \sup\limits_{i\in\mathbb{Z}} \lambda_i(\theta)$ and the Lagrange value of θ, $\ell(\theta)$ is $\ell(\theta) = \limsup\limits_{i\to\infty} \lambda_i(\theta)$. As was proved by Perron, the Markov spectrum is the set $M = \{m(\theta) < \infty : \theta \in (\mathbb{N}^*)^{\mathbb{Z}}\}$ and the Lagrange spectrum is the set $L = \{\ell(\theta) < \infty : \theta \in (\mathbb{N}^*)^{\mathbb{Z}}\}$.

Now, given $\varphi : S \to S$ a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and any differentiable function $f : S \to \mathbb{R}$. Following the dynamical characterizations of the classical spectra given by Perron, we defined the Lagrange spectrum of (φ, f, Λ) and also the Markov spectrum of (φ, f, Λ) as the sets

$$L_{\varphi,f}(\Lambda) = \{\ell_{\varphi,f}(x) = \limsup_{n\to\infty} f(\varphi^n(x)) : x \in \Lambda\}$$

and

$$M_{\varphi,f}(\Lambda) = \{m_{\varphi,f}(x) = \sup_{n\in\mathbb{Z}} f(\varphi^n(x)) : x \in \Lambda\}.$$

Moreira in [16] proved several results on the geometry of the classical Markov and Lagrange spectra, for example that

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) = \min\{1, 2D(t)\},$$

where $D(t) = HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is a continuous surjective function from $\mathbb{R}$ to $[0, 1)$. Even more, he proved the limit

$$\lim_{t\to\infty} HD(k^{-1}(t)) = 1.$$

In this chapter, we use that dynamical Markov and Lagrange spectra associated with conservative horseshoes in surfaces are natural generalizations of the classical Markov and Lagrange spectra. In fact, classical Markov and Lagrange spectra are not compact sets, so they cannot be dynamical spectra associated to horseshoes. However, in [9] is showed that for any $N \geq 2$ with $N \neq 3$, the initial segments of the classical spectra until $\sqrt{N^2+4N}$ (i.e., $M \cap (-\infty, \sqrt{N^2+4N}]$ and $L \cap (-\infty, \sqrt{N^2+4N}]$) coincide with the sets $M(N)$ and $L(N)$, given, in the notation we used in Perron's characterization of M and L by

$$M(N) = m(\Sigma(N)) = \{m(\theta) : \theta \in \Sigma(N)\}$$

and

$$L(N) = \ell(\Sigma(N)) = \{\ell(\theta) : \theta \in \Sigma(N)\}$$

where $\Sigma(N) = \{1, 2, \ldots, N\}^{\mathbb{Z}}$.

It is proved also that $M(N)$ and $L(N)$ are dynamical Markov and Lagrange spectra associated to a smooth real function f and to a horseshoe $\Lambda(N)$ defined by a smooth conservative diffeomorphism φ, and also that they are naturally associated to continued fractions with coefficients bounded by N.

Here we use this relation between classical and dynamical spectra in order to understand better the fractal geometry (Hausdorff dimension) of the preimage of half-lines by the function k. We can state our main result as:

Theorem 5.1.1. *For $t \geq 6$, the map D is strictly increasing and $D(t) = HD(k^{-1}(t))$ i.e.*

$$HD(k^{-1}((-\infty, t))) = HD(k^{-1}((-\infty, t])) = HD(k^{-1}(t)).$$

5.2 Preliminares

5.2.1 Continued fractions

Remember that the continued fraction expansion of a real number η is denoted by

$$\eta = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Given a finite sequence $\alpha = (a_1, a_2, \ldots, a_n) \in (\mathbb{N}^*)^n$, we defined the interval

$$I(\alpha) = I(a_1, a_2, \ldots, a_n) = \{x \in [0,1] : x = [0; a_1, a_2, \ldots, a_n, \alpha_{n+1}],\ \alpha_{n+1} \geq 1\}$$

that have length

$$|I(a_1, a_2, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})},$$

where $\frac{p_i}{q_i} = [0; a_1, \ldots, a_i] \in \mathbb{Q}$. Also, for $(a_0, a_1, \ldots, a_n) \in (\mathbb{N}^*)^{n+1}$ we set

$$I(a_0; a_1, \ldots, a_n) = \{x \in [0,1] : x = [a_0; a_1, a_2, \ldots, a_n, \alpha_{n+1}],\ \alpha_{n+1} \geq 1\}$$

and then, we have

$$|I(a_0; a_1, \ldots, a_n)| = |I(a_1, a_2, \ldots, a_n)|. \tag{5.2.1}$$

We will use the following lemmas stated in chapter 2:

Lemma 5.2.1. *Let* $\alpha = [a_0; a_1, \dots, a_n, a_{n+1}, \dots]$ *and* $\tilde{\alpha} = [a_0; a_1, \dots, a_n, b_{n+1}, \dots]$, *then:*

- $|\alpha - \tilde{\alpha}| < 1/2^{n-1}$,
- *If* $a_{n+1} \neq b_{n+1}$, $\alpha > \tilde{\alpha}$ *if and only if* $(-1)^{n+1}(a_{n+1} - b_{n+1}) > 0$.

Lemma 5.2.2. *If* $a_0, a_1, a_2 \dots, a_n, a_{n+1}, \dots$ *and* $b_{n+1}, b_{n+2}, \dots$ *are positive integers bounded by* $N \in \mathbb{N}$ *and* $a_{n+1} \neq b_{n+1}$ *then*

$$\begin{aligned} |[a_0; a_1, a_2 \dots, a_n, a_{n+1}, \dots] - [a_0; a_1, a_2 \dots, a_n, b_{n+1}, \dots]| &> c(N)/q_{n-1}^2 \\ &> c(N)|I(a_1, a_2, \dots, a_n)| \end{aligned}$$

for some positive constant $c(N)$.

Lemma 5.2.3. *For finite words* α *and* β

$$\frac{1}{2}|I(\alpha)||I(\beta)| < |I(\alpha\beta)| < 2|I(\alpha)||I(\beta)|.$$

For the sequel, the following application of lemma 5.2.1 also will be useful

Lemma 5.2.4. *Given* $R, N \in \mathbb{N}$, *let* $\beta^1, \beta^2, \beta^3 \in \Sigma(N)^+ := \{1, 2, \dots, N\}^{\mathbb{N}}$ *such that* $[0; \beta^1] < [0; \beta^2] < [0; \beta^3]$. *If for two sequences* $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ *and* $\tilde{\alpha} = (\tilde{\alpha}_n)_{n\in\mathbb{Z}}$ *in* $\Sigma(N)$ *it is true that* $\alpha_0, \dots, \alpha_{2R+1} = \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}$. *Then for all* $j \leq 2R+1$ *we have*

$$\begin{aligned} \lambda_0(\sigma^j(\dots, \alpha_{-2}, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^2)) < \max\{&m(\dots, \alpha_{-2}, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1), \\ &m(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}, \beta^3)\} + 1/2^{R-1}. \end{aligned}$$

Proof. It is just an application of lemma 5.2.1. Indeed, for $j \leq R+1$

$$\begin{aligned} &\lambda_0(\sigma^j(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^2)) < \lambda_0(\sigma^j(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1)) + 1/2^{R-1} \\ &\leq \max\{m(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1), m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}, \beta^3)\} + 1/2^{R-1}. \end{aligned}$$

For $R+1 < j \leq 2R+1$, if $[\alpha_j; \dots, \alpha_{2R+1}, \beta^2] < [\tilde{\alpha}_j; \dots, \tilde{\alpha}_{2R+1}, \beta^3]$

$$\begin{aligned} \lambda_0(\sigma^j(\dots, \alpha_{-2}, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^2)) &< \lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}, \beta^3)) + 1/2^R \\ &\leq \max\{m(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1), m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}, \beta^3)\} + 1/2^R. \end{aligned}$$

And for $R+1 < j \leq 2R+1$, if $[\alpha_j; \dots, \alpha_{2R+1}, \beta^2] < [\alpha_j; \dots, \alpha_{2R+1}, \beta^1]$

$$\begin{aligned} &\lambda_0(\sigma^j(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^2)) < \lambda_0(\sigma^j(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1)) \\ &\leq \max\{m(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_{2R+1}, \beta^1), m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_{2R+1}, \beta^1)\}. \end{aligned}$$

