Genuine deformations of Euclidean hypersurfaces in higher codimensions

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Abstract

In this thesis we study isometric rigidity problems from a local viewpoint. Mainly, we study local isometric deformations of submanifolds in higher codimensions.

Sbrana and Cartan locally classified the Euclidean hypersurfaces $M^n \subseteq \mathbb{R}^{n+1}$ which admit another isometric immersion in $\mathbb{R}^{n+1}$. In Chapter 3 we extend their classification to higher codimensions. Our main result is a complete description of the moduli space of genuine deformations of generic hypersurfaces of rank $(p+1)$ in $\mathbb{R}^{n+p}$ for $p \leq n-2$. As a consequence, we obtain an analogous classification to the ones by Sbrana and Cartan providing all local isometric immersions in $\mathbb{R}^{n+2}$ of a generic hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ for $n \geq 4$. We also show how the techniques developed here can be used to study conformally flat Euclidean submanifolds.

Despite that Sbrana and Cartan classifications go back a century, it took almost 90 years to find the first examples of an elusive discrete case; see [14]. We provide examples of hypersurfaces as above by extending the strategy used [14] for the classical theory.

Finally, we analyze Chern-Kuiper's inequalities for an Euclidean submanifold $g : M^n \to \mathbb{R}^{n+p}$. We prove that if the relative nullity does not coincides with the nullity of the curvature tensor then, in several circumstances, g is a composition.

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Summary

CHAPTER 1

Introduction

Nash's Theorem states that any Riemannian manifold M^n can be isometrically immersed into some Euclidean space. The isometric deformation problem is the uniqueness-related question. Namely, to describe the moduli space of isometric immersions $f: M^n \to \mathbb{R}^{n+q}$ that M^n can have for certain q. In the process, given such an f, we need to find a way to meaningfully distinguish it from another one $g: M^n \to \mathbb{R}^{n+p}$. The global case has been solved for $p+q < \min\{5, n\}$ in [17], [28], and [37]. On the other hand, the local problem has a satisfactory description only for $p = q = 1$ and is due to Sbrana [38] and Cartan [4] in the early 20^{th} century. This thesis is dedicated to analyze the isometric deformation problem from a local viewpoint.

Sbrana studied in [38] the local problem of classifying the Riemannian manifolds which possess at least two (locally) non-congruent isometric immersions $f, g: M^n \to \mathbb{R}^{n+1}$. He proved that, if M^n is nowhere flat, then f has rank 2 (that is, it has exactly two non-zero principal curvatures) and it belongs to one of four types. The two non-generic types, the *surface-like* and *ruled* ones, are highly deformable. Namely, a surface-like hypersurface is a product of a surface $L^2 \subseteq \mathbb{R}^3$ (or the cone of a surface $L^2 \subseteq \mathbb{S}^3$) with a Euclidean factor. In this case, the isometric immersions are given by deformations of the surface. A ruled hypersurface has a $(n-1)$-foliation by (open subsets of) affine subspaces of $\mathbb{R}^{n+1}$ and any deformation preserves this foliation. Moreover, straightforward computations show that the moduli space of deformations is in natural bijection with the set $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) := \{\lambda : \mathbb{R} \to \mathbb{R} : \lambda \text{ smooth}\}$. In contrast, a hypersurface f belonging to the *continuous type* has only a continuous one-parameter family of such immersions, while if f is of *discrete type* then it has exactly two (one aside from f). This description was given in terms of what is now called the Gauss parametrization, which parametrizes the hypersurface in terms of its Gauss map h and its support function $\gamma = \langle f, h\rangle$. Sbrana showed that in the continuous and discrete cases, h and γ are solutions of a linear hyperbolic (or elliptic) PDE, and the Gauss map defines a surface of what he called *first* and *second species*. The following table summarizes Sbrana's work.

Rank 2 deformable hypersurfaces	Geometric description of the hypersurface $f: M^n \to \mathbb{R}^{n+1}$	Moduli space of deformations $\mathcal{G} = \{g: M^n \to \mathbb{R}^{n+1}\}$
Surface-like	$\hat{f} \times \mathrm{Id}: L^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^{n+1}$, or $C(\hat{f}) \times \mathrm{Id}: C(L^2) \times \mathbb{R}^{n-3} \to \mathbb{R}^{n+1}$	$\mathcal{G} \cong \mathcal{G}_{L^2} = \{\hat{g}: L^2 \to \mathbb{R}^3\}$ $\mathcal{G} \cong \mathcal{G}_{L^2} = \{\hat{g}: L^2 \to \mathbb{S}^3\}$
Ruled	f is R^{n-1}-ruled	$\mathcal{G} \cong \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, any g preserves R^{n-1}
Continuous	$Q(h) = 0 = Q(\gamma)$ and h of first species	$\mathcal{G} \cong U \subseteq \mathbb{R}$ open subset
Discrete	$Q(h) = 0 = Q(\gamma)$ and h of second species	$\mathcal{G} \cong \{*\}$

For a detailed and modern approach to the problem see [14]. A few years later, Cartan in [4] gave an equivalent description to the one given by Sbrana in terms of envelopes of hyperplanes.

Chapter 2 is a basic review of tools needed for this work. Namely, flat bilinear forms, genuine rigidity, the Gauss parametrization, the Sbrana and Cartan classification, and Darboux-Manakov-Zakharov systems.

Chapter 3 is dedicated to study isometrically deformable generic hypersurfaces. For this, we use the relatively new concept of genuine rigidity which extends the one of isometric rigidity. This notion was introduced in [12] and extended in [28], and is more adequate for the study of rigidity in higher codimensions; see for example [15], [17] and [27].

Generic hypersurfaces in the Sbrana-Cartan classification have the property that both the Gauss map and the support function are solutions of the same linear hyperbolic or elliptic partial differential equation. In this work we will naturally associate to our problem a Darboux-Manakov-Zakharov (DMZ) system of PDEs which plays the role of such PDE when the codimension is bigger than one. Darboux introduced such systems to study the problem of triply orthogonal system of surfaces, which was a hot topic during the 19th century, to the point that Bianchi [3] wrote an 850 pages book on the subject. DMZ systems and $n-$orthogonal systems of hypersurfaces have gained attention more recently due to the strong relation with an n-dimensional generalization of the Euler equation in hydrodynamics, see [26] and [39].

Recall that $(u_0 \dots, u_p)$ is a *conjugate chart* of an immersed submanifold of the sphere $h : L^{p+1} \to \mathbb{S}^n$ if the associated Christoffel symbols satisfy $\Gamma_{ij}^k = 0$ for distinct indices and $\alpha^h(\partial_{u_i}, \partial_{u_j}) = 0$, where α^h is the second fundamental form of h. Equivalently, h as a map in $\mathbb{R}^{n+1}$ is a solution of the DMZ system

$$(Q(h))_{ij} := Q_{ij}(h) = \partial^2_{ij}h - \Gamma^i_{ji}\partial_i h - \Gamma^j_{ij}\partial_j h + g_{ij}h = 0, \quad \forall 0 \le i < j \le p.$$

Notice the similarity with Cartan submanifolds; see for example [33] and [35]. Despite the fact that Q depends on the choice of coordinates, the functions

$$m_{ij} = -\partial_i\Gamma^i_{ji} + \Gamma^i_{ji}\Gamma^j_{ij} - g_{ij},$$

$$m_{ijk} = \Gamma^i_{ji} - \Gamma^k_{jk},$$

called the (i, j) and $(i, j, k)-$*Laplace invariants of* Q, are invariants under natural change of coordinates.

Previously to this work there was no analogous classification to that of Sbrana and Cartan in higher codimensions, apart from certain restricted cases. Theorem 3.0.1 is the main result of Chapter 3 and is a natural extension of Sbrana and Cartan works for $p = 1$. We classify the generic hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p + 1) < n$ and its genuine deformations up to codimension p, where "generic" is the corresponding property in higher codimensions to being neither surface-like nor ruled for $p = 1$. For this, we have extended the notion of species by measuring the rank of the trivial holonomy component of what we call the *Sbrana bundle associated to* Q, that is, the rank of its maximal parallel flat subbundle. We say that a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p + 1) < n$ is of r^{th}-*type* if the moduli space of genuine deformations $g : M^n \to \mathbb{R}^{n+p}$ is naturally a union of at most $(p + 1)$ convex open subsets of $\mathbb{R}^r$. The following table describes Theorem 3.0.1, and notice that for $p = 1$ it recovers the two lowest rows of the last table.

Rank $p+1$ deformable generic hypersurfaces	Geometric description of the hypersurface $f : M^n \to \mathbb{R}^{n+1}$	Moduli space of deformations $\mathcal{G}_k = \{g : M^n \to \mathbb{R}^{n+k}\}$
p^{th}-type	$Q(h) = 0 = Q(\gamma)$ and h of 1^{st} species	$\mathcal{G}_k = \emptyset$ for $k < p$ and $\mathcal{G}_p \cong U \subseteq \mathbb{R}^p$ open subset
$\vdots$	$\vdots$	$\vdots$
$(p+1-k)^{th}$-type	$Q(h) = 0 = Q(\gamma)$ and h of k^{th} species	$\mathcal{G}_k = \emptyset$ for $k < p$ and $\mathcal{G}_p \cong U \subseteq \mathbb{R}^{p+1-k}$ open subset
$\vdots$	$\vdots$	$\vdots$
$0^{th}-$type	$Q(h) = 0 = Q(\gamma)$ and h of $(p+1)^{th}$ species	$\mathcal{G}_k = \emptyset$ for $k < p$ and $\mathcal{G}_p \cong \{*\}$

The codimension 2 case is particularly important since Theorem 3.0.1 closes all possible cases. In order for a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ to be genuinely deformable in codimension 2 its rank must be at most 3. Apart from flat submanifolds, classified in [7] and [27], Theorem 3.0.1 for $p = 2$ deals with the generic rank 3 case, and [15] together with [27] analyze the case in which the rank is 2. Theorem 3.0.2 summarizes those results and characterizes all generic Euclidean hypersurfaces which are isometrically deformable in $\mathbb{R}^{n+2}$.

We finish Chapter 3 by showing that our techniques can be used to study conformally flat Euclidean submanifolds. Theorem 3.0.3 is an extension of Theorem 5 of [7], and gives a description of such submanifolds.

Despite the fact that the local classification of deformable hypersurfaces goes back a century, it took almost 90 years to find examples of the elusive discrete case, since Sbrana and Cartan descriptions were not constructive. In [14] the authors showed that by intersecting generically two flat hypersurfaces $U_1, U_2 \subseteq \mathbb{R}^{n+2}$, the respective inclusions of $M^n := U_1 \cap U_2$ into U_1 and U_2 give the only two isometric immersions of M^n into $\mathbb{R}^{n+1}$. This simple geometric construction provided the first general examples of the discrete case in the Sbrana-Cartan classification. This construction also highlights the local nature of the classification by producing examples of connected locally deformable hypersurfaces of locally different types in the Sbrana-Cartan classification in complex ways. Recently, other examples have been found; see for example [13], [20] and [27].

We dedicate Chapter 4 to extend the techniques used in [14] by producing several examples of genuinely deformable submanifolds for $1 = q \leq p \leq (n-2)$ as the ones in Chapter 3. We characterize our examples by the vanishment of certain Laplace invariants of the associated DMZ system, just as in [14]. Furthermore, the discussion suggests how to naturally extend the concept of genuine rigidity in order to gain a transitivity property.

Sbrana and Cartan approach was to start with a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ and use f to describe the others. If we try to solve the same question in codimension 2, then the problem of *honest rigidity* arises, as shown in [27]. Namely, consider $\hat{f} : M^n \to \mathbb{R}^{n+1}$ a genuine deformation of f as above and an isometric immersion $h : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ with $\hat{f}(M^n) \subseteq U$. Then $h \circ \hat{f}$ is a genuine deformation of f, although it is not of $\hat{f}$. Honest rigidity discards this type of deformation. The problem becomes in some sense intrinsic.

In Chapter 5, we analyze Chern-Kuiper's inequalities. They relate the kernel Δ_g of the second fundamental form of $g : M^n \to \mathbb{R}^{n+p}$ with the kernel Γ of the curvature tensor of M^n. We show that, in many circumstances, they must coincide if we are interested in honest rigidity. In contrast, if $n > 2$ and f, $\hat{f}$, and h are as in the last paragraph, then $g = h \circ \hat{f}$ is a genuine deformation of f, but $\Delta_g \neq \Gamma$ generically.

We end this work with Appendices where we prove some technical lemmas.

CHAPTER 2

Preliminaries

2.1 A remark on complex and real vector spaces

Several of the tensors that we deal with in this work are more easily treatable in $(TM)_{\mathbb{C}}$, the complexification of the tangent bundle of some manifold M^n. In order to do this, we need to establish some identifications.

Given a (finite dimensional) real vector space $\mathbb{W}$ we denote by $\mathbb{W}_{\mathbb{C}} = \mathbb{W} \otimes \mathbb{C}$ its complexification. Conversely, let $\mathbb{V}$ be a complex vector space with an antilinear map $C : \mathbb{V} \to \mathbb{V}$, that is, $C(\lambda v) = \overline{\lambda} C(v)$ for $\lambda \in \mathbb{C}$, satisfying $C^2 = \mathrm{Id}$. Define $\mathrm{Re}(\mathbb{V}) = \mathrm{Re}_C(\mathbb{V}) = \{v \in \mathbb{V} : Cv = v\}$ and $\mathrm{Im}(\mathbb{V}) = \{v \in \mathbb{V} : Cv = -v\}$. Then $i : \mathrm{Re}(\mathbb{V}) \to \mathrm{Im}(\mathbb{V})$, $i(v) = iv$ is a real isomorphism, so $\dim_{\mathbb{R}}(\mathrm{Re}(\mathbb{V})) = \dim_{\mathbb{C}}(\mathbb{V})$, since $\mathbb{V} = \mathrm{Re}(\mathbb{V}) \oplus \mathrm{Im}(\mathbb{V})$ as real vector spaces. The map C is called a *conjugation map*. Notice that $\mathbb{W}_{\mathbb{C}}$ has a natural conjugation $v + iw \to \overline{v + iw} := v - iw$ for $v, w \in \mathbb{W}$.

Consider a complex basis $\{e_i\}_{i \in I}$ of $\mathbb{W}_{\mathbb{C}}$ closed under the conjugation, that is, for any index $i \in I$ there is a unique index $\bar{i} \in I$ such that $\overline{e_i} = e_{\bar{i}}$. The $\mathbb{C}$-antilinear map defined by $C(e_i) = e_{\bar{i}}$ is the natural conjugation and satisfies that $\mathbb{W} = \mathrm{Re}_C(\mathbb{W}_{\mathbb{C}})$. Hence any tensor in $\mathbb{W}_{\mathbb{C}}$ with the natural compatibility condition with respect to this basis automatically corresponds to a real tensor in $\mathbb{W}$.

2.2 Flat bilinear forms

Given a bilinear map $\beta : \mathbb{V} \times \mathbb{U} \to \mathbb{W}$ between real vector spaces, set

$$\mathcal{S}(\beta) = \mathrm{span}\{\beta(X, Y) : X \in \mathbb{V}, Y \in \mathbb{U}\} \subseteq \mathbb{W}.$$

The (left) *nullity* of β is the vector subspace

$$\Delta_\beta = \mathcal{N}(\beta) = \{X \in \mathbb{V} : \beta(X, Y) = 0\,, \forall Y \in \mathbb{U}\} \subseteq \mathbb{V}.$$

For each $Y \in \mathbb{U}$ we denote by $\beta^Y : \mathbb{V} \to \mathbb{W}$ the linear map defined by $\beta^Y(X) = \beta(X, Y)$. Let

$$\mathrm{Re}(\beta) = \{Y \in \mathbb{U} : \dim(\mathrm{Im}(\beta^Y)) \text{ is maximal}\}$$

be the set of (right) *regular elements of* β, which is open and dense in $\mathbb{U}$. There are similar definitions for left regular elements and right nullity.

Given $\beta_i : \mathbb{V}_i \times \mathbb{V}_i \to \mathbb{W}_i$ bilinear forms for $i = 1, 2$, call $\beta_1 \oplus \beta_2 : (\mathbb{V}_1 \times \mathbb{V}_2) \times (\mathbb{V}_1 \times \mathbb{V}_2) \to \mathbb{W}_1 \times \mathbb{W}_2$ the bilinear form given by

$$(\beta_1 \oplus \beta_2)(v_1 + v_2, \tilde{v}_1 + \tilde{v}_2) := \beta_1(v_1, \tilde{v}_1) + \beta_2(v_2, \tilde{v}_2) \quad \forall v_i, \tilde{v}_i \in \mathbb{V}_i.$$

Assume now that $\mathbb{W}$ has a non-degenerate inner product $\langle \cdot, \cdot \rangle : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$. We denote $\mathbb{W}^{p,q}$ to point out that the inner product in $\mathbb{W}$ has signature (p, q). We say that β is *flat* if

$$\langle \beta(X, Y), \beta(Z, W) \rangle = \langle \beta(X, W), \beta(Z, Y) \rangle \quad \forall X, Z \in \mathbb{V} \quad \forall Y, W \in \mathbb{U}.$$

For a symmetric bilinear map $\beta : \mathbb{V}\times\mathbb{V}\to\mathbb{W}$, we say that β is *diagonalizable* if there exists a basis $\{X_i\}_i$ of $\mathbb{V}_{\mathbb{C}}$ such that $\{X_i\}_i = \{\overline{X_i}\}_i$ and $\beta(X_i, X_j) = 0$ for all $i \neq j$, where we are extending β by $\mathbb{C}$-bilinearity $\beta : \mathbb{V}_{\mathbb{C}}\times\mathbb{V}_{\mathbb{C}}\to\mathbb{W}_{\mathbb{C}}$. We denote $\bar{j}$ the index such that $\overline{X_j} = X_{\bar{j}}$.

There are two results that we need in order to bound the dimension of the nullity of a flat bilinear form. The first one is due to Moore [36] and valid for not necessarily symmetric ones.