Then we have proved the result. $\square$

5.2.2 Results on Dynamical Markov and Lagrange spectra

Given $\varphi : S \to S$ a diffeomorphism of a C^∞ compact surface S with a mixing horseshoe Λ and $f : S \to \mathbb{R}$ differentiable. Fix a Markov partition $\{R_a\}_{a\in\mathcal{A}}$ with sufficiently small diameter consisting of rectangles $R_a \sim I_a^s \times I_a^u$ delimited by compact pieces I_a^s, I_a^u, of stable and unstable manifolds of certain points of Λ. It is possible define projections $\pi_a^u : R_a \to I_a^s \times \{i_a^u\}$ and $\pi_a^s : R_a \to \{i_a^s\} \times I_a^u$ of the rectangles into the connected components $I_a^s \times \{i_a^u\}$ and $\{i_a^s\} \times I_a^u$ of the stable and unstable boundaries of R_a, where $i_a^u \in \partial I_a^u$ and $i_a^s \in \partial I_a^s$ are fixed arbitrarily. In this way, we have the unstable and stable Cantor sets

$$K^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(\Lambda \cap R_a) \text{ and } K^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(\Lambda \cap R_a).$$

In fact K^u and K^s are $C^{1+\alpha}$ dynamically defined, associated to some expanding maps ψ_s and ψ_u. The stable and unstable Cantor sets, K^s and K^u, respectively, are closely related to the fractal geometry of the horseshoe Λ. For instance, it is well-known that $HD(\Lambda) = HD(K^s) + HD(K^u)$ and that in the conservative case $HD(K^s) = HD(K^u)$.

Given $t \in \mathbb{R}$ is of interest to us consider the set $\Lambda_t = \{x \in \Lambda : m_{\varphi,f}(x) = \sup\limits_{n\in\mathbb{Z}} f(\varphi^n(x)) \leq t\}$ and its projections on the stable and unstable Cantor sets of Λ

$$K_t^u = \bigcup_{a\in\mathcal{A}} \pi_a^s(\Lambda_t \cap R_a) \text{ and } K_t^s = \bigcup_{a\in\mathcal{A}} \pi_a^u(\Lambda_t \cap R_a).$$

In the previous chapter was shown the following result

Theorem 5.2.5. *Let $\varphi \in Diff^2(S)$ a conservative diffeomorphism preserving a smooth form ω and take Λ a mixing horseshoe of φ. If $f \in C^r(S,\mathbb{R})$ satisfies that $\forall\ z \in \Lambda,\ \nabla f(z) \neq 0$, then the functions*

$$t \mapsto HD(K_t^u) \text{ and } t \mapsto HD(K_t^s)$$

are equal and continuous. Even more, one has

$$HD(\Lambda_t) = 2HD(K_t^u).$$

5.2.3 The horseshoe $\Lambda(N)$

Given an integer $N \geq 2$, write $\tilde{C}_N = \{1, 2, ..., N\} + C_N$ and define

$$\Lambda(N) = C_N \times \tilde{C}_N.$$

If $x = [0; a_1, a_2, ...]$ and $y = [a_0; a_{-1}, a_{-2}, ...]$ then we take $\varphi : \Lambda(N) \to \Lambda(N)$ given by

$$\begin{aligned}\varphi(x, y) &= (G(x), a_1 + 1/y)\\ &= ([0; a_2, a_3, ...], a_1 + [0; a_0, a_{-1}, ...]).\end{aligned}$$

Also, equip $\Lambda(N)$ with the real map $f(x, y) = x+y$. We note that φ can be extended to a C^∞-diffeomorphism on a diffeomorphic copy of the 2-dimensional sphere $\mathbb{S}^2$.

Notice also that φ is conjugated to the restriction to $C_N \times C_N$ of the map $\psi : (0, 1) \times (0, 1) \to [0, 1) \times (0, 1)$ given by

$$\psi(x, y) = \left(G(x), \frac{1}{y + \lfloor 1/x \rfloor}\right)$$

and following [2] and [26] we know that ψ has an invariant measure equivalent to the Lebesgue measure, in particular, φ also has an invariant measure equivalent to the Lebesgue measure and then φ is conservative.

Indeed, if $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < 1/(1 + x)\}$ and $T : \mathcal{S} \to \mathcal{S}$ is given by

$$T(x, y) = (G(x), x - x^2 y),$$

then T preserves the Lebesgue measure in the plane. If $h : \mathcal{S} \to [0, 1) \times (0, 1)$ is given by $h(x, y) = (x, y/(1 - xy))$ then h is a conjugation between T and ψ (and thus ψ preserves the smooth measure $h_*(\text{Leb})$).

For $\Lambda(N)$ we have the Markov partition $\{R_a\}_{a\in\mathcal{A}}$ where $\mathcal{A} = \{1, 2, \ldots, N\}$ and R_a is such that $R_a \cap \Lambda(N) = C_N \times (C_N + a) = C_N \times C_N + (0, a)$. It is clear then that $\varphi|_{\Lambda_N}$ is topologically conjugated to $\sigma : \{1, 2, ..., N\}^{\mathbb{Z}} \to \{1, 2, .., N\}^{\mathbb{Z}}$; and that in sequences, f becomes $\tilde{f} : \{1, 2, ..., N\}^{\mathbb{Z}} \to \mathbb{R}$ given by

$$\tilde{f}(\theta) = [0; a_1(\theta), a_2(\theta), ...] + a_0(\theta) + [0; a_{-1}(\theta), a_{-2}(\theta), ...] = \lambda_0(\theta),$$

where $\theta = (a_i(\theta))_{i\in\mathbb{Z}}$, and so

$$L_{\varphi,f}(\Lambda(N)) = \{\ell_{\sigma,\tilde{f}}(\theta) : \theta \in \{1, 2, ..., N\}^{\mathbb{Z}}\} = L(N)$$

and

$$M_{\varphi,f}(\Lambda(N)) = \{m_{\sigma,\tilde{f}}(\theta) : \theta \in \{1, 2, ..., N\}^{\mathbb{Z}}\} = M(N).$$

In this context, let $\alpha = (a_{s_1}, a_{s_1+1}, ..., a_{s_2}) \in \mathcal{A}^{s_2-s_1+1}$ any word where $s_1, s_2 \in \mathbb{Z}$, $s_1 < s_2$ and fix $s_1 \le s \le s_2$. Define then

$$R(\alpha; s) = \bigcap_{m=s_1-s}^{s_2-s} \varphi^{-m}(R_{a_{m+s}}).$$

Finally, let us consider $A_N = [0; \overline{N, 1}]$ and $B_N = [0; \overline{1, N}]$. As

$$NA_N + A_N B_N = 1 \text{ and } B_N + B_N A_N = 1,$$

we have $A_N = \dfrac{B_N}{N}$. Thus $B_N = \frac{-N+\sqrt{N^2+4N}}{2}$, $A_N = \frac{-N+\sqrt{N^2+4N}}{2N}$ and then

$$\max f|_{\Lambda(N)} = 2B_N + N = \sqrt{N^2+4N},\ \min f|_{\Lambda(N)} = 2A_N + 1 = \frac{\sqrt{N^2+4N}}{N}.$$

5.3 Proof of the result

5.3.1 Connection of subhorseshoes

For the next, it will be useful to recall the following definition given in chapter 3. Here we fix some smooth diffeomorphism φ of some surface S possessing a mixing horseshoe Λ.

Definition 5.3.1. Given Λ^1 and Λ^2 subhorseshoes of Λ and $s \in \mathbb{R}$, we said that Λ^1 *connects* with Λ^2 or that Λ^1 and Λ^2 *connect* before s if there exist a subhorseshoe $\tilde{\Lambda} \subset \Lambda$ and some $q < s$ with $\Lambda^1 \cup \Lambda^2 \subset \tilde{\Lambda} \subset \Lambda_q$, where $\Lambda_q = \{x \in \Lambda : m_{\varphi,f}(x) \leq q\}$.

Among other properties, for a fixed $s \in \mathbb{R}$, the relation *"connect before s"* in the set of subhorseshoes of Λ satisfies the transitivity property. That is, consider Λ^1, Λ^2 and Λ^3 three subhorseshoes of Λ and $s \in \mathbb{R}$, if Λ^1 connects with Λ^2 before s and Λ^2 connects with Λ^3 before s. Then also Λ^1 connects with Λ^3 before s.

For our present purposes, the next criterion of connection proved in chapter 3, will be important

Proposition 5.3.2. *Suppose Λ^1 and Λ^2 are subhorseshoes of Λ and for some $x, y \in \Lambda$ we have $x \in W^u(\Lambda^1) \cap W^s(\Lambda^2)$ and $y \in W^u(\Lambda^2) \cap W^s(\Lambda^1)$. If for some $s \in \mathbb{R}$, it is true that*

$$\Lambda^1 \cup \Lambda^2 \cup \mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_s,$$

then for every $\epsilon > 0$, Λ^1 and Λ^2 connect before $s + \epsilon$.

5.3.2 Dimension estimates

Let $t \geq 6$ and take $m = \lfloor t \rfloor - 3$. Consider then the horseshoe

$$\Lambda := \Lambda(m+3) = C(m+3) \times \tilde{C}(m+3)$$

equipped with the diffeomorphism φ and the map f given in the previous section.

Given $\epsilon > 0$ such that

$$\epsilon < t - (\max f|_{\Lambda} - 2) = t + 2 - \sqrt{(m+3)^2 + 4(m+3)},$$

take $\ell(\epsilon) \in \mathbb{N}$ sufficiently large such that for the set

$$C_\epsilon = \{\alpha = (a_0, a_1 \cdots, a_{2\ell(\epsilon)}) \in \{1, 2, \cdots, m+3\}^{2\ell(\epsilon)+1} : R(\alpha; \ell(\epsilon)) \cap \Lambda_{t-\epsilon} \neq \emptyset\}$$

if $\alpha \in C_\epsilon$ and $z, y \in R(\alpha; \ell(\epsilon))$ then $|f(x) - f(y)| < \epsilon/2$. Set

$$P = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\bigcup_{\alpha\in C_\epsilon} R(\alpha; \ell(\epsilon))).$$

Note that by construction, $\Lambda_{t-\epsilon} \subset P \subset \Lambda_{t-\epsilon/2}$. Being P a hyperbolic set of finite type, by proposition A.0.3, it admits a decomposition

$$P = \bigcup_{x\in\mathcal{X}} \tilde{\Lambda}_x$$

where $\mathcal{X}$ is a finite index set and for $x \in \mathcal{X}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set i.e a set of the form $\tau = \{x \in P : \alpha(x) \subset \tilde{\Lambda}_{i_1}$ and $\omega(x) \subset \tilde{\Lambda}_{i_2}\}$ where $\tilde{\Lambda}_{i_1}$ and $\tilde{\Lambda}_{i_2}$ with $i_1, i_2 \in \mathcal{X}$ are subhorseshoes.

As for every transient τ set as before, we have

$$HD(\tau) = HD(K^s(\tilde{\Lambda}_{i_1})) + HD(K^u(\tilde{\Lambda}_{i_2}))$$

and for every subhorseshoe $\tilde{\Lambda}_i$, being φ conservative, one has

$$HD(\tilde{\Lambda}_i) = HD(K^s(\tilde{\Lambda}_i)) + HD(K^u(\tilde{\Lambda}_i)) = 2HD(K^u(\tilde{\Lambda}_i))$$

therefore

$$HD(P) = \max_{x\in\mathcal{X}} HD(\tilde{\Lambda}_x) = \max_{\substack{x\in\mathcal{X}:\ \tilde{\Lambda}_x\ is \\ subhorseshoe}} HD(\tilde{\Lambda}_x). \tag{5.3.1}$$

We will show that the subhorseshoe contained in P with the biggest dimension connects with $\Lambda(4) \subset \Lambda$ before any time bigger than $t - \epsilon/2$. To do that, take any $\delta > 0$ and write

$$\tilde{P} = \bigcup_{\substack{x\in\mathcal{X}:\ \tilde{\Lambda}_x\ is \\ subhorseshoe}} \tilde{\Lambda}_x = \bigcup_{i\in\mathcal{I}} \tilde{\Lambda}_i \cup \bigcup_{i\in\mathcal{J}} \tilde{\Lambda}_j$$

where

$$\mathcal{I} = \{i \in \mathcal{X} : \tilde{\Lambda}_i \text{ is a subhorseshoe and it connects with } \Lambda(4) \text{ before } t - \epsilon/2 + \delta\}$$

and

$$\mathcal{J} = \{j \in \mathcal{X} : \tilde{\Lambda}_j \text{ is a subhorseshoe and it doesn't connect with } \Lambda(4) \text{ before } t - \epsilon/2 + \delta\}.$$

We want to see that

$$HD(\bigcup_{j \in \mathcal{J}} \tilde{\Lambda}_j) < HD(\bigcup_{i \in \mathcal{I}} \tilde{\Lambda}_i). \tag{5.3.2}$$

In order to do that, we use the criterion given by proposition 5.3.2. That is, given $j \in \mathcal{J}$ as $\tilde{\Lambda}_j \cup \Lambda(4) \subset \Lambda_{t-\epsilon/2}$ we cannot have at the same time the existence of two points $x \in W^u(\tilde{\Lambda}_j) \cap W^s(\Lambda(4))$ and $y \in W^u(\Lambda(4)) \cap W^s(\tilde{\Lambda}_j)$ such that $\mathcal{O}(x) \cup \mathcal{O}(y) \subset \Lambda_{t-\epsilon/2+\delta/2}$. Without loss of generality suppose that there is no $x \in W^u(\tilde{\Lambda}_j) \cap W^s(\Lambda(4))$ with $m_{\varphi,f}(x) \leq t - \epsilon/2 + \delta/2$ (the argument for the other case is similar). We will show that this condition forces the possible letters that may appear in the sequences that determine the unstable Cantor set of $\tilde{\Lambda}_j$.

Let us begin fixing $R \in \mathbb{N}$ large enough such that $1/2^{R-1} < \delta/2$ and consider the set $\mathcal{C}_{2R+1} = \{I(a_0; a_1, \dots, a_{2R+1}) : I(a_0; a_1, \dots, a_{2R+1}) \cap K^u(\tilde{\Lambda}_j) \neq \emptyset\}$, clearly $\mathcal{C}_{2R+1}$ is a covering of $K^u(\tilde{\Lambda}_j)$. We will give a mechanism to construct coverings $\mathcal{C}_k$ with $k \geq 2R+1$ that can be used to *efficiently* cover $K^u(\tilde{\Lambda}_j)$ as k goes to infinity.

Indeed, if for some $k \geq 2R+1$, and $I(a_0; a_1, \dots, a_k) \in \mathcal{C}_k$, $(a_0, a_1, \dots, a_k)$ has continuations with forced first letter. That is, for every $\alpha = (\alpha_n)_{n \in \mathbb{Z}} \in \Pi(\tilde{\Lambda}_j)$ with $\alpha_0, \alpha_1, \dots, \alpha_k = a_0, a_1, \dots, a_k$ one has $\alpha_{k+1} = a_{k+1}$ for some fixed a_{k+1}; then we can refine the original cover $\mathcal{C}_k$, by replacing the interval $I(a_0; a_1, \dots, a_k)$ by the interval $I(a_0; a_1, \dots, a_k, a_{k+1})$.