Lemma 2.2.1. *Let $\beta : \mathbb{V}\times\mathbb{U}\to\mathbb{W}$ be a flat bilinear form. If $X \in \mathbb{U}$ is a right regular element, then*

$$\mathcal{S}(\beta|_{\ker(\beta^X)\times\mathbb{U}}) \subseteq \beta^X(\mathbb{V}) \cap \beta^X(\mathbb{V})^{\perp}.$$

In particular, if $\beta^X(\mathbb{V})$ is non-degenerate then $\Delta_\beta = \ker(\beta^X)$ and $\dim(\Delta_\beta) = \dim(\mathbb{V}) - \dim(\mathrm{Im}(\beta^X)) \geq \dim(\mathbb{V}) - \dim(\mathbb{W})$.

The second result is only valid for symmetric flat bilinear forms and is called the Main Lemma in the literature. We point out that the proof given in [10] has a gap for $\min\{p, q\} = 6$, as shown by counterexamples given in [11]. The correct statement for this case was given in [12].

Lemma 2.2.2 (Main Lemma)**.** *Let $\beta : \mathbb{V}^n\times\mathbb{V}^n\to\mathbb{W}^{p,q}$ be a flat symmetric bilinear form such that $\mathcal{S}(\beta) = \mathbb{W}^{p,q}$. If $\min\{p, q\} \leq 5$ then $\dim(\Delta_\beta) \geq n - p - q$.*

When $\beta : \mathbb{V}\times\mathbb{V}\to\mathbb{W}$ is symmetric, we can quotient out by its nullity. Namely, if $\pi : \mathbb{V}\to\overline{\mathbb{V}} := \mathbb{V}/\Delta_\beta$ is the quotient map then $\overline{\beta} : \overline{\mathbb{V}}\times\overline{\mathbb{V}}\to\mathbb{W}$ is the bilinear map determined by $\pi^*(\overline{\beta}) = \beta$. We say that two bilinear forms β_1 and β_2 are *equivalent*, and write $\beta_1 \cong \beta_2$, if they are isomorphic up to nullity, that is, when there is an isometry $I : \mathbb{W}_1\to\mathbb{W}_2$ and an isomorphism $T : \mathbb{V}_1/\Delta_{\beta_1}\to\mathbb{V}_2/\Delta_{\beta_2}$ such that $T^*\overline{\beta_2} = I\circ\overline{\beta_1}$.

2.3 Genuine rigidity

Genuine rigidity was introduced to better study the isometric rigidity of submanifolds in higher codimensions. This notion generalizes the classic ones of isometric rigidity and compositions. Here, we present a summary of the general concepts and results needed for this work.

Given a Riemannian manifold M^n and $x \in M^n$, the *nullity of M^n at x* is the nullity of its curvature tensor R at x, that is, the subspace of T_xM given by

$$\Gamma(x) = \mathcal{N}(R_x) = \{X \in T_xM : R(X, Y) = 0, \forall Y \in T_xM\}.$$

The *rank of M^n at x* is defined by $n - \mu$, where $\mu = \dim(\Gamma(x))$. As the results that we are looking for are of local nature and the subspaces that we deal with are all either kernels or images of smooth tensor fields, without further notice we will always work on each connected component of an open dense subset of M^n where all these dimensions are constant and thus give rise to smooth subbundles. In particular, we assume that μ is constant and hence the second Bianchi identity implies that Γ is a totally geodesic distribution, namely, $\nabla_\Gamma\Gamma \subseteq \Gamma$.

Given an isometric immersion $f : M^n\to\mathbb{R}^{n+q}$ we denote by $\alpha^f : TM\times TM\to T_f^{\perp}M$ its second fundamental form. We define the *relative nullity of f at x* as $\Delta_f(x) := \mathcal{N}(\alpha_x^f)$ and the *rank of f* as $n - \nu_f$, where $\nu_f = \dim(\Delta_f)$. Gauss equation implies that $\Delta_f \subseteq \Gamma$, while Codazzi equation implies that Δ_f is a totally geodesic distribution of M^n. Hence, to study $\Delta_f^{\perp}$, we make use of the *splitting tensor* $C_T : \Delta_f^{\perp}\to\Delta_f^{\perp}$ of $T \in \Delta_f$ given by

$$C_T(X) = -(\nabla_X T)_{\Delta_f^{\perp}},$$

where the subindex denotes the orthogonal projection onto $\Delta_f^{\perp}$. Notice that $C_T = 0$ for all T if and only if $\Delta_f^{\perp}$ is totally geodesic, in which case by the de Rham decomposition theorem M^n is locally a Riemannian product. Hence, the splitting tensor measures how far is M^n from being a product of integral leaves of Δ_f and $\Delta_f^{\perp}$ (in general $\Delta_f^{\perp}$ is not even integrable).

Definition 2.3.1. Let $f : M^n\to\mathbb{R}^{n+p}$ be a Riemannian submanifold of rank $r < n$. We say that f is *generic* (among the ones of rank r) if there exists $T \in \Delta_f$ such that the characteristic polynomial of C_T, $\psi_{C_T}(z) := \det(zI - C_T)$, has only simple roots over $\mathbb{C}$.

Given two isometric immersions $f : M^n\to\mathbb{R}^{n+q}$ and $g : M^n\to\mathbb{R}^{n+p}$, it is useful to work with the vector bundle $W = T_g^{\perp}M \oplus T_f^{\perp}M$, in which we endow with the natural semi-Riemannian metric with signature (p, q),

$$\langle(\xi_1, \eta_1), (\xi_2, \eta_2)\rangle = \langle\xi_1, \xi_2\rangle_{T_g^{\perp}M} - \langle\xi_1, \xi_2\rangle_{T_f^{\perp}M}.$$

The bilinear tensor $\beta = (\alpha^g, \alpha^f) : TM \times TM \to W$ is flat with respect to this metric by the Gauss equations of f and g. We also have the compatible connection in W induced by their normal connections, $\hat{\nabla} := (\nabla^{\perp g}, \nabla^{\perp f})$. Note that the flat bilinear form $\beta = (\alpha^g, \alpha^f)$ is different from $\alpha^g \oplus \alpha^f$ as defined in Section 2.2.

We say that the pair $\{f, g\}$ *extends isometrically* if there exists a Riemannian manifold N^{n+r}, an isometric embedding $j : M^n \to N^{n+r}$ and two isometric immersions $F : N^{n+r} \to \mathbb{R}^{n+q}$, $G : N^{n+r} \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $g = G \circ j$. That is, the following diagram commutes:

$$\begin{tikzcd}
& & \mathbb{R}^{n+p} \\
M^n \arrow[urr, "g"] \arrow[r, hook, "j"] \arrow[drr, "f"'] & N^{n+r} \arrow[ur, "G"'] \arrow[dr, "F"] & \\
& & \mathbb{R}^{n+q}
\end{tikzcd} \tag{2.1}$$

Observe that, in this situation, $\{(G_*\xi, F_*\xi) : \xi \in T_j^\perp M\} \subseteq \mathcal{S}(\beta)^\perp$ is a non-trivial null subbundle of W.

The pair $\{f, g\}$ is said to be *genuine*, or g is said to be a *genuine deformation of* f for a fixed f, if there is no open subset $U \subseteq M$ such that $\{f|_U, g|_U\}$ extends isometrically. Accordingly, an isometric immersion $f : M^n \to \mathbb{R}^{n+q}$ is *genuinely rigid* in $\mathbb{R}^{n+p}$ if there is no open subset $U \subseteq M^n$ such that $f|_U$ admits a genuine deformation in $\mathbb{R}^{n+p}$. When this is not the case, we say that f is *genuinely deformable* in $\mathbb{R}^{n+p}$. In particular, when f is a hypersurface, that $g : M^n \to \mathbb{R}^{n+p}$ is a genuine deformation of f means that there is no open subset $U \subseteq M^n$ along which g is a composition, that is, $g|_U = h \circ f|_U$, where $h : V \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+p}$ is some isometric immersion of an open subset V with $f(U) \subseteq V$.

The isometric immersion $f : M^n \to \mathbb{R}^{n+q}$ is said to be *R^d-ruled* (or *d-ruled*), if $R^d \subseteq TM$ is a d-dimensional totally geodesic distribution whose leaves are mapped by f onto (open subsets of) affine subspaces of $\mathbb{R}^{n+q}$. Theorem 1 of [12] asserts that a genuine pair $f : M^n \to \mathbb{R}^{n+q}$ and $g : M^n \to \mathbb{R}^{n+p}$ with $\min\{p, q\} \leq 5$ must be mutually R^d-ruled with $\Delta_\beta \subseteq R^d$, and it gives a sharp estimate for d. In particular, if $f : M^n \to \mathbb{R}^{n+1}$ is a rank $(p+1)$ hypersurface which is not $(n-p+3)$-ruled then f is genuinely rigid in $\mathbb{R}^{n+q}$ for all $q < p$. Notice that the condition of not being $(n-p+3)$-ruled is trivially satisfied for $p \leq 6$ by the following elementary fact.

Lemma 2.3.2. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear and symmetric map with respect to the Euclidean inner product. If there exists a d-dimensional subspace $R \subseteq \mathbb{R}^n$ such that $\langle A(R), R \rangle = 0$, then rank$(A) \leq 2(n-d)$.*

Therefore it is natural to study genuine deformations of hypersurfaces of rank $(p+1)$ in $\mathbb{R}^{n+p}$. Consider thus $g : M^n \to \mathbb{R}^{n+p}$ a genuine deformation of such an $f : M^n \to \mathbb{R}^{n+1}$. Let $\beta = \alpha^g \oplus \alpha^f$ and assume that $\mathcal{S}(\beta)$ is non-degenerate (this will be our case by Proposition 3.1.1). By the Main Lemma, we have

$$n - p - 1 \leq n - \dim S(\beta) \leq \dim(\Delta_\beta) \leq \nu_f = n - p - 1.$$

Hence, $S(\beta) = W^{p,1}$ and $\Delta_\beta = \Delta_f = \Gamma$. In particular, $\Delta_g \subseteq \Gamma = \Delta_\beta \subseteq \Delta_g$. We conclude that

$$\Delta_f = \Delta_g = \Delta_\beta = \Gamma.$$

We denote by $\mathbb{R}^N_\nu$ the semi-Euclidean space of index ν, that is, $\mathbb{R}^N$ with a non-degenerate inner product of index $\nu \leq N$. All the definitions of this subsection have their natural extensions to the semi-Riemannian context, and we will use them without further mention.

Recently, it has been shown that natural singularities must be allowed when studying certain rigidity phenomena. Following [28], we say that f and g *singular extend isometrically* if (2.1) holds for some embedding j and isometric maps F, G, with the set of points where F and G fail to be immersions (that may be empty) contained in $j(M^n)$. We say that f and g are *strongly genuine deformations* (of each other) if there is no open subset $U \subseteq M^n$ along which the restrictions $f|_U$ and $g|_U$ singularly extend isometrically.

2.4 The Gauss parametrization

An important step in our approach to characterize genuine deformations of hypersurfaces of rank $(p+1)$ is to reduce the problem to the quotient space of nullity leaves $\pi : M^n \to L^{p+1} = M/\Gamma$. Once this is done, we obtain a classification of the hypersurfaces themselves by means of the Gauss parametrization that we describe next. For a more detailed description see [16].

Let $f : M^n \to \mathbb{R}^{n+1}$ be an orientable Euclidean hypersurface with constant relative nullity ν_f. If $\rho : M^n \to \mathbb{S}^n$ is the Gauss map of f, then ρ is constant along the leaves of Δ_f. Hence, there is $h : L = M/\Delta_f \to \mathbb{S}^n$ such that $\rho = h \circ \pi$. This map h is in fact an immersion, so we always consider on L the metric induced by h. To give a complete local description of f in terms of h it is necessary to consider also its *support function* $\gamma : L \to \mathbb{R}$, which is defined by $\gamma \circ \pi = \langle f, \rho\rangle$. From h and γ we can recover $f(M^n)$ locally using the *Gauss parametrization* given by $\psi : T_h^\perp L \to \mathbb{R}^{n+1}$,

$$\psi(x, w) = (\gamma h + \nabla\gamma)(x) + w. \tag{2.2}$$

We also denote the Gauss parametrization of f simply by (h, γ). This useful tool was introduced by Sbrana in [38] precisely to study rigidity of hypersurfaces of rank 2, but since then it has had several applications in other contexts.

In particular, using the Gauss parametrization we have a local description of all flat hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$. By the Gauss equation, the rank of f is at most one. If $\nu_f = n$ then $f(M)$ is an open subset of some affine hyperplane. If $\nu_f = n - 1$, then $f(M)$ can be (locally) described by a regular curve $h(s)$ in $\mathbb{S}^n$ and a real function $\gamma(s)$. A deeper analysis can be done to classify flat submanifolds in codimension two by means of a different parametrization. This was recently fully understood in Corollary 18 of [27], and partially earlier in Theorem 13 of [7]. In [32] it is proved an analogous result for generic Euclidean flat submanifolds $M^n \subseteq \mathbb{R}^{n+p}$ and $p \leq n$.

2.5 The Sbrana-Cartan classification

The Sbrana-Cartan classification gives a local description of all hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ which possess genuine (namely, non-congruent) deformations in $\mathbb{R}^{n+1}$. To recall it we need a few definitions and results.

By the classical Beez-Killing rigidity theorem, in order for $f : M^n \to \mathbb{R}^{n+1}$ to have a genuine deformation in $\mathbb{R}^{n+1}$ it must have rank at most 2 everywhere. If the rank of f is 1 or 0, then M^n is flat and, as seen above, its genuine deformations can be easily understood by means of the Gauss parametrization. Hence, the interesting cases are among hypersurfaces of rank 2.

Definition 2.5.1. A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is called *surface-like* if there exists a surface $L^2 \subseteq \mathbb{R}^3$ (resp. $L^2 \subseteq \mathbb{S}^3$) such that $f(M^n) \subseteq L^2 \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^3 \times \mathbb{R}^{n-2}$ (resp. $f(M^n) \subseteq C(L^2) \times \mathbb{R}^{n-3} \subseteq \mathbb{R}^4 \times \mathbb{R}^{n-3}$ where $C(L^2)$ is the radial cone obtained from $L^2 \subseteq \mathbb{S}^3$).

In the Sbrana-Cartan classification, the family of surface-like hypersurfaces is the first one which has genuine deformations. Moreover, if f as above is surface-like, then any genuine deformation of f is given by a genuine deformation of L^2 in $\mathbb{R}^3$ (resp. in $\mathbb{S}^3$). However, a complete classification of the genuine deformations of surfaces is currently out of reach.

The second family of genuinely deformable hypersurfaces of rank 2 is that of $(n-1)$-ruled ones. It turns out that they all are highly deformable, any deformation preserves the rulings and the moduli space of genuine deformations is easily seen to be the set of smooth functions of one variable.

In order to describe the remaining deformable hypersurfaces, we need to recall some definitions.

Definition 2.5.2. Given a surface $h : L^2 \to \mathbb{S}^n$, we call a coordinate system $(u, v) \in \mathbb{R}^2$ *real conjugate* if its second fundamental form satisfies $\alpha^h(\partial_u, \partial_v) = 0$. Similarly, a coordinate system $z \in \mathbb{C}$ is called *complex conjugate* if $\alpha^h(\partial_u, \partial_v) = 0$, where $u = z = \overline{v}$. Accordingly, we say that h is of *real* (resp. *complex*) *type.*

Given a surface $h : L^2 \to \mathbb{S}^n$ with a real (resp. complex) conjugate system (u, v) and $\Gamma^u_{vu}, \Gamma^v_{uv}$ its Christoffel symbols, assume that the following system of PDE

$$\begin{cases} \partial_u\tau = 2\Gamma^v_{uv}\tau(1-\tau) \\ \partial_v\tau = 2\Gamma^u_{vu}(1-\tau), \end{cases} \tag{2.3}$$

has a solution $\tau : L^2 \to \mathbb{R}$ (resp. $\tau : L^2 \to \mathbb{S}^1 \subseteq \mathbb{C}$) other than the trivial one $\tau \equiv 1$. The integrability condition of this system is

$$(\partial_v\Gamma^v_{uv} - 2\Gamma^v_{uv}\Gamma^v_{vu})\tau = \partial_u\Gamma^u_{vu} - 2\Gamma^u_{uv}\Gamma^v_{vu}. \tag{2.4}$$

Then h is called of *first species* if the above equation is trivially satisfied, that is,

$$\partial_u\Gamma^u_{vu} = 2\Gamma^u_{uv}\Gamma^v_{vu} = \partial_v\Gamma^v_{uv}. \tag{2.5}$$

We say that h is of *second species* if $\partial_v\Gamma^v_{uv} \neq 2\Gamma^v_{uv}\Gamma^v_{vu}$, $\partial_u\Gamma^u_{vu} \neq 2\Gamma^v_{uv}\Gamma^v_{vu}$ and

$$\tau = \frac{\partial_v\Gamma^v_{uv} - 2\Gamma^u_{uv}\Gamma^v_{vu}}{\partial_u\Gamma^u_{vu} - 2\Gamma^u_{uv}\Gamma^v_{vu}} \neq 1 \tag{2.6}$$

is the necessarily unique solution of (2.3). For the real case, we also require τ to be positive.

Theorem 2.5.3 (Sbrana [38], Cartan [4])**.** *Let $f : M^n \to \mathbb{R}^{n+1}$ be a genuinely deformable hypersurface of rank 2. Assume further that f is nowhere either surface-like or $(n-1)$-ruled. Then, along connected components of an open dense subset, its Gauss map $h : L^2 \to \mathbb{S}^n$ is of first or second species, and, with respect to its conjugate coordinate system, the support function satisfies*

$$\partial^2_{uv}\gamma - \Gamma^u_{vu}\partial_u\gamma - \Gamma^v_{uv}\partial_v\gamma + \gamma g_{uv} = 0.$$

If h is of first species, then the moduli space of genuine deformations of f is naturally parametrized by the positive initial conditions of the solutions τ of (2.3). This set is $\mathbb{R}_{>0} \setminus \{1\} \cong \mathbb{R} \setminus \{0\}$ for the real type, while $\mathbb{S}^1 \setminus \{1\} \cong \mathbb{R}$ for the complex type. If h is of second species, the hypersurface f has a unique genuine deformation.