On the other hand, if $(a_0, a_1, \dots, a_k)$ has two continuations with different initial letter, said $\gamma_{k+1} = (a_{k+1}, a_{k+2}, \dots)$ and $\beta_{k+1} = (a^*_{k+1}, a^*_{k+2}, \dots)$ with $a_{k+1} \neq a^*_{k+1}$. Take $\alpha = (\alpha_n)_{n \in \mathbb{Z}} \in \Pi(\tilde{\Lambda}_j)$ and $\tilde{\alpha} = (\tilde{\alpha}_n)_{n \in \mathbb{Z}} \in \Pi(\tilde{\Lambda}_j)$, such that $\alpha = (\dots, \alpha_{-2}, \alpha_{-1}; a_0, a_1, \dots, a_k, \gamma_{k+1})$ and $\tilde{\alpha} = (\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; a_0, a_1, \dots, a_k, \beta_{k+1})$. If $a_{k+1} = i$ then, necessarily either $a^*_{k+1} = i+1$ or $a^*_{k+1} = i-1$ because if for example $a_{k+1} + 1 < a^*_{k+1}$ we can set $s = a_{k+1} + 1$ and therefore by lemma 5.2.4 as $[0; \beta_{k+1}] < [0; s, \overline{1}] < [0; \gamma_{k+1}]$, we would have for all $j \leq k$

$$\begin{aligned} \lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, s, \overline{1})) &\leq \max\{m(\dots, \alpha_{-1}; \alpha_0, \dots, \alpha_k, \gamma_{k+1}), \\ &\quad m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, \beta_{k+1})\} + 1/2^{R-2} \\ &< t - \epsilon/2 + \delta/2. \end{aligned}$$

For $j = k+1$,

$$\begin{aligned}
\lambda_0(\sigma^j(\ldots,\tilde{\alpha}_{-2},\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,s,\overline{1})) &= [0;\tilde{\alpha}_k,\ldots,\tilde{\alpha}_0,\tilde{\alpha}_{-1},\ldots]+s+[0;\overline{1}]\\
&< [0;\tilde{\alpha}_k,\ldots,\tilde{\alpha}_0,\tilde{\alpha}_{-1},\ldots]+s+1\\
&< [0;\tilde{\alpha}_k,\ldots,\tilde{\alpha}_0,\tilde{\alpha}_{-1},\ldots]+a^*_{k+1}\\
&\quad +[0;a^*_{k+2},a^*_{k+3},\ldots]\\
&= \lambda_0(\sigma^{k+1}(\ldots,\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,\beta_{k+1}))\\
&\leq m(\ldots,\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,\beta_{k+1})\\
&\leq t-\epsilon/2
\end{aligned}$$

and for $j > k+1$, clearly

$$\lambda_0(\sigma^j(\ldots,\tilde{\alpha}_{-2},\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,s,\overline{1})) < 3 < t-\epsilon/2.$$

Then taking $x = \Pi^{-1}((\ldots,\tilde{\alpha}_{-2},\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,s,\overline{1}))$ one would have

$$x \in W^u(\tilde{\Lambda}_j)\cap W^s(\Lambda(4)) \text{ and } m_{\varphi,f}(x) \leq t-\epsilon/2+\delta/2$$

that is a contradiction.

The case $a_{k+1}-1 > a^*_{k+1}$ is quite similar. Indeed if we set $s = a_{k+1}-1$ therefore by lemma 5.2.4 as $[0;\gamma_{k+1}] < [0;s,\overline{1}] < [0;\beta_{k+1}]$, we would have for all $j \leq k$

$$\begin{aligned}
\lambda_0(\sigma^j(\ldots,\alpha_{-2},\alpha_{-1};\alpha_0,\ldots,\alpha_k,s,\overline{1})) &\leq \max\{m(\ldots,\alpha_{-1};\alpha_0,\ldots,\alpha_k,\gamma_{k+1}),\\
&\quad m(\ldots,\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,\beta_{k+1})\}+1/2^{R-2}\\
&< t-\epsilon/2+\delta/2.
\end{aligned}$$

For $j = k+1$,

$$\begin{aligned}
\lambda_0(\sigma^j(\ldots,\alpha_{-2},\alpha_{-1};\alpha_0,\ldots,\alpha_k,s,\overline{1})) &= [0;\alpha_k,\ldots,\alpha_0,\alpha_{-1},\ldots]+s+[0;\overline{1}]\\
&< [0;\alpha_k,\ldots,\alpha_0,\alpha_{-1},\ldots]+s+1\\
&< [0;\alpha_k,\ldots,\alpha_0,\alpha_{-1},\ldots]+a_{k+1}\\
&\quad +[0;a_{k+2},a_{k+3},\ldots]\\
&= \lambda_0(\sigma^{k+1}(\ldots,\alpha_{-1};\alpha_0,\ldots,\alpha_k,\gamma_{k+1}))\\
&\leq m(\ldots,\alpha_{-1};\alpha_0,\ldots,\alpha_k,\gamma_{k+1})\\
&\leq t-\epsilon/2
\end{aligned}$$

and for $j > k+1$,

$$\lambda_0(\sigma^j(\ldots,\alpha_{-2},\alpha_{-1};\alpha_0,\ldots,\alpha_k,s,\overline{1})) < 3 < t-\epsilon/2$$

then taking $x = \Pi^{-1}((\dots, \alpha_{-2}, \alpha_{-1}; \alpha_0, \dots, \alpha_k, s, \overline{1}))$ one would have

$$x \in W^u(\tilde{\Lambda}_j) \cap W^s(\Lambda(4)) \text{ and } m_{\varphi,f}(x) \le t - \epsilon/2 + \delta/2$$

that is again a contradiction.

Now, suppose $a_{k+1} = i$ and $a^*_{k+1} = i + 1$. We affirm that $a_{k+2} = 1$ because in other case by lemma 5.2.4, as $[0; \beta_{k+1}] < [0; i, \overline{1}] < [0; \gamma_{k+1}]$, we would have again for all $j \le k$

$$\lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i, \overline{1})) < t - \epsilon/2 + \delta/2.$$

For $j > k + 1$, one more time

$$\lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i, \overline{1})) < t - \epsilon/2$$

and for $j = k + 1$,

$$\begin{aligned}
\lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i, \overline{1})) &= [0; \tilde{\alpha}_k, \dots, \tilde{\alpha}_0, \tilde{\alpha}_{-1}, \dots] + i + [0; \overline{1}] \\
&< [0; \tilde{\alpha}_k, \dots, \tilde{\alpha}_0, \tilde{\alpha}_{-1}, \dots] + i + 1 \\
&< [0; \tilde{\alpha}_k, \dots, \tilde{\alpha}_0, \tilde{\alpha}_{-1}, \dots] + a^*_{k+1} \\
&\quad + [0; a^*_{k+2}, a^*_{k+3}, \dots] \\
&= \lambda_0(\sigma^{k+1}(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, \beta_{k+1})) \\
&\le m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, \beta_{k+1}) \\
&< t - \epsilon/2.
\end{aligned}$$

Then for $x = \Pi^{-1}((\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i, \overline{1}))$ one would get the contradiction

$$x \in W^u(\tilde{\Lambda}_j) \cap W^s(\Lambda(4)) \text{ and } m_{\varphi,f}(x) \le t - \epsilon/2 + \delta/2.$$

In a similar way, we must have $a^*_{k+2} \in \{m+1, m+2, m+3\}$ because if $a^*_{k+2} = \ell \le m$, then $[0; \beta_{k+1}] < [0; i+1, \ell+1, \overline{1}] < [0; \gamma_{k+1}]$ and by lemma 5.2.4 we would have for all $j \le k$

$$\lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i+1, \ell+1, \overline{1})) < t - \epsilon/2 + \delta/2.$$

For $j = k + 1$,

$$\begin{aligned}
\lambda_0(\sigma^j(\dots, \tilde{\alpha}_{-2}, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, i+1, \ell+1, \overline{1})) &= [0; \tilde{\alpha}_k, \dots, \tilde{\alpha}_0, \tilde{\alpha}_{-1}, \dots] + i + 1 + \\
&\quad [0; \ell + 1, \overline{1}] \\
&< [0; \tilde{\alpha}_k, \dots, \tilde{\alpha}_0, \tilde{\alpha}_{-1}, \dots] + a^*_{k+1} \\
&\quad + [0; a^*_{k+2}, a^*_{k+3}, \dots] \\
&= \lambda_0(\sigma^{k+1}(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, \beta_{k+1})) \\
&\le m(\dots, \tilde{\alpha}_{-1}; \tilde{\alpha}_0, \dots, \tilde{\alpha}_k, \beta_{k+1}) \\
&\le t - \epsilon/2
\end{aligned}$$

and for $j > k+1$,

$$\lambda_0(\sigma^j(\ldots,\tilde{\alpha}_{-2},\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,i{+}1,\ell{+}1,\overline{1})) < m{+}1{+}2[0;\overline{1,m+3}] = \max f|_\Lambda{-}2 < t{-}\epsilon$$

then taking $x = \Pi^{-1}((\ldots,\tilde{\alpha}_{-2},\tilde{\alpha}_{-1};\tilde{\alpha}_0,\ldots,\tilde{\alpha}_k,i+1,\ell+1,\overline{1}))$ one would have

$$x \in W^u(\tilde{\Lambda}_j) \cap W^s(\Lambda(4)) \text{ and } m_{\varphi,f}(x) \le t - \epsilon/2 + \delta/2$$

that is again a contradiction.

In particular, in this case, we can refine the cover $\mathcal{C}_k$ by replacing the interval $I(a_0;a_1,\ldots,a_k)$ with the four intervals $I(a_0;a_1,\ldots,a_k,i,1), I(a_0;a_1,\ldots,a_k,i+1,m+1), I(a_0;a_1,\ldots,a_k,i+1,m+2)$ and $I(a_0;a_1,\ldots,a_k,i+1,m+3)$ for one and only one $i = 1,\ldots,m+2$.

Observe that, in fact, some of the intervals considered in the last paragraph, maybe not be possible. For example, if $t \in \mathbb{N}$ then $t = m+3$ and so $t-\epsilon/2 < m+3$; therefore the letter $m+3$ cannot appear in the kneading sequence of any point of $\tilde{\Lambda}_j$. But this will not affect our argument. Indeed, we affirm that this procedure doesn't increase the 0.55-sum, $H_{0.55}(\mathcal{C}_k) = \sum\limits_{I\in\mathcal{C}_k} |I|^{0.55}$ of the cover $\mathcal{C}_k$ of $K^u(\tilde{\Lambda}_j)$. That is, by 5.2.1 we need to prove that

$$|I(a_1,\ldots,a_k,i,1)|^{0.55} + \sum_{j=m+1}^{m+3} |I(a_1,\ldots,a_k,i+1,j)|^{0.55} < |I(a_1,\ldots,a_k)|^{0.55}$$

or

$$\left(\frac{|I(a_1,\ldots,a_k,i,1)|}{|I(a_1,\ldots,a_k)|}\right)^{0.55} + \sum_{j=m+1}^{m+3}\left(\frac{|I(a_1,\ldots,a_k,i+1,j)|}{|I(a_1,\ldots,a_k)|}\right)^{0.55} < 1 \tag{5.3.3}$$

where $i = 1,\ldots,m+2$.

In this direction, we have the following lemma

Lemma 5.3.3. *Given $a_0,a_1,\ldots,a_n,a,b \in \{1,\ldots,m+3\}$ we have*

$$\frac{|I(a_1,\ldots,a_n,a,b)|}{|I(a_1,\ldots,a_n)|} = \frac{1+r}{(1+ab+r)(1+(1+b)a+(b+1)r)}$$

where $r \in (0,1)$.

Proof. Recall that the length of $I(b_1, \ldots, b_m)$ is

$$|I(b_1, \ldots, b_m)| = \frac{1}{q_m(q_m + q_{m-1})},$$

where q_s is the denominator of $[0; b_1, \ldots, b_s]$. And that also, we have the recurrence formula

$$q_{s+2} = b_{s+2}q_{s+1} + q_s.$$

Using this two times, we have

$$|I(a_1, \ldots, a_n, a, b)| = \frac{1}{((1+ab)q_n + bq_{n-1})((1+(1+b)a)q_n + (b+1)q_{n-1})}$$

and then

$$\begin{aligned} \frac{|I(a_1, \ldots, a_n, a, b)|}{|I(a_1, \ldots, a_n)|} &= \frac{q_n(q_n + q_{n-1})}{((1+ab)q_n + bq_{n-1})((1+(1+b)a)q_n + (b+1)q_{n-1})} \\ &= \frac{1+r}{(1+ab+r)(1+(1+b)a+(b+1)r)} \end{aligned}$$

with $r = \frac{q_{n-1}}{q_n} \in (0,1)$. ∎

Using this lemma, we have for $i = 1, \ldots, m+2$ and some $r \in (0,1)$

$$\begin{aligned} &\left(\frac{|I(a_1, \ldots, a_k, i, 1)|}{|I(a_1, \ldots, a_k)|}\right)^{0.55} + \sum_{j=m+1}^{m+3} \left(\frac{|I(a_1, \ldots, a_k, i+1, j)|}{|I(a_1, \ldots, a_k)|}\right)^{0.55} = \\ &\left(\frac{1+r}{(1+i+r)(1+2i+2r)}\right)^{0.55} + \\ &\sum_{j=m+1}^{m+3} \left(\frac{1+r}{(1+(i+1)j+r)(1+(j+1)(i+1)+(1+j)r)}\right)^{0.55} < \\ &\left(\frac{2}{2 \times 3}\right)^{0.55} + 3\left(\frac{2}{(1+2(m+1))(1+2(m+2))}\right)^{0.55} < \\ &\left(\frac{1}{3}\right)^{0.55} + 3\left(\frac{2}{9 \times 11}\right)^{0.55} < 0.9 \quad \text{(because } m \geq 3) \end{aligned}$$

that proves 5.3.3 and so let us conclude that

$$HD(K^u(\tilde{\Lambda}_j)) < 0.55.$$

As $HD(K^u(\Lambda(4))) = HD(C_4) \geq 0.7862\ldots$ (see [4]) and we are in the conservative setting

$$HD(\tilde{\Lambda}_j) = 2HD(K^u(\tilde{\Lambda}_j)) < 2HD(K^u(\Lambda(4))) = HD(\Lambda(4)).$$

Also, because $\Lambda(4)$ is a subhorseshoe of Λ, $m \geq 3$ and $m+1+2[0;\overline{1, m+3}] < t-\epsilon$, we have $\Lambda(4) \subset \Lambda_{t-\epsilon}$ and then we can find some $i \in \mathcal{I}$ such that $\Lambda(4) \subset \tilde{\Lambda}_i$. That proves 5.3.2 because

$$HD(\bigcup_{j\in\mathcal{J}} \tilde{\Lambda}_j) = \max_{j\in\mathcal{J}} HD(\tilde{\Lambda}_j) < HD(\Lambda(4)) \leq HD(\bigcup_{i\in\mathcal{I}} \tilde{\Lambda}_i).$$

5.3.3 Putting unstable Cantor sets into $k^{-1}(t)$

Now, we will see that for every $i \in \mathcal{I}$ we can construct a local homeomorphism $\theta_i : K^u(\tilde{\Lambda}_i) \to k^{-1}(t)$ with local Holder inverse and exponent arbitrarily close to one.

For fixing ideas take $\delta = \epsilon/4$. By definition, for $i \in \mathcal{I}$ we can find some subhorseshoe $\tilde{\Lambda}(i) \subset \Lambda$ and some $q(i) < t - \epsilon/4$ with $\tilde{\Lambda}_i \cup \Lambda(4) \subset \tilde{\Lambda}(i) \subset \Lambda_{q(i)}$. Being $\tilde{\Lambda}(i)$ a mixing horseshoe (because $\Lambda(4)$ is), we can find some $c = c(\tilde{\Lambda}(i)) \in \mathbb{N}$ such that given two letters a and b in the alphabet $\mathcal{A}(\tilde{\Lambda}(i))$ of $\tilde{\Lambda}(i)$ there exists some finite word of size c: $(a_1, \ldots, a_c)$ (in the letters $\mathcal{A}(\tilde{\Lambda}(i))$) such that $(a, a_1, \ldots, a_c, b)$ is admissible; given a and b consider always a fixed $(a_1, \ldots, a_c)$ as before. Also, as $\tilde{\Lambda}(i)$ is a subhorseshoe of Λ, it is the invariant set in some rectangles determined for a set of words of size $2n(i)+1$ for some $n(i) \in \mathbb{N}$.