We say that a deformable hypersurface $f : M^n \to \mathbb{R}^{n+1}$ is of the *continuous type* (resp. *discrete type*) if it is described by the above theorem and the Gauss map is of the first species (resp. second species).

Remark 2.5.4. In the case that the Gauss map of the hypersurface f is of second species and real type but τ given by (2.6) is negative, we can associate with f an isometric immersion in the Lorentz space $\mathbb{R}^{n+1}_1$, as shown in Theorem 5 of [8]. In a similar way, when the Gauss map is of the first species, for each initial condition for τ negative we can associate an isometric immersion $g = g_\tau : M^n \to \mathbb{R}^{n+1}_1$. This is an important result for studying conformally flat submanifolds and one of the main reasons we will not restrict ourselves only to Riemannian ambient Euclidean spaces.

2.6 Darboux-Manakov-Zakharov (DMZ) systems

This subsection describes Darboux-Manakov-Zakharov (overdetermined) systems of PDEs which have a crucial role in the description of our geometric problem.

One of Darboux's many interests was on orthogonal systems of coordinates for $\mathbb{R}^{p+1}$. That is, coordinate systems $(u_0, \ldots, u_p)$ of $\mathbb{R}^n$ with respect to witch the Euclidean metric is expressed as

$$ds^2 = v_0^2 du_0^2 + \ldots + v_p^2 du_p^2,$$

for some smooth functions $v_i = v_i(u_0, \ldots, u_p)$. For $p = 2$ this problem is called the problem of triply orthogonal systems of surfaces. It is easy to verify that for such a coordinate system we have that, for three distinct indices, the Christoffel symbols satisfy $\Gamma^k_{ij} = 0$ and $\Gamma^i_{ji} = \frac{\partial_j v_i}{v_i}$. This naturally implies that for any indices $i \neq j < k \neq i$ we have that

$$\partial^2_{jk} v_i - \Gamma^j_{kj}\partial_j v_i - \Gamma^k_{jk}\partial_k v_i = 0. \tag{2.7}$$

Additional non-linear equations must be satisfied by the v_i's in order to obtain a flat metric.

Darboux proposed an associated system of PDEs to find solutions of the last equations and linearize the problem. Consider $(u_0, \ldots, u_p) = (z_0, \overline{z_0}, \ldots, z_{s-1}, \overline{z_{s-1}}, x_{2s}, \ldots, x_p) \in \mathbb{C}^{2s} \times \mathbb{R}^{p+1-2s}$ for some s, and denote by $\bar{\imath}$ the unique index which satisfies $\overline{u_i} = u_{\bar{\imath}}$. The collection $Q = (Q_{ij})_{i<j}$ of second order linear PDEs given by

$$(Q(\xi))_{ij} = Q_{ij}(\xi) = \partial^2_{ij}\xi + a^j_{ij}\partial_j\xi + a^i_{ji}\partial_i\xi + b_{ij}\xi = 0 \quad \forall\, 0 \le i < j \le p, \tag{2.8}$$

for $\partial_i = \partial_{u_i}$, and some smooth complex functions a^j_{ij}, b_{ij} satisfying $\overline{a^j_{ij}} = a^{\bar{\jmath}}_{\bar{\imath}\bar{\jmath}}$, $\overline{b_{ij}} = b_{\bar{\imath}\bar{\jmath}}$ is called a *Darboux-Manakov-Zakharov (DMZ) system*. Darboux only analyzed the case when $s = 0$ and $p = 2$, but this generalization is natural and is needed for this work. Notice the similarity between (2.7) and (2.8) with $b_{ij} = 0$ (for us the case $b_{ij} = 0$ is irrelevant, see Proposition 2.6.3).

We now provide the natural generalization of the notion of conjugate chart for higher dimensional submanifolds.

Definition 2.6.1. A coordinate system $(z_0, \ldots z_{s-1}, x_{2s}, \ldots, x_p) \in \mathbb{C}^s \times \mathbb{R}^{p+1-2s}$ of a submanifold $h : L^{p+1} \to \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is called *conjugate* if h is a solution of a DMZ system with respect to $(u_0, \ldots, u_p) = (z_0, \overline{z_0}, \ldots, z_{s-1}, \overline{z_{s-1}}, x_{2s}, \ldots, x_p)$, that is

$$Q_{ij}(h) = \partial^2_{ij}h - \Gamma^i_{ji}\partial_i h - \Gamma^j_{ij}\partial_j h + g_{ij}h = 0, \quad \forall i < j, \tag{2.9}$$

where $\{\partial_i = \partial_{u_i}\}_{i=0}^p$ is the local coordinate frame for $(TL)_{\mathbb{C}}$, $\Gamma^i_{ji}, \Gamma^j_{ij} : L^{p+1} \to \mathbb{C}$ are necessarily the Christoffel symbols associated with this frame, and $g_{ij} = \langle \partial_i h, \partial_j h \rangle$.

Remark 2.6.2. Notice that (2.9) is equivalent to $\alpha^h(\partial_i, \partial_j) = 0$ and $\Gamma_{ij}^k = 0$ for distinct indices. Then the Gauss equation of h for three distinct indices becomes

$$R(\partial_i, \partial_j)\partial_k = g_{jk}\partial_i - g_{ik}\partial_j,$$

which is equivalent to

$$\partial_i \Gamma_{kj}^j + \Gamma_{kj}^j \Gamma_{ij}^j - \Gamma_{kj}^j \Gamma_{ik}^k - \Gamma_{ij}^j \Gamma_{ki}^i + g_{ik} = 0. \tag{2.10}$$

These equations and the compatibility of the connection with the metric are precisely the integrability conditions for the DMZ system (2.9).

As proved in [33] we have the following.

Proposition 2.6.3. *Suppose that $h : L^{p+1} \to \mathbb{S}^n$ has a conjugate chart and that $\gamma \in C^\infty(L^{p+1})$ is a non-zero solution of the associated DMZ system, that is, $Q(\gamma) = 0$. Then the submanifold $H : L^{p+1} \to \mathbb{R}^{n+1}$ given by $H := \frac{h}{\gamma}$ satisfies*

$$\tilde{Q}_{ij}(H) = \partial_{ij}^2 H - \tilde{\Gamma}_{ji}^i \partial_i H - \tilde{\Gamma}_{ij}^j \partial_j H = 0, \quad \forall i < j, \tag{2.11}$$

for $\tilde{\Gamma}_{ji}^i = \Gamma_{ji}^i - \frac{\partial_j \gamma}{\gamma}$.

Conversely, let $0 \neq H : L^{p+1} \to \mathbb{R}^{n+1}$ be a submanifold satisfying (2.11). Define $\gamma := \frac{1}{\|H\|} \neq 0$ and assume that $h := \gamma H : L^{p+1} \to \mathbb{S}^n$ is an immersion. Then h solves (2.9) for $\Gamma_{ji}^i = \tilde{\Gamma}_{ji}^i + \frac{\partial_j \gamma}{\gamma}$ and $g_{ij} = \frac{2\partial_i \gamma \partial_j \gamma - \gamma \tilde{Q}_{ij}(\gamma)}{\gamma^2}$. In this case, $Q(\gamma) = 0$.

This shows that finding conjugate charts for submanifolds in the sphere is equivalent to the problem in the Euclidean space, that is, finding independent solutions to DMZ systems.

CHAPTER 3

Genuine deformations with maximal rank

Consider hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p+1) < n$. As proven in [21], and later generalized in [12], f is genuinely rigid in $\mathbb{R}^{n+q}$ for $q < p$ if f is not $(n-p+3)$-ruled for $p \geq 7$. For this reason, we focus on its isometric deformations in $\mathbb{R}^{n+p}$. Also, we add the hypothesis of being generic (in the sense of Definition 3.1.8) in order to discard the surface-like and ruled situations.

The following is the main result of this Chapter, which for $p = 1$ recovers the Sbrana-Cartan classification. For this, we have extended the notion of species that defines those families. Roughly, the species measures the trivial holonomy component, that is, the maximal parallel flat subbundle of what we call the *Sbrana bundle associated to* Q, where Q is the associated DMZ system as in (2.9); see Section 3.1.2. We say that a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p+1) < n$ is of r^{th}*-type* if the moduli space of its genuine deformations $g : M^n \to \mathbb{R}^{n+p}$ is naturally homeomorphic to a union of at most $(p+1)$ convex open subsets of $\mathbb{R}^r$ for $r \in \{0, 1, \ldots, p\}$. Also, we denote by $\mathbb{R}^N_\mu$ the Euclidean space $\mathbb{R}^N$ with a non-degenerate inner product of index $\mu \leq p$.

Theorem 3.0.1. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a simply connected hypersurface of rank $(p+1)$, with $1 \leq p \leq n-2$. If $p \geq 7$ assume in addition that f is not $(n-p+2)$-ruled. Then f is genuinely rigid in $\mathbb{R}^{n+q}$ for any $q < p$. Moreover, if f possesses a genuine deformation in $\mathbb{R}^{n+p}$ and is generic, then, along each connected component of an open dense subset of M^n, f is of r^{th}-type for some $r \in \{0, \ldots, p\}$. In this case, the Gauss map h of f has a unique conjugate chart of $(p+1-r)^{th}$-species, and its support function $\gamma = \langle f, h\rangle$ also satisfies $Q(\gamma) = 0$.*

Conversely, under the Gauss parametrization, (h, γ) as above gives rise to a Euclidean hypersurface genuinely deformable in $\mathbb{R}^{n+p}_\nu$ for some $\mu \leq p$. Furthermore, f is of r^{th}-type where M^n is generic.

We comment that the value of μ of the last result is easily determined by the trivial holonomy component of the Sbrana bundle of Q.

Although the Sbrana-Cartan work was done in 1908, it took almost a century to find explicit examples of hypersurfaces of the discrete type. The first examples, which are now called of *intersection type*, were found in [14] as intersection of two generic flat hypersurfaces $N_1^{n+1}, N_2^{n+1} \subseteq \mathbb{R}^{n+2}$, in which case Q is hyperbolic. This construction also shows the local nature of the classification by producing examples of connected locally deformable hypersurfaces of locally different types in the Sbrana-Cartan classification. Later, Dajczer-Florit in [13] gave a procedure to obtain the first examples of locally deformable hypersurfaces of discrete-type with Q elliptic.

Until now there is no analogous classification to that of Sbrana and Cartan in higher codimensions, not even in codimension 2, only classifications in certain restricted cases. As commented before, if $f : M^n \to \mathbb{R}^{n+1}$ is genuinely deformable in $\mathbb{R}^{n+2}$, then its rank must be at most three. If its rank is one or less the hypersurface is flat, and all its isometric immersions in $\mathbb{R}^{n+2}$ are described in Corollary 18 of [27]. Theorem 1 of [15] describes the rank two generic case in terms of their support function γ and a conjugate coordinate system for its Gauss map $h : L^2 \to \mathbb{S}^n$, just as in Theorem 3.0.1. Moreover, it computes the moduli space $\mathcal{C}_h$ of deformations of f in $\mathbb{R}^{n+2}$. Theorem 3.0.1 for $p = 2$ analyzes the generic rank three case. Thus, the following result summarizes the above discussion, and characterizes all generic Euclidean hypersurfaces which are genuinely deformable in $\mathbb{R}^{n+2}$ and the respective moduli space of their *honest* deformations, as defined in [27]. The concept of honest rigidity is the natural one for such a result and is slightly stronger than genuine rigidity. We point out that Theorem 1 of [15] has a gap for hypersurfaces of intersection type. However, Theorem 33 of [27] and an adaptation of that result for Lorentz ambient space (Theorem 3.2.1 bellow) allow us to fill this gap, describing the honest deformations for hypersurfaces of intersection type in codimension 2 in terms of its *shared dimension* I; see Section 3.2.

Theorem 3.0.2. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a genuinely deformable hypersurface in codimension 2. Then the rank of M^n is at most 3. Assume that M^n is generic and nowhere flat, in particular $n \geq 4$. Then each connected component U of an open dense subset of M^n falls in exactly one of these categories:*

1. *The rank of U is 3. The Gauss map $h : L^3 \to \mathbb{S}^n$ is of $(3-r)^{th}$-species for some $r \in \{0, 1, 2\}$ and the support function γ satisfies $Q(\gamma) = 0$. In this case, $f|_U$ is of r^{th}-type and all its genuine deformations in $\mathbb{R}^{n+2}$ are honest deformations;*

2. *The rank of U is 2 and $f|_U$ is not a Sbrana-Cartan hypersurface of intersection type. Then the Gauss map $h : L^2 \to \mathbb{S}^n$ of $f|_U$ has a conjugate chart and the support function γ satisfies $Q(\gamma) = 0$. In this case, the moduli space of honest deformations is naturally $\mathcal{C}_h$;*

3. *The rank of U is 2 and $f|_U$ is a Sbrana-Cartan hypersurface of intersection type, that is, U is obtained as an intersection of two flat Riemannian hypersurfaces on $\mathbb{R}^{n+2}_\nu$ for $\nu \leq 1$ and $f|_U$ is the inclusion in one of such hypersurfaces. Then $f|_U$ is honestly rigid in $\mathbb{R}^{n+2}$, unless $I = 2$. In the latter case, the moduli space of honest deformations of f in $\mathbb{R}^{n+2}$ is naturally an open interval of $\mathbb{R}$.*

The study of conformally flat Euclidean submanifolds in codimension 2, namely, submanifolds $M^n \subseteq \mathbb{R}^{n+2}$ which are conformally flat, is strongly linked to the Sbrana-Cartan theory. In fact, the description given in [7] for such submanifolds is similar to the one given for deformable hypersurfaces, and some examples can be found using intersections of flat submanifolds in a similar way as for deformable hypersurfaces; see [8]. However, in this case we must consider Riemannian hypersurfaces of the Lorentz space. This and the development of the proof of Theorem 3.0.1 led us to consider hypersurfaces and its genuine deformations in semi-Euclidean spaces.

It is therefore not surprising that the techniques developed in this work can be used also to study conformally flat submanifolds $g : M^n \to \mathbb{R}^{n+p+1}$. As proven in [7], if $p \leq n-4$, (locally) such manifolds M^n can be obtained as the intersection of some Riemannian hypersurface $F : N^{n+1} \to \mathbb{R}^{n+2}_1$ with the light cone, and N^{n+1} admits an isometric immersion $G : N^{n+1} \to \mathbb{R}^{n+p+1}$ such that $g = G|_{M^n}$. The hypersurface F must have rank at most $(p+1)$. The following result characterizes such Riemannian hypersurfaces of rank $(p+1)$. This extends Theorem 5 of [7] that deals with the case $p = 1$. As before, the hypothesis of being generic is to discard the surface-like situation.

Theorem 3.0.3. *Let $F : N^m \to \mathbb{R}^{m+1}_1$ be a Riemannian hypersurface of rank $(p+1) \geq 2$. Then N^m cannot be isometrically immersed in $\mathbb{R}^{m+q}$ for any $q < p$. Assume further that there exists an isometric immersion $G : N^m \to \mathbb{R}^{m+p}$. Then, the Gauss map h of F has a unique conjugate chart of the $k^{th}-$species for some $k \in \{1, \ldots, p+1\}$, and the support function $\gamma = \langle f, h \rangle$ also satisfies $Q(\gamma) = 0$.*

Conversely, under the Gauss parametrization, (h, γ) as above gives rise to a Riemannian hypersurface F genuinely deformable in $\mathbb{R}^{n+p}_\nu$ for some $\nu \leq p$. Furthermore, if N^m is generic, then F is of $(p+1-k)^{th}$-type.

In Chapter 4, we will provide examples of the hypersurfaces described in this work using the intersection techniques developed in [14]. In a future paper we will present an analogous result to Theorem 3.0.1 classifying the isometric deformations of Euclidean hypersurfaces of rank $(p+1)$ in $\mathbb{R}^{n+p+1}$, extending Theorem 1 in [15] to higher codimensions.

There are several results in the literature which are described in terms of surfaces with conjugate charts, and in several of them this surface is the leaf space of some umbilical distribution of codimension 2; besides the ones already cited, see for example [6], [18], [22], [23], [24], [25]. We believe that some of those results can be extended to dimensions bigger than 2 using the tools developed in this paper.

This Chapter is organized as follows. Section 3.1 is devoted to describe the rigidity problem and to prove Theorem 3.0.1. In Section 3.2 we demonstrate Theorem 3.0.2, while in Section 3.3 we analyze the conformal case and prove Theorem 3.0.3.

3.1 Description of the genuine deformations

Our purpose in this section is to find an intermediate analytical characterization for the genuine deformations of a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ with rank $(p+1) \geq 2$ in higher codimensions.

From now on, $A = A_\rho$ will denote the shape operator of f with respect to a fixed unit normal vector field ρ, $\alpha := \alpha^g$ the second fundamental form of another isometric immersion g of M^n, and $\beta = \alpha \oplus \alpha^f : TM \times TM \to T_g^\perp M \oplus T_f^\perp M$ the associated flat bilinear form. All sub-indices in this section will be in the range $\{0, 1, \ldots, p\}$.

Proposition 3.1.1. *Suppose that $f : M^n \to \mathbb{R}^{n+1}$ has rank $(p+1)$ and fix $\varepsilon \in \{0,1\}$. Let $g : M^n \to \mathbb{R}^{n+p+\varepsilon}$ be a genuine deformation of f with $p+1+\varepsilon < n$. For $p \geq 5-2\varepsilon$, assume in addition that f and g are not mutually $(n-p-\varepsilon+2)$-ruled. Then $\mathcal{S}(\beta)$ is non-degenerate on an open dense subset of M^n.*

Proof. First, observe that the condition of not being mutually $(n-p-\varepsilon+2)$-ruled is trivially satisfied for $p \leq 4-2\varepsilon$ by Lemma 2.3.2.