Fix $r_0 \in \mathbb{N}$ big enough such that $(r_0-1)! > n(i)$. Given $a = [a_0; a_1, a_2, \ldots] \in K^u(\tilde{\Lambda}_i)$ for $r \geq r_0$ set $a^{(r)} := (a_{(r-1)!+1}, \ldots, a_{r!})$, so one has

$$a = [a_0; a_1, a_2, \ldots] = [a_0; a_1, a_2, \ldots a_{(r_0-1)!}, a^{(r_0)}, a^{(r_0+1)}, \ldots, a^{(n)}, \ldots].$$

Also, take $n \in \mathbb{N}$ large enough such that $n > n(i)$ and $1/2^{n-3} < t - q(i)$ and consider also the family of words

$$\{h_r\}_{r\geq r_0} = \{(c_r, \underbrace{1, \ldots, 1}_{2n\ times}, \tilde{c}_r)\}_{r\geq r_0}$$

where c_r and $\tilde{c}_r$ are the words of size c in the original alphabet $\mathcal{A} = \{1, \ldots, m+3\}$ such that

$$\tilde{a} = [a_0; a_1, a_2, \ldots a_{(r_0-1)!}, a^{(r_0)}, h_{r_0}, a^{(r_0+1)}, h_{r_0+1}, \ldots, h_{n-1}, a^{(n)}, h_n, \ldots] \in K^u(\tilde{\Lambda}(i)).$$

Now, as $C_4 + C_4 = \{\alpha + \beta : \alpha, \beta \in C_4\} = [\sqrt{2} - 1, 4(\sqrt{2} - 1)]$, there are $n \in \{m + 2, m+3\}$ and $x = [0; x_1, x_2, x_3, \dots], y = [0; y_1, y_2, y_3, \dots] \in C_4$ such that $t = n + x + y$. Consider for $r \geq r_0$ the modification of h_r given by

$$\tilde{h}_r = (c_r, \underbrace{1, \dots, 1}_{n\ times}, x_r, \dots, x_1, n, y_1, \dots, y_r, \underbrace{1, \dots, 1}_{n\ times}, \tilde{c}_r)$$

Define then for $a = [a_0; a_1, a_2, \dots]$

$$\theta_i(a) := [a_0; a_1, a_2, \dots a_{(r_0-1)!}, a^{(r_0)}, \tilde{h}_{r_0}, a^{(r_0+1)}, \tilde{h}_{r_0+1}, \dots, \tilde{h}_{n-1}, a^{(n)}, \tilde{h}_n, \dots].$$

Using lemma 5.2.1 and the construction of θ_i it is easy to see that for every $a \in K^u(\tilde{\Lambda}_i)$, $k(\theta_i(a)) = t$, so we have defined the map

$$\begin{array}{rcl} \theta_i : K^u(\tilde{\Lambda}_i) & \to & k^{-1}(t) \\ a & \to & \theta_i(a) \end{array}$$

that is clearly continuous and injective.

On the other hand, given any small $\rho > 0$ because of the growth of the factorial map, we have $|a_1 - a_2| = O(|\theta_i(a_1) - \theta_i(a_2)|^{1-\rho})$ for any $a_1, a_2 \in K^u(\tilde{\Lambda}_i)$ and $|a_1 - a_2|$ small. Indeed, if a_1 and a_2 are such that the letters in their continued fraction expressions are equal up to the s-nth letter (we suppose $s > r_0!$) and $k \in \mathbb{N}$ is maximal such that $(k-1)! < s$ then because $|\tilde{h}_r| = 2n + 2c + 2r + 1$; $\theta_i(a_1)$ and $\theta_i(a_2)$ coincide exactly in their first

$$s + \sum_{r=r_0}^{k-1} 2n + 2c + 2r + 1 = s + p(k)$$

letters, where p is a fixed polynomial.

So if s is big such that $s/(s + p(k)) > 1 - \rho$, using lemmas 5.2.1, 5.2.2 and 5.2.3 we have for some constant $\tilde{C}(m + 3)$

$$\begin{array}{rcl} |\theta_i(a_1) - \theta_i(a_2)|^{1-\rho} & > & \dfrac{\tilde{C}(m+3)^{1-\rho}}{2^{(s+p(k))(1-\rho)}} \\ & > & \dfrac{\tilde{C}(m+3)^{1-\rho}}{2^s} \\ & > & \dfrac{\tilde{C}(m+3)^{1-\rho}}{2}|a_1 - a_2|. \end{array}$$

Therefore the map $\theta_i^{-1}:\theta_i(K^u(\tilde\Lambda_i))\to K^u(\tilde\Lambda_i)$ is locally a Holder map with exponent $1-\rho$ and then

$$\begin{aligned} HD(K^u(\tilde\Lambda_i)) = HD(\theta_i^{-1}(\theta_i(K^u(\tilde\Lambda_i)))) &\leq 1/(1-\rho)HD(\theta_i(K^u(\tilde\Lambda_i)))\\ &\leq 1/(1-\rho)HD(k^{-1}(t)). \end{aligned}$$

Letting ρ go to zero, we obtain

$$HD(K^u(\tilde\Lambda_i)) \leq HD(k^{-1}(t)).$$

Now, in [16] was proved for $s\leq \max f|_\Lambda$

$$HD(k^{-1}(-\infty,s]) = HD(K_s^u)$$

and by theorem 5.2.5, we have that

$$HD(K_s^u) = \frac{1}{2}HD(\Lambda_s).$$

Therefore, if $i_0\in\mathcal{I}$ is such that $HD(M)=HD(\tilde M)=HD(\tilde\Lambda_{i_0})$, one has

$$\begin{aligned} HD(k^{-1}(-\infty,t-\epsilon]) = \frac{1}{2}HD(\Lambda_{t-\epsilon}) &\leq \frac{1}{2}HD(M) = \frac{1}{2}HD(\tilde\Lambda_{i_0})\\ &= HD(K^u(\tilde\Lambda_{i_0})) \leq HD(k^{-1}(t)). \end{aligned}$$

Letting ϵ tend to zero we have

$$HD(k^{-1}(-\infty,t]) \leq HD(k^{-1}(t))$$

and as the other inequality is clearly true, the second part of the result is proven.

For the first part of the theorem, recall that the pressure of ψ and a potential ϕ is given by

$$P(\psi,\phi) = \sup\{h_\mu(\psi) + \int \phi d\mu : \mu \text{ is an invariant measure}\}. \tag{5.3.4}$$

Moreover, by the Ergodic Decomposition Theorem and Jacobson's theorem, the last supremum can be taken on the ergodic invariant measure. We say that a measure m is an *equilibrium state* if the supremum is attained in 5.3.4. When ψ is $C^{1+\alpha}$ and the potential $\phi=-s\log|\psi'|$, we know that (cf. [23, theorem 20.1]) there exists a unique equilibrium measure and it is equivalent to the d-dimensional Hausdorff measure, where d is the Hausdorff dimension of Cantor set defined by ψ.

The proof of the following lemma is essentially the same as the proof of lemma 2.5 of [10]. We include it here for completeness.

Lemma 5.3.4. *Given $(K, \mathcal{P}, \psi)$ a C^s-regular Cantor set, if $\mathcal{P}' \neq \mathcal{P}$ is a finite sub collection of $\mathcal{P}$ that is also a Markov partition of ψ, then the Cantor set determined by ψ and $\mathcal{P}'$*

$$\tilde{K} = \bigcap_{n \geq 0} \psi^{-n} \left(\bigcup_{I \in \mathcal{P}'} I \right)$$

satisfies $HD(\tilde{K}) < HD(K)$.

Proof. Let $d = HD(K)$ and m_d be the Hausdorff dimension of K and the d-dimensional Hausdorff measure, respectively. We know that $m_d(K) > 0$ and there exists $c > 0$ such that, for all $x \in K$ and $0 < r \leq 1$ (cf. [25, proposition 3]),

$$c^{-1} \leq \frac{m_d((x-r, x+r) \cap K)}{r^d} \leq c.$$

Moreover, if μ denotes the unique equilibrium measure corresponding to the Holder continuous potential $-s \log|\psi'|$, we have that m_d is equivalent to μ (cf. [23, pag. 203]).

By uniqueness, μ is an ergodic invariant measure for ψ. Consider for $x \in K$

$$\tau\Big(\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K; x\Big) = \lim_{n \to \infty} \frac{\left| \left\{ 0 \leq j \leq n-1 : \psi^j(x) \in \bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K \right\} \right|}{n}.$$

From the Birkhoff's Ergodic Theorem, $\tau(\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K; x) = \mu(\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K)$ for μ-almost every $x \in K$. Take $y \in \bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K$ and any interval $\tilde{I} \subset \bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I$ centered at y. Note that $\mu(\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K) > 0$ because

$$m_d\Big(\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I \cap K\Big) \geq m_d(\tilde{I} \cap K) \geq \frac{c^{-1}}{2^d} |\tilde{I}|^d > 0.$$

This implies that the set of points in K which never visit $\bigcup_{I \in \mathcal{P} \setminus \mathcal{P}'} I$ has measure zero. Note that this set contains $\tilde{K}$. Since μ is equivalent to m_d, we have $m_d(\tilde{K}) = 0$. On the other hand, if $\tilde{d} = HD(\tilde{K})$ the one has $m_{\tilde{d}}(\tilde{K}) > 0$. So $HD(\tilde{K}) < d$.

$\square$

Corollary 5.3.5. *Let Λ be a mixing horseshoe associated with a C^2-diffeomorphism $\varphi : S \to S$ of some surface S. Then for any proper mixing subhorseshoe $\tilde{\Lambda} \subset \Lambda$*

$$HD(\tilde{\Lambda}) < HD(\Lambda).$$

Proof. Refine the original Markov partition $\mathcal{P}$ of Λ in such a way that some $\mathcal{P}' \subset \mathcal{P}$, $\mathcal{P}' \neq \mathcal{P}$ is a Markov partition for $\tilde{\Lambda}$. Use the lemma 5.3.4 with the maps ψ_s and ψ_u that define the stable and unstable Cantor sets, in order to obtain

$$HD(\tilde{\Lambda}) = HD(K^s(\tilde{\Lambda})) + HD(K^u(\tilde{\Lambda})) < HD(K^s(\Lambda)) + HD(K^u(\Lambda)) = HD(\Lambda).$$

□

Now, consider one more time the subhorseshoe $\tilde{\Lambda}(i_0) \subset \Lambda_{q(i_0)} \subset \Lambda_{t-\epsilon/4}$. Because $t \geq 6$ we have

$$m(\Pi^{-1}(\dots, x_2 x_1; n, y_1, y_2, \dots)) = t.$$

Calling $x = \Pi^{-1}(\dots, x_2 x_1; n, y_1, y_2, \dots)$, as $\Lambda(4) \subset \tilde{\Lambda}(i_0)$ one has

$$x \in W^s(\tilde{\Lambda}(i_0)) \cap W^u(\tilde{\Lambda}(i_0)),$$

as also $\mathcal{O}(x) \cup \tilde{\Lambda}(i_0) \subset \Lambda_t$, then by proposition 5.3.2 we have the existence of some subhorseshoe $\tilde{\Lambda}$ with

$$\mathcal{O}(x) \cup \tilde{\Lambda}(i_0) \subset \tilde{\Lambda} \subset \Lambda_{t+\epsilon}.$$

So, as $\tilde{\Lambda}(i_0)$ is a proper subhorseshoe of $\tilde{\Lambda}$

$$\begin{aligned} HD(k^{-1}(-\infty, t-\epsilon]) = \frac{1}{2} HD(\Lambda_{t-\epsilon}) &\leq \frac{1}{2} HD(\tilde{\Lambda}(i_0)) < \frac{1}{2} HD(\tilde{\Lambda}) \\ &\leq \frac{1}{2} HD(\Lambda_{t+\epsilon}) = HD(k^{-1}(-\infty, t+\epsilon]). \end{aligned}$$

And then the map is locally strictly monotone, which ends the proof of the result.

Final Considerations

Now, we discuss some questions related to our previous work which remain open and which can lead to future research.

The behavior of the map L close to the first accumulation point of the Lagrange spectra remains unknown. We think that generically there is some neighborhood of that point where the map L is continuous. Even more, we also think that the hypothesis of being close to a conservative diffeomorphism is unnecessary and then, that generically close to every diffeomorphism with a mixing horseshoe it is true that the map L has finitely many discontinuities.

In the same direction, we proved proposition 3.3.1, showing that we can take δ depending only on the value of ϵ and c_0 and not on the compact φ-invariant set. We think that this kind of result can help to prove finiteness of discontinuities of the map L far away from the first accumulation point in the general setting. Then, it is natural to ask for a similar computation that let us estimate the modulus of continuity (i.e. the value of δ in terms of ϵ and c_0) when the horseshoe has Hausdorff dimension greater than one. That is, when we use the notion of *critical windows* instead of the notion of *good positions*.

The study of the discontinuities of the map M is of great interest also. The methods that we used to study the map L seem don't have natural extensions in this case because it is not possible to express the Hausdorff dimension of the set of Markov values attained in a transient set in term of the Hausdorff dimension of the associated subhorseshoes and also because we couldn't find a satisfactory notion of "one-side connection" that would allow us to establish some analog result of proposition 3.3.12 in this case. We hope that similar results to those for L can be established for M in the future.

A natural question we can ask is the validity of the theorem proved in the last chapter for $t \geq c$ where c is the Freiman constant $c = \frac{2221564096+283748\sqrt{462}}{491993569} \simeq 4.52782956\ldots$. We don't know how to solve that question. However, we think it is possible to improve theorem 5.1.1 by modifying our arguments and show the same for $t \geq 4 + [0; \overline{1,3}] + [0; 3, 4, \overline{1,3}] = 4 + \frac{1}{2}(\sqrt{21} - 3) + \frac{1}{50}(\sqrt{21} + 11) = 5.102939\ldots$. We hope to publish that in future work.

Appendix A

Hyperbolic sets of finite type

Given a horseshoe Λ, we know that there is a complete subshift of finite type $\Sigma(\mathcal{B})$ and a homeomorphism $\Pi : \Lambda \to \Sigma(\mathcal{B})$ such that $\varphi \circ \Pi = \Pi \circ \sigma$ as explained before. Take a finite collection X of finite admissible words $\alpha = (a_0, a_1, \dots, a_n)$, we said that the maximal invariant set

$$M(X) = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\bigcup_{\alpha\in X} R(\alpha; 0))$$

is a *hyperbolic set of finite type*. Even more, it is said to be a *subhorseshoe* of Λ if it is nonempty and $\varphi|_{M(X)}$ is transitive. Observe that a subhorseshoe need not be a horseshoe; indeed, it could be a periodic orbit in which case it will be called of trivial.

By definition, hyperbolic sets of finite type have local product structure. In fact, any hyperbolic set of finite type is a locally maximal invariant set of a neighborhood of a finite number of elements of some Markov partition of Λ:

If X is as before and $n(X) = \max_{\alpha\in X}|\alpha|$, then the set $\tilde{X}$ of admissible words $\tilde{\alpha} = (a_{-n(X)}, \dots, a_0, \dots, a_{n(X)})$ such that $\tilde{\alpha} = \alpha_1\alpha_2\alpha_3$ where the words $\alpha_1, \alpha_2, \alpha_3$ are admissible and for some n, $\alpha_2 = (a_0, a_1, \dots, a_n) \in X$, satisfies

$$M(X) = \bigcap_{n\in\mathbb{Z}} \varphi^{-n}(\bigcup_{\tilde{\alpha}\in\tilde{X}} R(\alpha; 0)).$$

Suppose then, without loss of generality, that $X \subset \mathcal{A}$. We set $A = A(X)$ as the matrix with entries $A_{\alpha,\beta}$ defined by $A_{\alpha,\beta} = 1$ if the letters of X, α and β are such that $\alpha\beta$ is admissible and $A_{\alpha,\beta} = 0$ otherwise.

Let $\Sigma_X = \{\underline{\alpha} = (\alpha_n)_{n\in\mathbb{Z}} : \alpha_n \in X \text{ for all } n \in \mathbb{Z}\}$ equipped with the usual shift $\sigma : \Sigma_X \to \Sigma_X$. Consider $\Sigma_A = \{\underline{\alpha} = (\alpha_n)_{n\in\mathbb{Z}} \in \Sigma_X : A_{\alpha_n,\alpha_{n+1}} = 1\}$, this set is closed and σ-invariant subspace of Σ_X. The pair (Σ_A, σ) is the two-side subshift of finite type associated to A in the alphabet X.