Suppose that there is an open subset $U \subseteq M$ where $\mathcal{S}(\beta)$ is degenerate. Since $W^{p+\varepsilon,1} = T_g^\perp M \oplus T_f^\perp M$ is Lorentzian, there is a smooth unit normal section $\xi \in T_g^\perp U$ such that

$$\operatorname{span}\{(\xi,\rho)\} = \mathcal{S}(\beta) \cap \mathcal{S}(\beta)^\perp. \tag{3.1}$$

Consider $\gamma : TU \times TU \to E$ the orthogonal projection of α^g onto $E = \{\xi\}^\perp \subseteq T_g^\perp M$. By (3.1), γ is flat. Theorems 11 and 14 of [12] imply that f and g are simultaneously R^d-ruled, where $R^d = \mathcal{N}(\alpha^g_{L^\perp}) \cap \mathcal{N}(\alpha^f_{\hat{L}^\perp})$, $L \subseteq \operatorname{span}\langle\xi\rangle$, $\hat{L} \subseteq \operatorname{span}\langle\rho\rangle$, $0 \leq \ell = \dim(L) = \dim(\hat{L}) \leq 1$ and

$$d \geq n-p-\varepsilon-1+3\ell. \tag{3.2}$$

As f and g are not simultaneously $(n-p-\varepsilon+2)$-ruled we have that $L = \hat{L} = \{0\}$ and $R = \Delta_\beta$. By the construction of L in Theorem 11 of [12], this happens only when either $\Delta_\gamma = \Delta_\beta$ or if there is $Z_0 \in \Delta_\gamma$ such that $\nabla^\perp_{Z_0}\xi \neq 0$. If $\Delta_\gamma = \Delta_\beta$, by the Main Lemma for γ we have that

$$n-p-\varepsilon+1 \leq \dim(\Delta_\gamma) = \dim(\Delta_\beta) \leq \nu_f = n-p-1,$$

a contradiction. Hence, assume the existence of such $Z_0 \in \Delta_\gamma$.

Call $\phi : TU \times (TU \oplus \operatorname{span}\{\xi\}) \to E$ the map given by

$$\phi(X,v) = (\tilde{\nabla}_X v)_E,$$

where $\tilde{\nabla}$ denotes the connection of $\mathbb{R}^{n+p+\varepsilon}$ and the sub-index E denotes the orthogonal projection onto E. An easy computation shows that ϕ is flat and satisfies Codazzi equation. By the above $\Delta_\phi \subsetneq \Delta_\gamma$. Take $W \in \Delta_\phi$ and $Y \in TU$. Codazzi equation $(\nabla^E_{Z_0}\phi)(W,Y) = (\nabla^E_W\phi)(Z_0,Y)$ reduces to

$$\phi([Z_0,W],Y) = \langle AW,Y\rangle \nabla^\perp_{Z_0}\xi.$$

Using the flatness of ϕ and the above relation we get

$$\langle AW,Y\rangle \|\nabla^\perp_{Z_0}\xi\|^2 = \langle \phi([Z_0,W],Y), \phi(Z_0,\xi)\rangle = \langle \phi(Z_0,Y), \phi([Z_0,W],\xi)\rangle = 0.$$

This proves that $\langle AW,Y\rangle = 0$ for all $Y \in TU$, since $\nabla^\perp_{Z_0}\xi \neq 0$. Then, $\Delta_\phi \subseteq \Delta_f$, and by Lemma 2.2.1, we have that $\nu_f \geq \dim(\Delta_\phi) \geq n-p-\varepsilon+1$, which is also a contradiction. ☐

Remark 3.1.2. For $p \in \{5-2\varepsilon, 6-2\varepsilon\}$ we can prove a weaker version of Proposition 3.1.1 without the hypotheses of not being $(n-p-\varepsilon+2)$-ruled. In this case, we can conclude that either $\mathcal{S}(\beta)$ is non-degenerate, or f and g are mutually R^d ruled with $d = n-p-\varepsilon+2$ and $\Delta_g = \Gamma \subseteq R^d$. Indeed, if we follow the steps of the proof we see that the only problem is when $l = 1$. In this case, if $\dim(\Gamma + R^d) \geq n-p-\varepsilon+3$ using Lemma 2.3.2 for $(\Gamma + R^d)$ we get a contradiction. Then, using (3.2) we get that $\Delta_\beta = \Gamma \subsetneq R^d$ and $d = n-p-\varepsilon+2$. Finally, just notice that $\Gamma = \Delta_\beta \subseteq \Delta_g \subseteq \Gamma$.

The Main Lemma gives us the next corollary.

Corollary 3.1.3. *If f and g are as in Proposition 3.1.1 with $\varepsilon = 0$, then $\Delta_g = \Delta_f = \Gamma$ and $\mathcal{S}(\beta) = W^{p,1}$.*

For our purposes, it is more natural and fruitful to classify the deformations in semi-Euclidean spaces, that is, $\mathbb{R}^{n+p}$ with a non-degenerate inner product, which satisfies the same formal properties as the ones in the Euclidean case. In this case, we denote the ambient space as $\mathbb{R}^{n+p}_\mu$, where μ is the index of the inner product. In particular, $\mathbb{R}^{n+p} = \mathbb{R}^{n+p}_0$.

Definition 3.1.4. Consider $f : M^n \to \mathbb{R}^{n+q}_\eta$ and $g : M^n \to \mathbb{R}^{n+p}_\mu$ two isometric immersions of a Riemannian manifold M^n. We say that g is a *non-degenerate deformation* of f if there exists $X \in \operatorname{Re}(\beta)$ such that $\beta^X(TM) \subseteq W = T_g^\perp M \oplus T_f^\perp M$ is a non-degenerate subspace, where $\beta = \alpha^g \oplus \alpha^f$.

Corollary 3.1.3 and Corollary 2 of [36] imply the following.

Corollary 3.1.5. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a rank $p+1 < n$ hypersurface. If $p \geq 5$ assume further that f is not $(n-p+2)$-ruled. Then any genuine deformation $g : M^n \to \mathbb{R}^{n+p}$ of f is non-degenerate.*

Remark 3.1.6. By Lemma 2.2.1, for any non-degenerate deformation $g : M^n \to \mathbb{R}^{n+p}_\mu$ of a nowhere flat hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p+1)$ we have that $\mathcal{S}(\beta) = W$ and $\Delta_g = \Gamma$, as in Corollary 3.1.3.

The splitting tensor is important in the Sbrana-Cartan classification to differentiate the families of deformable hypersurfaces of rank 2. We will use it in an analogous way.

Definition 3.1.7. Consider M^n a Riemannian manifold. For $T \in \Gamma$ we define the *splitting tensor with respect to T* as the endomorphism $C_T : \Gamma^\perp \to \Gamma^\perp$ given by

$$C_T X = -(\nabla_X T)^h,$$

where h denotes the orthogonal projection on $\Gamma^\perp$.

For a non-degenerate deformation $g : M^n \to \mathbb{R}^{n+p}_\mu$ of f (for some $0 \leq \mu \leq p$), Remark 3.1.6 and Codazzi equation imply that

$$\beta(C_S X, Y) = \beta(X, C_S Y), \quad \forall S \in \Gamma, \quad \forall X, Y \in \Gamma^\perp. \tag{3.3}$$

We introduce the following definition to discard the ruled and surface-like types of situations.

Definition 3.1.8. We call M^n *generic* it there exists $T \in \Gamma$ such that the characteristic polynomial of C_T, $\psi_{C_T}(z) := \det(zI - C_T)$, has only simple roots over $\mathbb{C}$.

Throughout this section, we assume that $g : M^n \to \mathbb{R}^{n+p}_\mu$ is a non-degenerate deformation of f and that M^n is generic. We will classify all such deformations.

Corollary 3.1.9. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a generic hypersurface of rank $2 \leq p+1 < n$ and $g : M^n \to \mathbb{R}^{n+p}_\mu$ a non-degenerate deformation. Then, there exists a unique basis (up to order and scalar multiplication) $\{X_i\}_{i=0}^p \in \Gamma^\perp_{\mathbb{C}}$, such that $C_T X_i = \lambda_i(T) X_i$ $\forall T \in \Gamma$. Moreover, for every non-degenerate deformation $g : M^n \to \mathbb{R}^{n+p}_\mu$ of f, we have that $\beta(X_i, X_j) = 0$ for $i \neq j$.*

Proof. Take $T_0 \in \Gamma$ such that the eigenvalues of C_{T_0} are distinct, and $C_{T_0} X_i = \lambda_i X_i$. By (3.3), $\beta(X_i, X_j) = 0$ for $i \neq j$ and again by (3.3) we get that $C_T X_i = \lambda_i(T) X_i$ for some 1-forms λ_i on Γ. This proves that this frame is intrinsic and unique. Moreover, by (3.3) this frame must diagonalize β for all genuine deformations. ☐

If $\{X_i\}_{i=0}^p$ are the diagonalizing directions of β as above, then after a re-scaling factor, the frame $\{X_i\}$ projects at L^{p+1} as coordinate vectors. More precisely, there exists a chart $(z_1, \ldots, z_s, x_{2s}, \ldots, x_p) \in \mathbb{C}^s \times \mathbb{R}^{p+1-2s}$ (where $2s$ is the number of non-real eigenvectors of the splitting tensor) such that for the variables $(u_0, \ldots, u_p) = (z_1, \overline{z_1}, \ldots, z_s, \overline{z_s}, x_{2s}, \ldots, x_p)$ they satisfy

$$\partial_i \circ \pi := \partial_{u_i} \circ \pi = \pi_* X_i. \tag{3.4}$$

For a proof of this fact, see Proposition A.3.4 in the Appendix. This chart will be extensively used throughout this work. These directions also define a conjugation of indices: we denote by $\bar{i}$ the unique index such that $\overline{X_i} = X_{\bar{i}}$. This conjugation will be used without further mention. Notice also that this coordinate system is unique (up to order and rescale of variables).

Observe now that the set $\{\beta(X_j, X_j)\}_j$ is pointwise a $\mathbb{C}$-basis of $W_{\mathbb{C}}$. We extend the metrics and the connections of the tangent and normal bundles to their complexifications by $\mathbb{C}$-bilinearity. Then

$$\langle \beta(X_i, X_i), \beta(X_i, X_i) \rangle \neq 0, \quad \forall i.$$

Indeed, if $\langle \beta(X_i, X_i), \beta(X_i, X_i) \rangle = 0$ for some i, by flatness, $\langle \beta(X_i, X_i), \beta(X_j, X_j) \rangle = 0$ for all j. Since $\mathcal{S}(\beta) = W$ we obtain that $\beta(X_i, X_i) = 0$, which is a contradiction. Recalling that $\alpha^f(X_i, X_i) = \langle AX_i, X_i \rangle \rho \neq 0$, set

$$\varphi_i := \frac{\langle \alpha^f(X_i, X_i), \alpha^f(X_i, X_i) \rangle}{\langle \beta(X_i, X_i), \beta(X_i, X_i) \rangle} \neq 0, \tag{3.5}$$

and

$$\eta_i := \frac{\alpha^g(X_i, X_i)}{\langle AX_i, X_i \rangle} \in \Gamma(T_g^\perp M \otimes \mathbb{C}). \tag{3.6}$$

Notice that φ_i and η_i do not change if we replace X_i by $\mu_i X_i$ for any $\mu_i \neq 0$. By the flatness of β,

$$d_{ij} := \langle \eta_i, \eta_j \rangle = 1 + \frac{\delta_{ij}}{\varphi_i}, \tag{3.7}$$

where δ_{ij} is the Kronecker symbol. Since the $p+1$ vectors η_i generate the normal space of g which has dimension p, the matrix $(D_\varphi)_{ij} = d_{ij}$ must be singular. By Lemma A.1.1 this is equivalent to

$$\varphi_* := -(\varphi_0 + \ldots + \varphi_p + 1) = 0. \tag{3.8}$$

With this, we can verify that

$$\varphi_0\eta_0 + \ldots + \varphi_p\eta_p = 0, \tag{3.9}$$

since $\langle \sum_j \varphi_j\eta_j, \eta_k\rangle = \sum_j \varphi_j(1 + \frac{\delta_{jk}}{\varphi_k}) = 0$ for all k.

Definition 3.1.10. We call a tuple $\varphi = (\varphi_i)_{i=0}^p$ *admissible* if $\overline{\varphi_i} = \varphi_{\bar{i}} \neq 0$ for all i and satisfies $\varphi_* = 0$. In this case we denote by $2s$ and P the cardinalities of the sets $\{i \in \{0,\ldots,p\} | i \neq \bar{i}\}$ and $\{i \in \{0,\ldots,p\} | i = \bar{i}$ and $\varphi_i > 0\}$ respectively. We call $p-(s+P)$ the *index* of φ.

Thus, the collection of functions $\varphi = (\varphi_i)_{i=0}^p$ defined by (3.5) is admissible. Moreover, Proposition A.1.2 of the Appendix shows that the index of φ is precisely the index μ of the metric in the ambient space of $g: M^n \to \mathbb{R}^{n+p}_\mu$.

By Codazzi equation for α and A, we have that

$$\nabla^\perp_T \eta_i = 0, \quad \forall T \in \Gamma. \tag{3.10}$$

Indeed,

$$\begin{aligned}\nabla^\perp_T\eta_i &= \frac{\langle AX_i, X_i\rangle(\alpha([T,X_i],X_i) + \alpha(\nabla_T X_i, X_i)) - (\langle A[T,X_i],X_i\rangle + \langle A\nabla_T X_i, X_i\rangle)\alpha(X_i,X_i)}{\langle AX_i,X_i\rangle^2}\\ &= \frac{\langle AX_i, X_i\rangle(\langle A[T,X_i],X_i\rangle + \langle A\nabla_T X_i, X_i\rangle)\eta_i - (\langle A[T,X_i],X_i\rangle + \langle A\nabla_T X_i, X_i\rangle)\langle AX_i,X_i\rangle\eta_i}{\langle AX_i,X_i\rangle^2} = 0.\end{aligned}$$

As a consequence of (3.10) and (3.7), $T(\varphi_i) = 0$ for all i and $T \in \Gamma$.

For each $\eta \in (T_g^\perp M)_\mathbb{C}$ we define

$$D_\eta = A^{-1}A_\eta : \Gamma^\perp_\mathbb{C} \to \Gamma^\perp_\mathbb{C}, \tag{3.11}$$

where A is the second fundamental form of f restricted to $\Gamma^\perp_\mathbb{C}$ and A_η is the shape operator of g in the η direction also restricted to $\Gamma^\perp_\mathbb{C}$. Since $0 = \langle A_\eta X_i, X_j\rangle = \langle AD_\eta X_i, X_j\rangle$ for $i \neq j$, D_η is diagonalizable with the same basis $\{X_i\}$. In particular, for $D_i := D_{\eta_i}$ the Gauss equation implies that

$$D_i X_j = d_{ij} X_j,$$

where d_{ij} is defined in (3.7).

As shown in Lemma 15 of [15] we have

$$\nabla_T D_i = [D_i, C_T] = 0 \quad \forall T \in \Gamma \quad \forall i. \tag{3.12}$$

This motivates the following definition.

Definition 3.1.11. Consider a Riemannian manifold M^n of rank $(p+1) \geq 2$. We call a set of smooth tensors $D_i : \Gamma^\perp_\mathbb{C} \to \Gamma^\perp_\mathbb{C}$, $i = 0,\ldots,p$, a *D-system* if there is a conjugation of indices such that $\overline{D_i} = D_{\bar{i}}$ and the following conditions are satisfied:

i) $\dim_\mathbb{C} \ker(D_i - I) = p$, where I is the identity. We denote by $(\frac{1}{\varphi_i} + 1) \neq 1$ the remaining eigenvalue of D_i and X_i an associated eigenvector;

ii) $X_j \in \ker(D_i - I)$ for all $j \neq i$;

iii) $\nabla_T D_i = [D_i, C_T] = 0 \quad \forall T \in \Gamma \quad \forall i.$

Remark 3.1.12. Whenever convenient, we will consider $D_i : (TM)_\mathbb{C} \to (TM)_\mathbb{C}$ by extending it as zero on $\Gamma_\mathbb{C}$.

Remark 3.1.13. There may be several D-systems on M^n, but if M^n is generic, then the directions of the corresponding frame $X_0,\ldots,X_p$ are uniquely determined since the X_i's must also be eigenvectors of the splitting tensor by condition *iii)*. However, we still have some freedom on the φ_i's which determine the D-system.

Let ϕ_{ij} be the associated normal connection 1-forms

$$\phi_{ij}(X) = \langle \nabla^\perp_X \eta_i, \eta_j \rangle. \tag{3.13}$$

Clearly $\phi_{ii} = \frac{1}{2} d(\frac{1}{\varphi_i})$ and $\phi_{ij} = -\phi_{ji}$ for $i \neq j$. We denote by $\phi = (\phi_{ij})$ the matrix of 1-forms whose components are ϕ_{ij}. We can express the normal connection as

$$\nabla^\perp_X \eta_i = \sum_j \phi_{ij}(X) \varphi_j \eta_j. \tag{3.14}$$

Indeed, this is a consequence of (3.9) and

$$\Big\langle \sum_j \phi_{ij}(X)\varphi_j\eta_j, \varphi_k \Big\rangle = \varphi_{ik}(X) + \Big\langle \nabla^\perp_X \eta_i, \sum_j \varphi_j \eta_j \Big\rangle = \phi_{ik}(X), \quad \forall k.$$

The next result gives a bijection between the set of non-degenerate deformations of f in codimension p and the set of pairs (D, ϕ) satisfying certain equations.

Proposition 3.1.14. *Consider a simply connected generic hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $2 \leq p+1 < n$. Let $g : M^n \to \mathbb{R}^{n+p}_\mu$ be a non-degenerate deformation of f (for some $0 \leq \mu \leq p$). Then there exists a D-system and a $(p+1) \times (p+1)$ matrix of 1-forms $\phi = (\phi_{ij})$ satisfying:*

a) φ *is admissible of index* μ*;*

b) $\overline{\phi_{ij}(X)} = \phi_{\bar{i}\bar{j}}(\overline{X})$*;*

c) $AD_i = D_i^t A$*;*

d) $\sum_k \varphi_k \phi_{ik} = 0$, $\forall i$*;*

e) $\phi_{ij} + \phi_{ji} = 0$ *for* $i \neq j$ *and* $\phi_{ii} = \frac{1}{2} d(\frac{1}{\varphi_i})$*;*

f) $\phi_{ij}(T) = d\phi_{ij}(Z, T) = 0$ *for any* Z *and* $T \in \Gamma$*;*

g) $\nabla_X (AD_i)Y - \nabla_Y (AD_i)X = A\Big(\sum_j \varphi_j (\phi_{ij} \wedge D_j)(X, Y)\Big), \quad \forall i, X, Y \in TM$*;*

h) $\langle [AD_i, AD_j]X, Y \rangle = d\phi_{ij}(X, Y) + \Omega_{ij}(X, Y)$, $\forall i, j$ *and* $X, Y \in TM$*, where* $\Omega = (\Omega_{ij})$ *is the matrix of 2-forms given by* $\Omega_{ij} = \sum_k \varphi_k (\phi_{ik} \wedge \phi_{jk})$*.*

Conversely, suppose that we have a D-system and a $(p+1) \times (p+1)$ matrix of 1-forms $\phi = (\phi_{ij})$ satisfying the conditions a) to h) above. Then, there exists an isometric immersion $g = g_{(D,\phi)} : M^n \to \mathbb{R}^{n+p}_\mu$ which is a genuine deformation of f determined by D and ϕ. Moreover, given two pairs $(D, \phi), (\hat{D}, \hat{\phi})$ that satisfy the above properties, then $g_{(D,\phi)}$ and $\hat{g}_{(\hat{D},\hat{\phi})}$ are congruent if and only if $(D, \phi) = (\hat{D}, \hat{\phi})$.

Proof. We have already proved that if $g : M^n \to \mathbb{R}^{n+p}_\mu$ is a deformation for f, then there is such a pair (D, ϕ) satisfying all the above properties. Indeed, observe that $AD_i = A_{\eta_i}$ is a symmetric tensor, g) is Codazzi equation for A_{η_i}, and h) is just Ricci equation expressed as

$$\langle R^\perp(X, Y)\eta_i, \eta_j \rangle = X\langle \nabla^\perp_Y \eta_i, \eta_j \rangle - Y\langle \nabla^\perp_X \eta_i, \eta_j \rangle - \langle \nabla^\perp_{[X,Y]} \eta_i, \eta_j \rangle + \langle \nabla^\perp_X \eta_i, \nabla^\perp_Y \eta_j \rangle - \langle \nabla^\perp_Y \eta_i, \nabla^\perp_X \eta_j \rangle.$$

Moreover, if $g : M^n \to \mathbb{R}^{n+p}_\mu$ and $\hat{g} : M^n \to \mathbb{R}^{n+p}_\mu$ are two isometric immersions with the same associated pair (D, ϕ), then they are congruent. Indeed, define $t : (T^\perp_g M)_\mathbb{C} \to (T^\perp_{\hat{g}} M)_\mathbb{C}$ by $t(\eta_i) = \hat{\eta}_i$, where the η_i's are defined by (3.6), and similarly for the $\hat{\eta}_i$'s. It is easy to verify that t is a well-defined parallel bundle isometry which preserves the respective second fundamental form, $t \circ \alpha^g = \alpha^{\hat{g}}$. By the Fundamental Theorem of submanifolds this map induces an isometry $T : \mathbb{R}^{n+p}_\mu \to \mathbb{R}^{n+p}_\mu$ such that $\hat{g} = T \circ g$.

Let us prove the converse. The main idea is to consider the bundle $E = \mathbb{C}^{p+1}/\ker(D_\varphi) \to M^n$ as a candidate to be the complexification of the normal bundle for g and use the pair (D, ϕ) to define a second fundamental form, a metric and a connection on E. Then, the Fundamental Theorem of submanifolds will imply the existence of g. We denote the elements of $E \to M$ with brackets to distinguish them from those of $\mathbb{C}^{p+1}$.

Consider on E the bilinear product defined by $\langle [e_i], [e_j] \rangle = d_{ij} = 1 + \frac{\delta_{ij}}{\varphi_i}$. By Proposition A.1.2 of the Appendix, this defines a non-degenerate inner product on the real bundle $\mathrm{Re}_C(E) \to M^n$ of index μ, where the conjugation is given on the canonical basis by $C([e_i]) = [e_{\bar{i}}]$.

Equation (3.14) induces the connection $\tilde{\nabla}_X e_i = \sum_j \varphi_j \phi_{ij}(X) e_j$ on the trivial bundle $\mathbb{C}^{p+1} \to M$. This connection descends to the quotient E. Indeed, using d), e) and (A.1) we get

$$\tilde{\nabla}_X\Big(\sum_j \varphi_j e_j\Big) = \sum_k \Big(X(\varphi_k) + \varphi_k\Big(\sum_j \varphi_j \phi_{jk}(X)\Big)\Big)e_k = \sum_k \Big(X(\varphi_k) + \varphi_k(2\varphi_k \phi_{kk}(X))\Big)e_k = 0.$$

Thus, $\nabla^E_X[e_i] = \sum_j \varphi_j \phi_{ij}(X)[e_j]$ is a well-defined connection on $E \to M^n$. By e), this connection is compatible with the product induced by D_φ. Indeed, notice that

$$\langle \nabla^E_X[e_i], [e_j]\rangle = \sum_k \varphi_k \phi_{ik}(X) d_{kj} = \phi_{ij}(X) + \sum_k \varphi_k \phi_{ik}(X) = \phi_{ij}(X),$$

and then $\langle \nabla^E_X[e_i], [e_j]\rangle + \langle [e_i], \nabla^E_X[e_j]\rangle = \phi_{ij}(X) + \phi_{ji}(X) = X(d_{ij}) = X\langle [e_i], [e_j]\rangle$.

For $X, Y \in (T_xM)_{\mathbb{C}}$ we define the linear map $\ell_{X,Y} : \mathbb{C}^{p+1} \to \mathbb{C}$ by $\ell_{X,Y}(e_i) = \langle AD_iX, Y\rangle$. Then, by (A.1),

$$\ell_{X,Y}\Big(\sum_j \varphi_j e_j\Big) = \Big\langle A\Big(\sum_j D_j \varphi_j\Big)X, Y\Big\rangle = 0.$$

Thus there exists a unique $\gamma(X,Y) \in E$ such that $\langle \gamma(X,Y), [e_i]\rangle = \langle AD_iX, Y\rangle$ for all i. This tensor γ is symmetric by c) and by definition $\Gamma \subseteq \Delta_\gamma$. Observe that

$$\gamma(X_i, X_i) = \langle AX_i, X_i\rangle [e_i] \quad \forall i, \tag{3.15}$$

$$\gamma(X_i, X_j) = 0 \quad \forall i \neq j, \tag{3.16}$$

since

$$\langle \langle AX_i, X_i\rangle [e_i], [e_k]\rangle = \langle AX_i, X_i\rangle d_{ik} = \langle \gamma(X_i, X_i), [e_k]\rangle \quad \forall k,$$
$$\langle \gamma(X_i, X_j), [e_k]\rangle = \langle AD_kX_i, X_k\rangle = d_{ki}\langle AX_i, X_j\rangle = 0 \quad \forall k.$$

Equations (3.15) and (3.16) show that $\Delta_\gamma = \Gamma$, $\{X_i\}_{i=0}^p$ diagonalizes γ, and $\mathcal{S}(\beta) = E \oplus T_f^\perp M$ where $\beta = \gamma \oplus \alpha^f$. Notice that

$$\langle \gamma(X_i, X_i), \gamma(X_j, X_j)\rangle = \langle AX_i, X_i\rangle \langle AX_j, X_j\rangle d_{ij} = \langle AX_i, X_i\rangle\langle AX_j, X_j\rangle, \quad \forall i \neq j.$$

This proves that γ satisfies Gauss equation on $(TM)_{\mathbb{C}}$ since all the other Gauss equations are trivially satisfied since $\{X_0, \ldots, X_p\}$ is a basis of $\Gamma^\perp_{\mathbb{C}}$ which simultaneously diagonalizes γ and α^f.

To verify that γ is a Codazzi tensor, just observe that, for all X, Y, Z, we have

$$\begin{aligned}\langle (\nabla^E_X \gamma)(Y,Z), [e_i]\rangle &= X(\langle \gamma(Y,Z), [e_i]\rangle) - \langle \gamma(\nabla_X Y, Z), [e_i]\rangle - \langle \gamma(Y, \nabla_X Z), [e_i]\rangle - \langle \gamma(Y,Z), \nabla^E_X[e_i]\rangle \\ &= X(\langle AD_iY, Z\rangle) - \langle AD_i\nabla_XY, Z\rangle - \langle AD_iY, \nabla_XZ\rangle - \sum_j \varphi_j\phi_{ij}(X)\langle AD_jY, Z\rangle \\ &= \langle \nabla_X(AD_i)Y, Z\rangle - \sum_j \varphi_j \phi_{ij}(X)\langle AD_jY, Z\rangle.\end{aligned}$$

This expression is symmetric for X, Y by g).

Lastly, Ricci equation follows from

$$\begin{aligned}\langle R(X,Y)[e_i], [e_j]\rangle &= X(\langle \nabla^E_Y[e_i], [e_j]\rangle) - Y(\langle \nabla^E_X[e_i], [e_j]\rangle) - \langle \nabla^E_{[X,Y]}[e_i], [e_j]\rangle \\ &\quad - \langle \nabla^E_Y[e_i], \nabla^E_X[e_j]\rangle + \langle \nabla^E_X[e_i], \nabla^E_Y[e_j]\rangle \\ &= d\phi_{ij}(X,Y) + \Big\langle \sum_k \nabla^E_X[e_i], \sum_k \varphi_k\phi_{jk}(Y)[e_k]\Big\rangle - \Big\langle \nabla^E_Y[e_i], \sum_k \varphi_k \phi_{jk}(X)[e_k]\Big\rangle \\ &= d\phi_{ij}(X,Y) + \Omega_{ij}(X,Y) = \langle [AD_i, AD_j]X, Y\rangle.\end{aligned}$$

We conclude from the Fundamental Theorem of submanifolds that there exists an isometric immersion $g = g_{(D,\phi)} : M^n \to \mathbb{R}^{n+p}_\mu$ such that the complexification of the normal bundle is (E, ∇^E) and the second fundamental form of g is γ, up to a parallel isometry of vector bundles. Moreover, g is a non-degenerate deformation, since $X = \sum_i X_i \in TM$ is such that $\beta^X : \Gamma^\perp \to W$ is an isomorphism. ☐

3.1.1 Projecting to the nullity leaf space

Since we now have a description of the genuine deformations in terms of pairs (D, ϕ), we proceed to reduce the problem to the nullity leaf space $L^{p+1} = M^n/\Gamma$, and characterize each condition of Proposition 3.1.14 in terms of φ and the Gauss parametrization data (h, γ) of the hypersurface f.

First, we translate Proposition 3.1.14 to the leaf space, which is a crucial point in our argument. We denote by $\langle \cdot, \cdot \rangle'$ and ∇' the metric and the connection induced by the Gauss map $h : L^{p+1} \to \mathbb{S}^n$.

Proposition 3.1.15. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a rank $(p+1)$ hypersurface. Consider the nullity leaf space $\pi : M^n \to L^{p+1} = M^n/\Gamma$, and $\gamma \in C^\infty(L^{p+1})$, $h : L^{p+1} \to \mathbb{S}^n$ the Gauss parametrization data of f. If (D, ϕ) is a pair on M^n as in Proposition 3.1.14, then there is an induced pair $(\hat{D}, \hat{\phi})$ on L^{p+1} such that*

$$\hat{\varphi}_i \circ \pi = \varphi_i, \quad \hat{D}_i \circ \pi_* = \pi_* \circ D_i, \quad \hat{\phi}_{ij} \circ \pi_* = \phi_{ij}.$$

In addition, $(\hat{D}, \hat{\phi})$ satisfies for $\pi_* X = \hat{X}, \pi_* Y = \hat{Y} \in TL$:

i) $\hat{\varphi}$ is admissible of index μ;

ii) $\overline{\hat{\phi}_{ij}(X)} = \hat{\phi}_{\bar{i}\bar{j}}(\overline{X})$;

iii) $(Hess_\gamma + \gamma I)\hat{D}_i = \hat{D}_i^t(Hess_\gamma + \gamma I)$;

iv) $\alpha^h(\hat{D}_i \hat{X}, \hat{Y}) = \alpha^h(\hat{X}, \hat{D}_i \hat{Y})$;

v) $\sum_k \hat{\varphi}_k \hat{\phi}_{ik} = 0, \quad \forall i$;

vi) $\hat{\phi}_{ij} + \hat{\phi}_{ji} = 0$ for $i \neq j$ and $\hat{\phi}_{ii} = \frac{1}{2} d(\frac{1}{\hat{\varphi}_i})$;

vii) $(\nabla'_{\hat{X}} \hat{D}_i)\hat{Y} - (\nabla'_{\hat{Y}} \hat{D}_i)\hat{X} = \sum_j \hat{\varphi}_j (\hat{\phi}_{ij} \wedge \hat{D}_j)(\hat{X}, \hat{Y}), \quad \forall i$;

viii) $\langle \hat{D}_j \hat{X}, \hat{D}_i \hat{Y} \rangle' - \langle \hat{D}_i \hat{X}, \hat{D}_j \hat{Y} \rangle' = d\hat{\phi}_{ij}(\hat{X}, \hat{Y}) + \hat{\Omega}_{ij}(\hat{X}, \hat{Y})$ where $\hat{\Omega}_{ij} \circ \pi_* = \Omega_{ij}$.

Conversely, if (h, γ) and $(\hat{D}, \hat{\phi})$ satisfy i)-viii) above, then they give rise, via the Gauss parametrization, to a hypersurface f and a pair (D, ϕ) satisfying Proposition 3.1.14.

Proof. From Corollary 12 of [15], we know that D_i, φ, ϕ and Ω descend to the quotient by definition of a D-system and f) of Proposition 3.1.14.

Let ρ be the Gauss map of f. Then $f_* AX = -\rho_* X = -h_* \pi_* X$. Take X, Y projectable vector fields on M^n, $\hat{X} \circ \pi = \pi_* X$, $\hat{Y} \circ \pi = \pi_* Y$. We see that viii) comes from c) and h) of Proposition 3.1.14 since

$$\langle AD_j X, AD_i Y \rangle - \langle AD_i X, AD_j Y \rangle = \langle \hat{D}_j \pi_* X, \hat{D}_i \pi_* Y \rangle' - \langle \hat{D}_i \pi_* X, \hat{D}_j \pi_* Y \rangle'.$$

Notice that

$$\begin{aligned} f_* \nabla_X AD_i Y &= \widetilde{\nabla}_X f_* AD_i Y - \langle AX, AD_i Y \rangle \rho = -\widetilde{\nabla}_X h_* \pi_* D_i Y - \langle h_* \pi_* X, h_* \pi_* D_i Y \rangle h \circ \pi \\ &= -h_* \nabla'_{\hat{X}} \hat{D}_i \hat{Y} - \alpha^h(\hat{X}, \hat{D}_i \hat{Y}). \end{aligned}$$

Hence, using this in g) of Proposition 3.1.14 we obtain iv) and vii). By the Gauss parametrization $\Phi : U \subseteq T_h^\perp L \to M$ and $\psi(w) = f\Phi(w) = \gamma h + h_* \nabla \gamma + w$, $w \in T_h^\perp L$, we get

$$\psi_* X = h_* P \hat{\pi}_* X + \alpha^h(\hat{\pi}_* X, \nabla' \gamma),$$

where $\hat{\pi} : T_h^\perp L \to L$ is the bundle projection, $X \in T_w(T_h^\perp L)$ is a horizontal vector, and P is the symmetric tensor

$$P = P_w = \text{Hess}_\gamma + \gamma I - B_w : TL \to TL, \tag{3.17}$$

where B_w is the shape operator of h in the $w-$direction. This implies that

$$-\langle AD_i \Phi_* X, \Phi_* Y \rangle = \langle h_* \hat{D}_i \pi_* \Phi_* X, h_* P \pi_* Y \rangle = \langle \hat{D}_i \hat{\pi}_* X, P \hat{\pi}_* Y \rangle'.$$

Therefore $\hat{D}_i^t P = P \hat{D}_i$ and as $D_i^t B_w = B_w D_i$ by iv), we conclude iii).

The converse follows easily by defining $D_i(\Gamma) = 0$ and $\pi_* D_i X = \hat{D}_i \pi_* X$ for $X \in \Gamma^\perp$. ∎

From now on, we will drop the hat over variables and the prime for the metric and connection of $h : L^{p+1} \to \mathbb{S}^n$, since we now focus on the leaf space and not on the manifold M^n.

The main idea will be to express Proposition 3.1.15 in terms of the coordinate system given by (3.4). As $(\partial_j := \partial_{u_j})_j$ is a basis on $(TL)_{\mathbb{C}}$, all the indices will be with respect to this basis between 0 and p. Notice that, since the coordinate vectors are the eigenvectors of the D_i's, they are completely determined by φ.