Given $\alpha, \beta \in X$, we said that α is related to β if for some $k, \ell > 0$, $(A^k)_{\alpha,\beta} > 0$ and $(A^\ell)_{\beta,\alpha} > 0$. This corresponds to having a path from α to β and a path from β to α in the graph G_A that have as vertex set, the set X and as transition matrix, the matrix A. We said $\alpha \in X$ is a transient state if α is not related to itself, i.e there is no path from it to itself. In this context, the set Σ_A can be identified as the set of infinite paths in the graph G_A.

We said that A is irreducible if for every $\alpha, \beta \in X$ there exists some $\ell \in \mathbb{N}$ with $(A^\ell)_{\alpha,\beta} > 0$. Equivalently, the matrix A is irreducible when it is possible to connect by a path each pair of vertex in the graph G_A.

Using the above relation, we can divide X into transients states and a collection of disjoint classes that determine some submatrices of A. More precisely, it follows from theorem 1.3.10 of [11] that there is a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} A_1 & * & * & \dots & * \\ 0 & A_2 & * & \dots & * \\ 0 & 0 & A_3 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_m \end{bmatrix}$$

where each A_i is a transition irreducible matrix that we call of transitive component of A or the one by one matrix $[0]$ corresponding to a transient state. If $A_{i_1}, \dots, A_{i_r}$ are the transitive components of A, we said that $\Sigma_{A_{i_1}}, \dots, \Sigma_{A_{i_r}} \subset \Sigma_A$ are the transitive components of Σ_A.

Next, we write observation 5.1.2 of [11]

Proposition A.0.1. *For $x \in \Sigma_A$, the following holds:*

- *the positive and negative limit sets of x, $\omega(x)$ and $\alpha(x)$ are each contained in a transitive component of Σ_A,*
- *x is nonwandering if and only if it belongs to a transitive component of Σ_A,*
- *x is nonwandering if and only if $\omega(x) \cup \alpha(x)$ is a subset of some transitive component of Σ_A.*

By proposition A.0.1, if some $x \in \Sigma_A$ doesn't belong to any transitive component of Σ_A, then x is nonwandering and there are different transitive components $\Sigma_{A_{i_a}}$ and $\Sigma_{A_{i_b}}$ such that $\omega(x) \subset \Sigma_{A_{i_a}}$ and $\alpha(x) \subset \Sigma_{A_{i_b}}$.

Definition A.0.2. Any $\tau \subset M(X)$ for which there are two different subhorseshoes Λ_1 and Λ_2 of Λ such that

$$\tau = \{x \in M(X): \ \omega(x) \subset \Lambda_1 \text{ and } \alpha(x) \subset \Lambda_2\}$$

will be called a transient set or transient component of $M(X)$.

Note that by the local product structure, given a transient set τ as before,

$$HD(\tau) = HD(K^s(\Lambda_2)) + HD(K^u(\Lambda_1)). \tag{A.0.1}$$

From the last discussion, we can recover a decomposition of the set $\Pi(M(X))$ and then for the set $M(X)$

Proposition A.0.3. *Any hyperbolic set of finite type $M(X)$, associated with a finite collection of finite admissible words X, can be written as*

$$M(X) = \bigcup_{i \in \mathcal{I}} \tilde{\Lambda}_i$$

where $\mathcal{I}$ is a finite index set (that may be empty) and for $i \in \mathcal{I}$, $\tilde{\Lambda}_i$ is a subhorseshoe or a transient set.

Bibliography

[1] T. W. Cusick and M. E. Flahive. *Markoff and Lagrange Spectra*, volume 30. Mathematical Surveys and Monographs, Amer. Math. Soc., 1989.

[2] P. Arnoux, *Le codage du flot géodésique sur la surface modulaire*, Enseign. Math. (2) 40, no. 1-2, 1994, 29-48.

[3] A. Cerqueira, C. Matheus, C. G. Moreira. *Continuity of Hausdorff dimension across generic dynamical Lagrange and Markov spectra. Journal of Modern Dynamics*, 12:151–174, 2018.

[4] M. Pollicott and P. Vytnova, *Hausdorff dimension estimates applied to Lagrange and Markov spectra, Zaremba theory, and limit sets of Fuchsian groups*, arXiv:2012.07083v3 [math.DS] 17 Jan 2022

[5] M. Hall, *On the sum and product of continued fractions*, Annals of Math., Vol. 48, (1947), pp. 966–993.

[6] C. G. Moreira, *Sums of regular Cantor sets, dynamics and applications to number theory*, Periodica Mathematica Hungarica, v.37, n.1, p.55-63, 1998.

[7] C. G. Moreira, *Conjuntos de Cantor, Dinâmica e Aritmética*, 22o. Colóquio Brasileiro de Matemática, 1999.

[8] D. Lima, C. Matheus, C. G. Moreira and S. Romaña *Classical and Dynamical Markov and Lagrange spectra*, *World Scientific*, 2020.

[9] D. Lima and C. G. Moreira, *Dynamical characterization of initials segments of the Markov and Lagrange spectra*

[10] D. Lima and C. G. Moreira. *Phase transtitions on the Markov and Lagrange dynamical spectra.* Annales de L'Institute Henri Poincaré (C), Analyse nonlineaire, 1–31,2020.

[11] B. P. Kitchens. *Symbolic Dynamics: One-sided, Two-sided and Countable State Markov Shifts*, Universitext, Springer, 1997.

[12] H. McCluskey and A. Manning. *Hausdorff dimension for horseshoes.* Ergodic Theory and Dynamical Systems, 3:251–260, 1983.

[13] G.A. Freiman, *Diophantine approximation and geometry of numbers (The Markoff spectrum)*, Kalininskii Gosudarstvennyi Universitet, Moscow, 1975.

[14] C. G. Moreira. *Geometric properties of images of cartesian products of regular Cantor sets by differentiable real maps,* Preprint (2016) available at arXiv:1611.00933

[15] A. Markoff. *Sur les formes quadratiques binaires indéfinies.* Math.Ann., 15:381–406, 1879.

[16] C. G. Moreira. *Geometric properties of Markov and Lagrange spectra.* Annals of Math., 188: 145–170, 2018.

[17] C. G. Moreira, *On the minima of Markov and Lagrange Dynamical Spectra*, Astérisque No. 415, Quelques aspects de la théorie des systèmes dynamiques: un hommage à Jean-Christophe Yoccoz. I (2020), 45–57.

[18] C. G. Moreira and S. Romaña. *On the Lagrange and Markov Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature.* Ergodic Theory and Dynamical Systems, 133:77–101, 2015.

[19] C. Matheus, C. G. Moreira, M. Pollicott and P. Vytnova, *Hausdorff dimension of Gauss–Cantor sets and two applications to classical Lagrange and Markov spectra*. https://arxiv.org/abs/2106.06572

[20] C. G. Moreira and S. Romaña. *On the Lagrange and Markov dynamical spectra.* https://arxiv.org/pdf/1505.05178.pdf

[21] T. Cusick and M. Flahive, *The Markoff and Lagrange spectra*, *Mathematical Surveys and Monographs*, **30**. American Mathematical Society, Providence, RI, 1989. x+97 pp.

[22] C. G. Moreira and J.-C. Yoccoz. *Tangences homoclines stables pour des ensembles hyperboliques de grande dimension fractale.* Annales Scientifiques de l'École Normale Supérieure, 43:1–68, 2010.

[23] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, University of Chicago Press, 1998.

[24] O. Perron, *Über die Approximation irrationaler Zahlen durch rationale II*, S.-B. Heidelberg Akad. Wiss., Abh. 8, 1921, 12 pp.

[25] J. Palis and F. Takens. *Hyperbolicity and Sensitive chaotic dynamics at homoclinic biifurcations: fractal dimensios and infinitely many attractors*. Cambridge Univ. Press, 1993.

[26] S. Ito, *Number Theoretic expansions, Algorithms and Metrical observations.* Séminaire de Théorie des Nombres de Bordeaux, 1–27,1984.

[27] M. Shub. *Global Stability of Dinamical Systems*. Springer-Verlag, 1986.

[28] A. and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995.

[29] K. Falconer. *Fractal geometry. Mathematical foundations and applications*. John Wiley & Sons Ltd., 1990.