As was shown in the proof of Proposition 3.1.14 φ is used to define the second fundamental form and the metric of the normal bundle of g. On the other hand, ϕ is used to define the normal connection. Since Codazzi equation relates the second fundamental form with the normal connection, we expect ϕ to be related with φ. In fact, φ determines ϕ completely:

Lemma 3.1.16. *Let (D, ϕ) be a pair as in Proposition 3.1.15. Then, vii) (Codazzi equation) and vi) (compatibility of the connection with the metric) are equivalent to ϕ being uniquely determined by φ by the followings conditions:*

$$\phi_{is}(\partial_r) = 0 \quad \forall r \neq i \neq s \neq r, \tag{3.18}$$

$$\phi_{is}(\partial_i) = -\frac{\Gamma^s_{is}}{\varphi_i}, \quad \forall s \neq i, \tag{3.19}$$

$$\phi_{is}(\partial_s) = \frac{\Gamma^i_{si}}{\varphi_s}, \quad \forall s \neq i, \tag{3.20}$$

and

$$\Gamma^k_{ij} = 0 \quad \forall i \neq j \neq k \neq i, \tag{3.21}$$

$$\partial_j \varphi_i = 2\Gamma^i_{ji}\varphi_i \quad \forall i \neq j. \tag{3.22}$$

Proof. Take in vii) of Proposition 3.1.15 $X = \partial_r$, $Y = \partial_s$ with $s \neq r$. Then

$$\partial_r(d_{is}) + (d_{is} - d_{ir})\Gamma^s_{rs} = \sum_j \phi_{ij}(\partial_r)d_{js}\varphi_j = \phi_{is}(\partial_r)(d_{ss} - 1)\varphi_s + \sum_j \phi_{ij}(\partial_r)d_j\varphi_j = \phi_{is}(\partial_r), \tag{3.23}$$

$$(d_{is} - d_{ir})\Gamma^t_{rs} = 0, \quad \forall t \neq r, s,$$

and symmetric equations interchanging r with s. In particular, for $i = s$, we get (3.21) and

$$\partial_r(d_{ii}) + 2\big(d_{ii} - 1\big)\Gamma^i_{ri} = 0, \quad \forall r \neq i,$$

which is an equivalent form of (3.22). Using (3.22) in (3.23) we get (3.18) and (3.19). Equation vi) of Proposition 3.1.15, for $X = \partial_s$ and $j = s$ implies (3.20). □

By i) of Proposition 3.1.15, we can use (3.22) to get

$$\partial_i \varphi_i = -2\sum_{j \neq i} \Gamma^j_{ij}\varphi_j. \tag{3.24}$$

This implies the following.

Corollary 3.1.17. *The pair (D, ϕ) is determined by an admissible function $\varphi = (\varphi_i)_{i=0}^p$ satisfying*

$$\partial_i \varphi_j = 2\Gamma^j_{ij}\varphi_j \text{ for } i \neq j, \text{ and } \partial_i \varphi_i = -2\sum_{j \neq i} \Gamma^j_{ij}\varphi_j.$$

In particular, the moduli space of genuine deformations of f has finite dimension at most p.

Remark 3.1.18. Since φ is admissible, the matrix of 1-forms ϕ defined by (3.18), (3.19) and (3.20) immediately satisfies ii), v) and vi) of Proposition 3.1.15. Indeed, by (3.22)

$$\sum_s \phi_{is}(\partial_r)\varphi_s = \phi_{ii}(\partial_r)\varphi_i + \phi_{ir}(\partial_r)\varphi_r = \frac{1}{2}\partial_r(\varphi_i^{-1})\varphi_i + \Gamma^i_{ri} = 0,$$

and by (3.8) we get

$$\sum_s \phi_{is}(\partial_i)\varphi_s = \frac{1}{2}\partial_i(\varphi_i^{-1})\varphi_i - \sum_{s \neq i} \frac{\Gamma^s_{is}\varphi_s}{\varphi_i} = \frac{1}{2\varphi_i}\partial_i\Big(\sum_s \varphi_s\Big) = 0.$$

From now on, whenever we work with ϕ we will assume that it is defined by φ by (3.18), (3.19) and (3.20).

Lemma 3.1.19. *Condition iv) of Proposition 3.1.15 is equivalent to*

$$\alpha^h(\partial_j, \partial_k) = 0 \quad \forall j \neq k. \tag{3.25}$$

In particular, the chart is a conjugate chart. Moreover, condition iii) of Proposition 3.1.15 is equivalent to the support function γ satisfying $Q(\gamma) = 0$.

Proof. Take $X = \partial_j$ and $Y = \partial_k$ in iv) for $j \neq k$. Then $(d_{ij} - d_{ik})\alpha^h(\partial_j, \partial_k) = 0$ for all i. We obtain (3.25) from this for $i = j$. Using (3.21) and Remark 2.6.2 we conclude that the chart is conjugate.

The last assertion follows by evaluating the bilinear map given by iii) of Proposition 3.1.15 on the coordinates fields $X = \partial_j$ and $Y = \partial_k$ for $j \neq k$. □

The only remaining condition to analyze is viii), Ricci equation. We see now that, by Remark 2.6.2 and Lemmas 3.1.16 and 3.1.19, this is trivially satisfied.

Lemma 3.1.20. *Assume that (D, ϕ) satisfies conditions ii) to vii) of Proposition 3.1.15. Then viii) of Proposition 3.1.15 is satisfied if and only (2.10) holds.*

Proof. By (3.18), the only non-zero equations of viii) are when $X = \partial_j, Y = \partial_r$ for $r \neq j$. First, for $r \neq i, j$ we get

$$d\phi_{ij}(\partial_j, \partial_r) = \partial_j(\phi_{ij}(\partial_r)) - \partial_r(\phi_{ij}(\partial_j)) = -\partial_r\Big(\frac{\Gamma^i_{ji}}{\varphi_j}\Big) = \frac{-\partial_r\Gamma^i_{ji} + 2\Gamma^j_{rj}\Gamma^i_{ji}}{\varphi_j}, \tag{3.26}$$

and

$$\begin{aligned}\Omega_{ij}(\partial_j, \partial_r) &= (\phi_{ij}(\partial_j)\phi_{jj}(\partial_r))\varphi_j + (-\phi_{ir}(\partial_r)\phi_{jr}(\partial_j))\varphi_r + (-\phi_{ii}(\partial_r)\phi_{ji}(\partial_j))\varphi_i \\ &= \frac{\Gamma^i_{ji}\partial_r(\varphi_j^{-1})}{2} + \frac{\Gamma^i_{ri}\Gamma^r_{jr}}{\varphi_j} + \frac{\partial_r(\varphi_i^{-1})\Gamma^i_{ji}\varphi_i}{2\varphi_j}.\end{aligned}$$

Therefore, by (3.22) we have

$$\Omega_{ij}(\partial_j, \partial_r) = \frac{-\Gamma^i_{ji}\Gamma^j_{rj} + \Gamma^i_{ri}\Gamma^r_{jr} - \Gamma^i_{ri}\Gamma^r_{jr}}{\varphi_j}. \tag{3.27}$$

Adding (3.26) and (3.27) we get (2.10).

For $X = \partial_j$ and $Y = \partial_i$, first notice that

$$\begin{aligned}\sum_k \varphi_k(d\phi_{ik} + \Omega_{ij}) &= \sum_k \varphi_k d\phi_{ik} + \sum_l \varphi_l\phi_{il} \wedge \Big(\sum_k \varphi_k\phi_{kl}\Big) = \sum_k \varphi_k d\phi_{ik} + \sum_l \varphi_l\phi_{il} \wedge (2\varphi_l\phi_{ll}) \\ &= \sum_k \varphi_k d\phi_{ik} - \sum_l \phi_{il} \wedge d\varphi_l = d\Big(\sum_k \varphi_k\phi_{ik}\Big) = 0.\end{aligned}$$

Then using that $\sum_k \varphi_k D_k = 0$ we conclude that

$$\begin{aligned}0 = \sum_k \varphi_k(d\phi_{ik} + \Omega_{ik})(\partial_j, \partial_i) &= \varphi_j[d\phi_{ij}(\partial_j, \partial_i) + \Omega_{ij}(\partial_j, \partial_i)] + \sum_{k\neq j} \varphi_k[\langle D_k\partial_j, D_i\partial_i\rangle - \langle D_i\partial_j, D_k\partial_i\rangle] \\ &= \varphi_j[d\phi_{ij}(\partial_j, \partial_i) + \Omega_{ij}(\partial_j, \partial_i)] - \varphi_j[\langle D_j\partial_j, D_i\partial_i\rangle - \langle D_i\partial_j, D_j\partial_i\rangle].\end{aligned}$$

This shows that all Ricci equations are satisfied. □

Remark 3.1.21. Equation (2.10) can also be expressed as

$$Q_{ij}(\xi_k) = 0 \quad \forall i \neq j \neq k \neq i,$$

where ξ_k is a (possibly complex) local smooth square root of φ_k.

The last results motivate the following definition.

Definition 3.1.22. Given $h : L^{p+1} \to \mathbb{S}^n$ with a conjugate chart, let

$$\mathcal{S}^*_\mu = \{\varphi \text{ is admissible of index } \mu \text{ and } \partial_i\varphi_j = 2\Gamma^j_{ij}\varphi_j, \forall i \neq j\},$$

$$\mathcal{S}^* = \bigcup_{\mu=0}^{p} \mathcal{S}^*_\mu = \{\varphi \text{ is admissible and } \partial_i\varphi_j = 2\Gamma^j_{ij}\varphi_j, \forall i \neq j\}.$$

Remark 3.1.23. The moduli space $\mathcal{C}_h$ described in Theorem 1 of [15] is naturally related to our moduli space $\mathcal{S}^*$. Suppose that $H : L^3 \to \mathbb{S}^n$ has a conjugate chart (u_0, u_1, u_2) centered at the origin with u_2 real and $\mathcal{S}^*_0 \neq \emptyset$. Let $L^2 = \{u = 0\} \subseteq L^3$ and $h = H|_{L^2}$. Then there is an injection $\mathcal{S}^*_0 \to \mathcal{C}_h$ given by

$$\varphi = (\varphi_0, \varphi_1, \varphi_2) \to \Big(\frac{1}{2}\varphi_0|_{u_1=u_2=0}, \frac{1}{2}\varphi_1|_{u_0=u_2=0}\Big), \quad \text{if } (u_0, u_1) \text{ are real,}$$

$$\varphi = (\varphi_0, \varphi_1, \varphi_2) \to \frac{1}{2}\varphi_0|_{u_1=u_2=0}, \quad \text{if } (u_0, u_1) \text{ are complex.}$$

Indeed, using the notation in [15], the condition $Q(\rho^{UV}) = 0$ in the real case is just Remark 3.1.21 for $p = k = 2$. The complex case is analogous.

All the previous results can be summarized in the following.

Theorem 3.1.24. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a generic simply connected hypersurface of rank $2 \leq p+1 < n$. Suppose that f has a non-degenerate deformation in codimension p. Then the Gauss map $h : L^{p+1} \to \mathbb{S}^n$ possesses a conjugate chart with $\mathcal{S}^* \neq \emptyset$ and the support function satisfies $Q(\gamma) = 0$. Moreover, the set $\mathcal{S}^*_\mu \subseteq \mathcal{S}^*$ naturally parametrizes the moduli space of non-degenerate genuine deformations of f in $\mathbb{R}^{n+p}_\mu$.*

Conversely, any pair (h, γ) satisfying these properties is the Gauss data of a hypersurface $f : M^n \to \mathbb{R}^{n+1}$ which possesses non-degenerate genuine deformations in codimension p.

Remark 3.1.25. In the converse, the parametrized hypersurface may not be generic and then the set $\mathcal{S}^*$ parametrizes the non-degenerate deformations such that β is diagonalizable by the vectors $X_i \in \Gamma^\perp_\mathbb{C}$ given by (3.4). To verify if f is generic, we express the splitting tensor in terms of the Gauss data. Using [16], we see that for $(y, w) \in M^n = T^\perp_h L$, the splitting tensor is given by $C_\xi = B_\xi P_w^{-1}$ where $\xi \in T^\perp_h L(y) = \Delta(y, w)$ and P_w was defined in (3.17). Thus, the hypersurface is generic precisely in the open subset

$$U = U_{h,\gamma} = \{w \in T^\perp_h L : P_w \text{ is invertible and } \exists \xi \text{ such that } B_\xi P_w^{-1} \text{ is semisimple over } \mathbb{C}\}.$$

3.1.2 The moduli space $\mathcal{S}^*$

In this subsection we introduce the notion of species of a conjugate chart. This concept will characterize $\mathcal{S}^*$ and will also give a geometric description of it.

Suppose that $h : L^{p+1} \to \mathbb{S}^n$ has a conjugate chart $(u_0, \ldots, u_p)$. By Corollary 3.1.17 any section φ over the trivial $\mathbb{C}$-bundle $\mathbb{C}^{p+1} \to L^{p+1}$ that is also in $\mathcal{S}^*$ must satisfy

$$d\varphi + \omega\varphi = 0,$$

where $\omega : TL \to \operatorname{End}(\mathbb{C}^{p+1})$ is the bundle map $\omega_i(e_j) := \omega(\partial_i)(e_j) = \sum_k \omega^k_{ij} e_k$, where

$$\omega^k_{ij} = \begin{cases} -2\Gamma^j_{ij} & \text{if } k = j \neq i, \\ 2\Gamma^j_{ij} & \text{if } k = i \neq j, \\ 0 & \text{in other case.} \end{cases} \tag{3.28}$$

In other words, this element $\varphi \in \mathcal{S}^*$ is a parallel section of the connection $\tilde{\nabla} : \mathfrak{X}(L^{p+1}) \times \Gamma(\mathbb{C}^{p+1}) \to \Gamma(\mathbb{C}^{p+1})$ on the trivial bundle given by

$$\tilde{\nabla}\xi = d\xi + \omega\xi. \tag{3.29}$$

Notice that the conjugation $C(e_i) = e_{\bar{i}}$ is parallel with respect to $\tilde{\nabla}$ since C commutes with ω_i for all i. This motivates the following definition.

Definition 3.1.26. Consider a DMZ system

$$Q_{ij} = \partial^2_{ij} - \Gamma^i_{ji}\partial_i - \Gamma^j_{ij}\partial_j + g_{ij}, \quad \forall 0 \leq i < j \leq p,$$

defined on $L^{p+1} \subseteq \mathbb{R}^{p+1}$. We call the real affine bundle $F = (\operatorname{Re}_C(\mathbb{C}^{p+1}) \to L^{p+1}, \tilde{\nabla} = d + \omega)$ the *Sbrana bundle associated with Q*, where ω is defined by (3.28).

Remark 3.1.27. Whenever $h : L^{p+1} \to \mathbb{S}^n$ has a conjugate coordinate system, that is $Q(h) = 0$, then the Sbrana bundle is assumed to be associated with this DMZ system Q.

Any parallel section φ of the Sbrana bundle satisfies $\partial_i\big(\sum_j \varphi_j\big) = 0$ for any i, so the sum of the coordinates is constant. Thus, if $\varphi(q)$ is admissible and has index μ for some $q \in L^{p+1}$, then $\varphi \in \mathcal{S}^*_\mu$ in the neighborhood of q where $\varphi_i \neq 0$ for all i.

For completeness, we describe next a procedure to find all parallel sections of an affine bundle E, that is, the trivial holonomy component of the Sbrana bundle which is the maximal parallel flat subbundle of E. This procedure is an integration of the Ambrose-Singer Theorem. Since this result is local, we fix a trivialization and assume that $E = \mathbb{C}^N \to L^{p+1}$. Denote by

$$\omega = \tilde{\nabla} - d \in \Gamma(T^*L \otimes \operatorname{End}(\mathbb{C}^N)),$$

the connection 1-form and

$$\Omega_0 := \Omega = d\omega + [\omega, \omega] \in \Gamma(T^*L \otimes T^*L \otimes \operatorname{End}(\mathbb{C}^N)),$$

the curvature 2-form. Fix any connection ∇ for L^{p+1}, and define inductively

$$\Omega_k = \nabla\Omega_{k-1} - \Omega_{k-1} \circ \omega \in \Gamma\Big(\big(\bigotimes_{k=0}^{k+1} T^*L\big) \otimes \operatorname{End}(\mathbb{C}^N)\Big).$$

Consider the sets

$$\Delta_k := \{\varphi \in E : \Omega_k(X_0, \dots, X_k)\varphi = 0, \forall X_i \in TL\},$$

$$\mathcal{N}_k = \bigcap_{j=0}^{k} \Delta_j.$$

As usual, we assume that $\mathcal{N}_k$ is a smooth vector bundle of E for $k = 0, \dots, (N-1)$ since this is true along each connected component of an open dense subset of L^{p+1}.

Proposition 3.1.28. *Assume that $\mathcal{N}_k$ is a smooth subbundle of $E = (\mathbb{C}^N \to L^{p+1}, \tilde{\nabla})$ for $k = 0, \dots, N-1$. Then $\mathcal{N}_{N-1}$ is the maximal parallel flat subbundle of E. In particular, given any initial condition $\varphi_q \in \mathcal{N}_{N-1}(q)$ for some $q \in L^{p+1}$, there exists a unique parallel section φ of E such that $\varphi(q) = \varphi_q$ and $\varphi \in \Gamma(\mathcal{N}_{N-1})$.*

Proof. Suppose that $\varphi \in \Gamma(E)$ is a parallel section. Then as

$$0 = d(d\varphi + \omega\varphi) = d(\omega\varphi) = (d\omega + [\omega, \omega])\varphi,$$

we have that $\varphi \in \mathcal{N}_0$. If $\varphi \in \mathcal{N}_{k-1}$, then

$$\begin{aligned} 0 &= \nabla_{X_{k+1}}(\Omega_{k-1}(X_0, \dots, X_k)\varphi) = \nabla_{X_{k+1}}\Omega_{k-1}(X_0, \dots, X_k)\varphi - \Omega_{k-1}(X_0, \dots, X_k) \circ \omega(X_{k+1})\varphi \\ &= \Omega_k(X_0, \dots, X_{k+1})\varphi, \quad \forall X_i \in \Gamma(TL), \end{aligned}$$

which proves that $\varphi \in \mathcal{N}_k$ and inductively $\varphi \in \mathcal{N}_j$ for all j. Thus, any parallel flat subbundle is contained in $\mathcal{N}_{N-1}$. In particular, if $\mathcal{N}_{N-1} = 0$ there are no non-trivial parallel sections.

Assume that $\mathcal{N}_{N-1} \neq 0$, and consider the inclusions of $\mathbb{C}$-vector bundles

$$0 \neq \mathcal{N}_{N-1} \subseteq \ldots \subseteq \mathcal{N}_0 \subseteq \mathbb{C}^N =: \mathcal{N}_{-1}.$$

Let $k \in \{0, \dots, N-1\}$ be the first index such that $\mathcal{N}_k = \mathcal{N}_{k-1}$. If $k = 0$ this means that E is flat and all flat bundles possess (local) parallel sections given any initial condition. Notice that in this case $\mathcal{N}_{N-1} = \mathcal{N}_0 = \mathbb{C}^N$ since $\Omega_j = 0$ for all j. Assume that $k \geq 1$. For any section ξ of $\mathcal{N}_{k-1}$ and any $1 \leq j \leq k$ we have that

$$0 = \nabla(\Omega_{j-1}\xi) - \Omega_j\xi = \Omega_{j-1} \circ (\nabla\xi + \omega\xi),$$

which shows that $\tilde{\nabla}\xi = \nabla\xi + \omega\xi \in \Gamma(T^*L \otimes \Delta_{j-1})$. Hence $\tilde{\nabla}\xi \in \Gamma(T^*L \otimes \mathcal{N}_{k-1})$, but by the choice of k, this proves that $\mathcal{N}_k \subseteq E$ is a parallel subbundle and then $\mathcal{N}_k \subseteq \mathcal{N}_{N-1}$ by the maximality property. Therefore $\mathcal{N}_k = \mathcal{N}_{N-1}$ is a flat parallel subbundle, which concludes the proof. ☐

Using the above for the connection (3.29), we can give a description of the moduli space $\mathcal{S}^*$. First, we notice that for $i \neq j$ the i^{th}-row of $\Omega_0(\partial_j, \partial_i)$ is the same as its j^{th}-row up to sign, and the remaining rows are zero. Thus, we can collect the non-trivial information of Ω_0 in a single matrix. Let $B : \mathbb{C}^{p+1} \to \mathbb{C}^{\binom{p+1}{2}}$ whose coefficients for $0 \leq i < j \leq p$ are given by

$$B_{ijk} = \partial_i \Gamma^k_{jk} + 2\Gamma^k_{ik}\Gamma^k_{jk} - 2\Gamma^k_{ik}\Gamma^i_{ji} - 2\Gamma^k_{jk}\Gamma^j_{ij} \text{ for } k \notin \{i, j\},$$

$$B_{iji} = \partial_i \Gamma^i_{ji} - 2\Gamma^i_{ji}\Gamma^j_{ij},$$
$$B_{ijj} = \partial_j \Gamma^j_{ij} - 2\Gamma^i_{ji}\Gamma^j_{ij}.$$

Then the i^{th}-row of $\Omega(\partial_j, \partial_i)$ is $2B_{ij}$ for $i < j$. Notice that the last two coefficients are precisely the ones that appear in the Sbrana-Cartan classification, yet the first one is new. In the same way as before, to Ω_k we can associate a matrix B_k which contains its non-trivial data. Let $B_0 = B$ and inductively

$$B_{n+1} = \begin{pmatrix} \partial_0 B_n - B_n\omega_0 \\ \vdots \\ \partial_p B_n - B_n\omega_p \end{pmatrix} : \mathbb{C}^{p+1} \to \mathbb{C}^{\binom{p+1}{2}(p+1)^{n+1}}.$$

We conclude that

$$\mathcal{N}_p = \bigcap_{i=0}^{p} \ker(B_i).$$

Notice that the conjugation $C(e_i) = e_{\bar{i}}$ is parallel with respect to this connection and then, $\hat{\mathcal{N}}_p = \mathrm{Re}_C(\mathcal{N}_p)$ is the maximal parallel flat subbundle of the Sbrana bundle, i.e., its trivial holonomy component.

Definition 3.1.29. Let $h : L^{p+1} \to \mathbb{S}^n$ be a submanifold with a conjugate chart. We say that h is of the k^{th}-*species* for $1 \leq k \leq p+1$ if the trivial holonomy component of the Sbrana bundle $\hat{\mathcal{N}}_p \subseteq F$ has rank $(p+2-k)$ and is generic in the sense that it intersects the open dense subset $\{v \in F : v_i \neq 0 \text{ and } \sum_i v_i \neq 0\} \subseteq F$.

Remark 3.1.30. Our definition of species has a slight difference with the one in the Sbrana-Cartan to include semi-Riemannian ambient spaces. The condition that τ in (2.6) has to be positive when the conjugate directions are real guarantees that the unique element in $\mathcal{U}$ has index 0, in order to obtain a deformation in the Euclidean space, and not in the Lorentz space.

Remark 3.1.31. Given $h : L^{p+1} \to \mathbb{S}^n$ a submanifold with a conjugate chart, we call L^j a *slice* of L^{p+1} if L^j is obtained after fixing some of the conjugate coordinate variables to some values. In this case, the slice naturally has a conjugate chart for $H = h|_{L^j}$ by restricting the original coordinates to L^j. If $h : L^{p+1} \to \mathbb{S}^n$ is of k^{th}-species, with $\min\{2, k\} < p+1$, generically we can construct new submanifolds of some species by taking slices. Indeed, let $L^j \subseteq L^{p+1}$ a slice with $k \leq j$, then the trivial holonomy component of the Sbrana bundle of $H = h|_{L^j}$ is at most $(p+2-k)$. Indeed, the rank of the Sbrana bundle of H is $(p+2-k)$ if and only if the matrix

$$\mathcal{B} = \mathcal{B}_{L^{p+1}} = (B_0^T \, B_1^T \, \dots \, B_p^T)^T,$$

has rank $(k-1)$. Notice that the matrix $\mathcal{B}_{L^j}$ appears as a submatrix of the original $\mathcal{B}_{L^{p+1}}$, so it has less or equal rank. The condition of the trivial holonomy component being generic is generically satisfied and in that case, H is of l^{th}-species for some $l \leq k$.

Assume now that h is of the k^{th}-species for $1 \leq k \leq p+1$, fix $q \in L^{p+1}$ and let

$$\mathcal{U} = \{u \in \hat{\mathcal{N}}_p(q) : u \text{ is admissible}\} \subseteq \hat{\mathcal{N}}_p(q) \cap \{u = (u_i)_i : 1 + \sum_i u_i = 0\} \cong \mathbb{R}^{p+1-k}. \tag{3.30}$$

We have the natural bijection $u \to \varphi_u$ between $\mathcal{U}$ and $\mathcal{S}^*$, where $\varphi_u \in \Gamma(\hat{\mathcal{N}}_p)$ is the parallel section which satisfies $\varphi_u(q) = u$. Naturally, the open subset

$$\mathcal{U}_\mu = \{u \in \mathcal{U} : u \text{ is admissible and has index } \mu\} \subseteq \mathcal{U}, \tag{3.31}$$

is in bijection with $\mathcal{S}^*_\mu$. We conclude:

Theorem 3.1.32. *Suppose that $h : L^{p+1} \to \mathbb{S}^n$ is of k^{th}-species for some $k \in \{1, \dots, p+1\}$. Then $\mathcal{S}^*$ and $\mathcal{S}^*_\mu$ are naturally diffeomorphic to a finite union of open and convex subsets of $\mathbb{R}^{p+1-k}$ for all $0 \leq \mu \leq p$. Moreover, $\mathcal{S}^*_0 \cong \mathcal{U}_0 \subseteq \mathbb{R}^{p+1-k}$ has at most $(p+1)$ connected components.*

Proof. By the above discussion, we only need to bound the number of connected components of $\mathcal{S}_0^*$. Proposition A.1.2 bounds the number of connected components of the set $U = \{\varphi : \varphi \text{ is admissible of index } 0\}$. If the conjugate chart has a complex conjugate chart this set is convex. If the conjugate coordinates are real then U has $(p+1)$ convex components determined by the choice of which coordinate is negative. Thus, $\mathcal{U}_0 = U \cap \mathcal{N}_p$ has at most $(p+1)$ components that are convex since each one is an intersection of convex subsets. □

In order to recover the discrete and continuous types of hypersurfaces in the Sbrana-Cartan classification, we introduce the following concept. The last remark also let us bound the number of connected components.

Definition 3.1.33. We say that a generic hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $2 \le p+1 < n$ is of the r^{th}*-type*, for $r \in \{0, \dots, p\}$ if the set of genuine deformations $g : M^n \to \mathbb{R}^{n+p}$ is naturally an union of at most $(p+1)$ convex open subsets of $\mathbb{R}^r$.

Finally, we can prove Theorem 3.0.1.

Proof of Theorem 3.0.1. As discussed in the preliminaries, any hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $2 < p+1 < n$ is genuinely rigid in $\mathbb{R}^{n+q}$ for any $q < p$ if we add the hypotheses of not being $(n-p+3)$-ruled for $p \ge 7$. To conclude the proof, by Theorem 3.1.24 and Theorem 3.1.32 we only need to show that any genuine deformation $g : M^n \to \mathbb{R}^{n+p}$ of f is non-degenerate.

First, observe that the cases $p \le 4$ and $p \ge 7$ are immediate by Corollary 3.1.5.

For $p \in \{5, 6\}$ we use Remark 3.1.2. Assume that $\mathcal{S}(\beta)$ degenerates. Then by Remark 3.1.2, $\Delta_g = \Gamma \subsetneq R^d$, where R^d is some mutual ruling for f and g. Denote $\tilde{R} = R \cap \Gamma^\perp$. As R^d is totally geodesic, $C_T(\tilde{R}) \subset \tilde{R}$ for all $T \in \Gamma$, and then by the generic condition we get $X_i \in \tilde{R}$ where X_i is some eigenvector of the semisimple endomorphism C_{T_0}. However, this implies that $X_i \in \Gamma$, since the eigenvectors of C_{T_0} diagonalize β by (3.3) and $\beta(X_i, X_i) = 0$ as $X_i \in R^d$, which is a contradiction. Thus, $\mathcal{S}(\beta)$ is non-degenerate. The Main Lemma and Corollary 2 of [36] imply that g is non-degenerate. □

3.2 Deformations of generic hypersurfaces in codimension 2

In this section, we apply Theorem 3.0.1 to prove Theorem 3.0.2. It is an analogous description to the one given by Sbrana and Cartan, and characterizes all the deformable generic hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ in codimension 2 and its moduli space of deformations. As already observed, the hypothesis of being generic is to discard the surface-like and ruled type of situation.

We start by recalling Sbrana-Cartan hypersurfaces of *intersection type*, as named in [27]. They are Riemannian submanifolds M^n obtained by intersecting two flat hypersurfaces $F : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}_\nu$ and $G : U_1 \subseteq \mathbb{R}^{n+1}_\mu \to \mathbb{R}^{n+2}_\nu$ in general position. Then

$$M^n = F_1(U_1) \cap F_2(U_2) \subseteq \mathbb{R}^{n+2}_\nu,$$

f, g stands for the inclusions of M^n into U_1 and U_2 respectively, and $H := F \circ f = G \circ g$. They were introduced in [14] for $(\mu, \nu) = (0, 0)$ and studied in [8] for $(\mu, \nu) = (1, 1)$. The case $(\mu, \nu) = (0, 1)$ is new and necessary to present the deformations of hypersurfaces in codimension 2.

A hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of intersection type is determined by the conjugate chart $(u, v) \in \mathbb{R}^2$ of its Gauss map $h : L^2 = M^n/\Gamma \to \mathbb{S}^n$. In fact, the Christoffel symbols satisfy

$$\partial_v \Gamma^v_{uv} - \Gamma^u_{vu} \Gamma^v_{uv} + g_{uv} = 0. \tag{3.32}$$

Namely, if Q is the hyperbolic linear operator

$$Q := \partial^2_{uv} - \Gamma^u_{vu} \partial_u - \Gamma^v_{uv} \partial_v + g_{uv},$$

for which $Q(h) = Q(\gamma) = 0$ where γ is the support function of f, then one of its Laplace invariants vanishes. Moreover, if (3.32) holds, then any non-degenerate deformation of f is obtained as an intersection. In fact, in [14] they show that if g is any such deformation of f given by $\varphi = (\varphi_0, \varphi_1) \in \mathcal{S}^*$ with $\varphi_1 < -1$ then the index of φ is $\mu = 0$ and the intersection is in $\mathbb{R}^{n+2}$. If $\varphi_1 \in (-1, 0)$ then the index of φ is $\mu = 1$ and they intersect in $\mathbb{R}^{n+2}_1$ as in [8]. Similarly, if $\varphi_1 > 0$ then the index of φ is $\mu = 0$ and the intersection is in $\mathbb{R}^{n+2}_1$.

By Theorem 1 of [12], in order for a generic hypersurface $f : M^n \to \mathbb{R}^{n+1}$ to have a genuine deformation in codimension 2, its rank must be at most 3. If it is less than 2, then M^n is flat, and all the local immersions are described in Corollary 18 of [27]. Theorem 3.0.1 characterizes the rank 3 case. Theorem 1 of [15] describes when the rank is 2, but this result has a gap that we discuss next.

If $f : M^n \to \mathbb{R}^{n+1}$ is a Sbrana-Cartan hypersurface, $g : M^n \to U \subseteq \mathbb{R}^{n+1}$ a genuine deformation of f and $j : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ an isometric immersion with $\alpha^j \neq 0$, then $\hat{g} = j \circ g$ is generically also a genuine deformation of f which is not considered in Theorem 1 of [15]. In particular, for rank two generic Sbrana-Cartan hypersurfaces, there are more genuine deformations than the moduli space $\mathcal{C}_h$ described in that paper. As defined in [27], we say that a genuine deformation g of f is *honest* if g is not a composition as before. The set $\mathcal{C}_h$ measures the honest deformations of f except for Sbrana-Cartan hypersurfaces of intersection type. For such hypersurfaces, some deformations described by $\mathcal{C}_h$ are not honest. Indeed, let $\hat{g} : M^n \to \mathbb{R}^{n+2}$ be a genuine deformation of a rank 2 generic hypersurface f associated with some element in $\mathcal{C}_h$ and assume that $\hat{g} = j \circ g$ for some isometric immersions $g : M^n \to U \subseteq \mathbb{R}^{n+1}$ and $j : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$. If $\Gamma \subseteq TM$ is elliptic (that is, some splitting tensor C_T has non-real eigenvalues), then

$$\alpha^j(u, u) + \alpha^j(v, v) = 0,$$

for some basis $u, v \in \Gamma^\perp$. This and the flatness of α^j imply that $\alpha^{\hat{g}} = \alpha^g$. This is a contradiction since the deformations described by $\mathcal{C}_h$ satisfy that $\dim(\mathcal{S}(\alpha^{\hat{g}})) = 2$. Then $\Gamma \subseteq TM$ is hyperbolic (that is, some splitting tensor C_T is semisimple over $\mathbb{R}$), and let $(u, v) \in \mathbb{R}^2$ be the conjugate chart of the Gauss map $h : L^2 = M^n/\Gamma \to \mathbb{S}^n$ of f satisfying (3.4) for $X_0, X_1 \in \Gamma^\perp$ the eigenvectors of the splitting tensors of $\Gamma \subseteq TM$. Then g and $\hat{g}$ are genuine deformations of f associated to some $\varphi = (\varphi_0, \varphi_1) \in \mathcal{S}^\#$ and $(U, V) \in \mathcal{C}_h$ respectively. Define

$$\hat{\varphi}_0(u, v) := 2U(u)e^{\int_0^v 2\Gamma^u_{vu}(u,s)ds}, \quad \hat{\varphi}_1(u, v) := 2V(v)e^{\int_0^u 2\Gamma^v_{uv}(s,v)ds} \quad \text{and} \quad \hat{\varphi}_* := -(1 + \hat{\varphi}_0 + \hat{\varphi}_1) \neq 0.$$

By the definition of $\mathcal{C}_h$ we have that $\hat{\varphi} = (\hat{\varphi}_0, \hat{\varphi}_1, \hat{\varphi}_*)$ is admissible of index 0 and $Q(\sqrt{|\hat{\varphi}_*|}) = 0$. Codazzi equation implies that

$$\alpha^g(X_0, X_1) = \alpha^{\hat{g}}(X_0, X_1) = \alpha^j(X_0, X_1) = 0,$$

and for $i = 0, 1$, let

$$\hat{\eta}_i := \frac{\alpha^{\hat{g}}(X_i, X_i)}{\langle AX_i, X_i\rangle} = \frac{\alpha^g(X_i, X_i)}{\langle AX_i, X_i\rangle} + \frac{\alpha^j(X_i, X_i)}{\langle AX_i, X_i\rangle} =: \eta_i + \varepsilon_i.$$

By flatness of j and dimension reasons, we can assume that $\varepsilon_1 \neq 0$ and $\varepsilon_0 = 0$. Thus,

$$1 + \frac{1}{\hat{\varphi}_0} = \langle \hat{\eta}_0, \hat{\eta}_0\rangle = \langle \eta_0, \eta_0\rangle = 1 + \frac{1}{\varphi_0}.$$

Here we used the geometric interpretation of $(U, V) \in \mathcal{C}_h$. Then $\hat{\varphi}_* = \varphi_1 - \hat{\varphi}_1$ and by (3.22) we have that $\partial_u \hat{\varphi}_* = 2\Gamma^v_{uv}\hat{\varphi}_*$, but in this case,

$$0 = Q(\sqrt{|\hat{\varphi}_*|}) = (\partial_v \Gamma^v_{uv} - \Gamma^u_{vu}\Gamma^v_{uv} + g_{uv})\sqrt{|\hat{\varphi}_*|}.$$

Thus, all the genuine deformations described by $\mathcal{C}_h$ are honest except when (3.32) is satisfied, that is, when the hypersurface is of intersection type.

Theorem 1 of [15] for Sbrana-Cartan hypersurfaces of intersection type only says that the moduli space of honest deformations is a subset of $\mathcal{C}_h$. However, Theorem 33 of [27] classifies all the honest deformations in codimension 2 for hypersurfaces obtained as intersections in $\mathbb{R}^{n+2}$. Thus, we need to extend some concepts and results of [27] to describe the honest deformations of Sbrana-Cartan hypersurfaces which are intersections in $\mathbb{R}^{n+2}_1$. Almost all the ideas are analogous, so we will leave the details to the reader.

Let $H : M^n \to \mathbb{R}^{n+2}_\mu$ be a generic Riemannian submanifold of rank 2, $\mathcal{S}(\alpha^H) = T^\perp_H M$. Then we can construct a polar surface in a similar way as in [27] or [9]. If $\Delta_H \subseteq TM$ is hyperbolic (the eigenvectors of the splitting tensors are real), then the polar surface is an immersion $g : L^2 = M^n/\Delta_H \to \mathbb{R}^{n+2}_1$ such that $g_*(TL) = T^\perp_H M$ and has conjugate coordinates $(u, v) \in \mathbb{R}^2$. Namely, it satisfies a hyperbolic linear differential equation

$$\tilde{Q}(g) = \partial^2_{uv} g - \tilde{\Gamma}^u_{vu}\partial_u g - \tilde{\Gamma}^v_{uv}\partial_v g = 0.$$

Let $f : M^n \to \mathbb{R}^{n+1}$ be a Sbrana-Cartan hypersurface obtained as an intersection of two flat Riemannian hypersurfaces of $\mathbb{R}^{n+2}_1$. The inclusion $H : M^n \to \mathbb{R}^{n+2}_1$ satisfies $\Delta_H = \Gamma$. Thus it has a polar surface. Moreover, as discussed in Section 9 of [27], this surface is the sum of two curves

$$g(u, v) = \alpha_1(u) + \alpha_2(v),$$

with $\alpha_1', \alpha_1'', \alpha_2', \alpha_2''$ being pointwise linearly independent, $\langle \alpha_1', \alpha_1'\rangle = \langle \alpha_2', \alpha_2'\rangle = -1$ and $\cosh(\theta) := -\langle \alpha_1', \alpha_2'\rangle$. This characterizes the hypersurfaces of intersection type obtained as the intersection of two Riemmanian flat hypersurfaces in $\mathbb{R}^{n+2}_1$.

Similarly to Theorem 32 of [27], the Sbrana-Cartan hypersurface of intersection type is of discrete type if $I(H) \geq 2$ and continuous if $I(H) = 1$, where $I(H) := I(\alpha_1, \alpha_2)$ is the shared dimension of α_1 and α_2 as defined in Section A.2.

The following result is an adaptation of Theorem 33 of [27] for Lorentz ambient space.

Theorem 3.2.1. *Let $f : M^n \to \mathbb{R}^{n+1}$ be a Sbrana-Cartan hypersurface obtained as an intersection of two flat Riemannian hypersurfaces of $\mathbb{R}^{n+2}_1$, and let $H : M^n \to \mathbb{R}^{n+2}_1$ be the inclusion. Then f is honestly rigid in $\mathbb{R}^{n+2}$, unless $I(H) = 2$. In the latter case, the moduli space of honest deformations is an open interval of $\mathbb{R}$.*

Proof. Since the proof is analogous to Theorem 33 of [27], we will only point out the slight differences. Using the notations in [27], we have in particular $s := -\sinh(\theta)^2$, and our analogous functions $U = U(u)$ and $V = V(v)$ must satisfy

$$U, V > -\frac{1}{s}, \quad \text{and} \quad (U+1)(V+1) < (1-s)UV. \tag{3.33}$$

The hypersurface f is honestly rigidity in $\mathbb{R}^{n+2}$ for $I(H) \neq 2$, for analogous reasons. When $I(H) = 2$, [27] uses the geometric characterization of this index to project into the shared space, which may not be possible in Lorentz ambient space. However, since $\operatorname{span}(\alpha_1)$, $\operatorname{span}(\alpha_2) \subseteq \mathbb{R}^{n+2}_1$ are Lorentzian subspaces, let $\mathbb{V}^l \subseteq \mathbb{R}^{n+2}_1$ be the Lorentz subspace given by Lemma A.2.1, with $l \leq 2$. If $l = 1$ then $I(H) = 1$, so $l = 2$. Define $\overline{\alpha}_i$ as the orthogonal projection of α_i in $\mathbb{V}^2$ for $i = 1, 2$. Then $\overline{\alpha}_1$, $\overline{\alpha}_2$ are light-like curves of $\mathbb{V}^2$ and

$$\langle \overline{\alpha}_1', \overline{\alpha}_1' \rangle \langle \overline{\alpha}_2', \overline{\alpha}_2' \rangle < \langle \overline{\alpha}_1', \overline{\alpha}_2' \rangle^2 = \langle \alpha_1', \alpha_2' \rangle^2 = \cosh(\theta)^2 = 1 - s. \tag{3.34}$$

Here we used the Cauchy Schwarz inequality for time-like vectors. Those curves work as the curves defined in [27] with the same notations.

Following the steps in the proof given in [27], we see that the moduli space of honest deformations is in bijection with

$$(U_t, V_t) = \Big((t^{-1} \langle \overline{\alpha}_1', \overline{\alpha}_1' \rangle - 1)^{-1}, (t \langle \overline{\alpha}_2', \overline{\alpha}_2' \rangle - 1)^{-1} \Big),$$

for $0 \neq t \in \mathbb{R}$ such that (3.33) is satisfied. That and (3.34) give us that t must satisfy

$$\langle \overline{\alpha}_1'(u), \overline{\alpha}_1'(u) \rangle < t < \langle \overline{\alpha}_2'(v), \overline{\alpha}_2'(v) \rangle^{-1}. \tag{3.35}$$

This is possible since $\langle \overline{\alpha}_1', \overline{\alpha}_1' \rangle \langle \overline{\alpha}_2', \overline{\alpha}_2' \rangle > \langle \alpha_1', \alpha_1' \rangle \langle \alpha_2', \alpha_2' \rangle = 1$. If t satisfies the above inequality for $(u, v) = (u_0, v_0)$, then (3.35) holds for (u, v) in a neighborhood of (u_0, v_0). Hence the honest deformations in $\mathbb{R}^{n+2}$ are in natural bijection with the open subset $\Big(\langle \overline{\alpha}_1'(u_0), \overline{\alpha}_1'(u_0) \rangle, \langle \overline{\alpha}_2'(v_0), \overline{\alpha}_2'(v_0) \rangle^{-1} \Big) \subseteq \mathbb{R}$. ◻

3.3 Riemannian hypersurfaces in Lorentz ambient space

All our analysis above can be translated for Riemannian hypersurfaces of the Lorentz space, that is, for generic hypersurfaces $f : M^n \to \mathbb{R}^{n+1}_1$ of rank $(p+1)$. In this section, we provide some remarks about this together with an application for studying conformally flat Euclidean submanifolds. As the analysis is similar to the Euclidean case, we leave the details to the reader.

Analogously to the Euclidean case, there is a Gauss parametrization (h, γ) for Riemannian submanifolds $F : M^n \to \mathbb{R}^{n+1}_1$ of rank $(p+1)$, where $h : L^{p+1} \to \mathbb{H}^n$ and $\gamma : L \to \mathbb{R}$ (see [16]). This parametrization can be used in the same way as before to study deformations of Lorentzian hypersurfaces.

Suppose that there is a non-degenerate deformation $g : M^n \to \mathbb{R}^{n+p}_\mu$ of f. If M^n is generic, then we can define φ_i and η_i as in (3.5) and (3.6), but in this case $\langle \eta_i, \eta_j \rangle = -(1 + \frac{\delta_{ij}}{\varphi_i})$, instead of (3.7). This shows that the index of φ is $p - \mu$, instead of being μ as in the Riemannian case. The diagonalizing directions also define a conjugate chart for $h : L^{p+1} \to \mathbb{H}^n$, in the same way as for submanifolds of the sphere, but in this case

$$Q_{ij}(h) = \partial^2_{ij} h - \Gamma^i_{ji} \partial_i h - \Gamma^j_{ij} \partial_j h - g_{ij} h = 0, \quad \forall i \neq j.$$

We define $\mathcal{S}^*_\mu$ and $\mathcal{S}^*$ as in Definition 3.1.22. Theorem 3.1.24 holds as for Euclidean hypersurfaces, but $\mathcal{S}^*_\mu$ parametrize the non-degenerate deformations of f in $\mathbb{R}^{n+p}_{p-\mu}$. Moreover, the concept of species can also be used to give an interpretation of $\mathcal{S}^*$.

This can be used to study conformally flat Euclidean submanifolds, namely, submanifolds $f : M^n \to \mathbb{R}^{n+p+1}$ that are conformally flat. It is known that a simply connected manifold M^n with $n \geq 3$, is conformally flat if and only if it can be realized as a hypersurface of the light cone $V^{n+1} = \{X \in \mathbb{R}^{n+2}_1 : \langle X, X\rangle = 0, X \neq 0\} \subseteq \mathbb{R}^{n+2}_1$ (see for example [2], [7]). Thus, to obtain examples of conformally flat manifolds of $\mathbb{R}^{n+p+1}$ ($p \geq 1$), we can take a Riemannian manifold N^{n+1} which has isometric immersions $F : N^{n+1} \to \mathbb{R}^{n+2}_1$ and $G : N^{n+1} \to \mathbb{R}^{n+p+1}$, and take M^n as the intersection $F(N^{n+1}) \cap V^{n+1}$ and $g = G|_{M^n}$. The first main result of [7] states that this procedure generates all the simply connected examples for $p \leq n-4$.

Consider $F : N^{n+1} \to \mathbb{R}^{n+2}_1$ a nowhere flat hypersurface of rank $(p+1) \geq 2$. Let $G : N^{n+1} \to \mathbb{R}^{n+q+1}$ be an isometric immersion. The Main Lemma for $\beta = \alpha^G \oplus \alpha^F$ proves that $q \geq p$, and if $q = p$ then $\mathcal{S}(\beta) = W = T^\perp_G M \oplus T^\perp_F M$. Assume that $q = p$. Notice that G is always a non-degenerate deformation of F since W has positive signature. The techniques of this work can be used in this context analogously. In this case, the existence of the diagonalizing directions $X_i \in \Gamma^\perp$ for $\beta = \alpha^G \oplus \alpha^F : TN \times TN \to W^{p+1,0}$ comes from Theorem 2 of [36]. Thus, the condition of being generic is not necessary in this context.

The proof of Theorem 3.0.1 can be easily adapted to prove Theorem 3.0.3. When N^{n+1} in Theorem 3.0.3 is also generic, the conjugate chart is uniquely determined up to order and re-scaling factors of the basis. In this case, all the isometric immersions $G : N^{n+1} \to \mathbb{R}^{n+1+p}$ are in bijection some $\varphi = \varphi_G \in \mathcal{S}^*_p$. We can define the type of the hypersurface F in the same way as in Definition 3.1.33. Thus, if the hypersurface F is of the r^{th}-type, the set of such G's is in bijection with $\mathcal{U}_p \subseteq \mathbb{R}^r$ as in (3.31). In this case, $\mathcal{U}_p$ is actually diffeomorphic to $\mathbb{R}^r$. Indeed, since the index of φ must be 0, Proposition A.1.2 guarantees that $\varphi_i \in (-1, 0)$ for all i, which is a convex set, thus $\mathcal{U}_p$ it is also convex.

CHAPTER 4

Intersections

In order for $f : M^n \to \mathbb{R}^{n+1}$ to be genuinely deformable in $\mathbb{R}^{n+p}$ its rank must be at most $p+1$. However, even for maximal rank $p+1$, the problem of describing all genuinely deformable hypersurfaces and their deformations is in general too complex even for $p = 1$; for example, it is currently out of reach for surfaces. For this reason, we need to restrict ourselves to *generic* hypersurfaces to discard surface-like situations.

The main results of Chapter 3 described the generic hypersurfaces f of rank $p+1$ which are genuinely deformable in $\mathbb{R}^{n+p}$. Moreover, we characterized the moduli space of all genuine deformations of f up to codimension p: it is empty for $q < p$, and in $\mathbb{R}^{n+p}$ it is homeomorphic to certain locally convex open subset of $\mathbb{R}^r$ for some $r \leq p$. Specifically, any genuine deformation $g : M^n \to \mathbb{R}^{n+p}$ is determined by a parallel section φ of what we called the Sbrana bundle, denoting this dependence as $g = g^\varphi$. Furthermore, we use of the Gauss parametrization to give an analytical description of those hypersurfaces.

We dedicate the first part of this Chapter to extend the techniques used in [14] to produce a large set of examples of genuinely deformable submanifolds for $1 = q \leq p \leq (n-2)$ as in Chapter 3. Furthermore, the discussion will suggest to naturally extend the concept of genuine rigidity in order to gain the transitivity property.

Before we present our next result, we recall a few concepts. We denote by $\mathbb{R}^{n+p}_\nu$ the Euclidean space with a non-degenerate inner product of index ν. The *rank* of a submanifold $F : N^n \to \mathbb{R}^{n+p}_\nu$ is the codimension of the nullity $\Delta_F \subseteq TM$ of its second fundamental form α^F. We say that F is *full* when the image of α^F spans $T^\perp_F M$. The strategy to obtain examples of genuinely deformable hypersurfaces will be the following. Let $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+p+1}_\nu$ and $F_2 : U_2 \subseteq \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+1}_\nu$ be full flat submanifolds of ranks p and 1 respectively. Assume that they intersect transversally on $M^n := F_1(U_1) \cap F_2(U_2)$, and call f_1, f_2 the inclusions of M^n into U_1 and U_2, respectively. The next diagram describes our situation:

$$
\begin{tikzcd}
 & U_2 \subseteq \mathbb{R}^{n+p} \arrow[dr, "F_2"] & \\
M^n \arrow[ur, "f_2"] \arrow[rr, "f"] \arrow[dr, "f_1"'] & & \mathbb{R}^{n+p+1}_\nu \\
 & U_1 \subseteq \mathbb{R}^{n+1} \arrow[ur, "F_1"'] &
\end{tikzcd}
\tag{4.1}
$$

Theorem 4.0.1 extends the intersection of two flat hypersurfaces as in [14], and it shows that generically f_2 is a genuine deformation of f_1. Just as in [14], we characterize analytically the resulting hypersurfaces with the vanishment of certain Laplace invariants of Q.

Theorem 4.0.1. *Using the notations as above, assume that F_1 and F_2 intersect generically. Then f_1 has rank $(p+1) < n$, is generic, and f_2 is a genuine deformation of f_1. Moreover, the Laplace invariants m_{ip} of the associated DMZ system Q vanish for all $i < p$.*

Conversely, let $f_1 : M^n \to \mathbb{R}^{n+1}$ be a generic hypersurface of rank $(p+1) < n$ genuinely deformable in $\mathbb{R}^{n+p}$. If $p \geq 7$ assume further that f_1 is not $(n-p+2)$-ruled. Suppose that the Laplace invariants m_{ip} vanish for all $i < p$. Then, locally, f_1 and any genuine deformation $f_2 : M^n \to \mathbb{R}^{n+p}$ of it are obtained as an intersection of flat submanifolds as above.

The index ν of the Theorem 4.0.1 is determined by f_2. If $f_2 = f_2^{\varphi}$ then $\nu = \frac{\operatorname{sign}(\varphi_p)+1}{2} \in \{0,1\}$, where $\operatorname{sign}(x) = x/|x|$ for $x \neq 0$.

We can study new types of intersections using these ideas. Consider $F : \hat{M}^{n+q} \to \mathbb{R}_\nu^{n+q+1}$ a rank $(p+1) < n-q$ hypersurface with a non-degenerate deformation $G : \hat{M}^{n+q} \to \mathbb{R}^{n+q+p}$ (a non-degenerate deformation is a deformation slightly stronger than a genuine one). Assume that $F_2 := F$ intersects generically with a full flat submanifold $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_\nu^{n+q+1}$ of rank q. Denote the intersection by $M^n := F_1(U_1) \cap F_2(\hat{M}^{n+q})$ and f_1, f_2 the inclusions of M^n into U_1 and $\hat{M}^{n+q}$, respectively. The following diagram describes our situation:

$$\begin{tikzcd}
 & & \mathbb{R}^{n+q+p} \\
 & \hat{M}^{n+q} \arrow[ur, "G"] \arrow[dr, "F=F_2"] & \\
M^n \arrow[ur, "f_2"] \arrow[rr, "f"] \arrow[dr, "f_1"'] \arrow[uurr, bend left, "g"] & & \mathbb{R}_\nu^{n+q+1} \\
 & U \subseteq \mathbb{R}^{n+1} \arrow[ur, "F_1"'] &
\end{tikzcd} \tag{4.2}$$

Our next result shows that $g = G \circ f_2$ is generically a genuine deformation of f_1. Again, we are able to characterize this construction as the vanishment of certain Laplace invariants. This allows us to find new examples of genuinely deformable hypersurfaces from old ones. Observe that the indices have a natural conjugation, that is, we denote by $\bar{i}$ the unique index such that $\overline{u_i} = u_{\bar{i}}$ (if $u_i \in \mathbb{R}$ then $\bar{i} = i$).

Theorem 4.0.2. *Using the notations as above, assume that F_1 and F_2 intersect generically. Then g is a non-degenerate deformation of the rank $(p+q+1)$ generic hypersurface $f_1 : M^n \to \mathbb{R}^{n+1}$. Moreover, there is a subset $I \subseteq \{0,\dots,p+q\}$ with $\overline{I} = I$ and $|I| = p+1$ such that the Laplace invariants $m_{j\alpha}$ and $m_{i\alpha j}$ of the associated DMZ system Q vanish for all $i,j \in I$ and $\alpha \in I^c = \{0,\dots,p+q\} \setminus I$.*

Conversely, let $f_1 : M^n \to \mathbb{R}^{n+1}$ be a rank $(p+q+1) < n$ generic hypersurface, and $g : M^n \to \mathbb{R}^{n+q+p}$ be a genuine deformation of f_1. For $(p+q) \geq 7$ assume that f is not $(n-p-q+2)$-ruled. Suppose that the Laplace invariants $m_{j\alpha}$ and $m_{i\alpha j}$ of the associated DMZ system Q vanish for some $I = \overline{I} \subseteq \{0,\dots,p+q\}$. Then, locally, f_2 and any genuine deformation $g : M^n \to \mathbb{R}^{n+q+p}$ of it are obtained as an intersection as described in (4.2).

As before, if $g = g^{\varphi}$ then the index ν is given by $\nu = \frac{\operatorname{sign}(\varphi_\#)+1}{2} \in \{0,1\}$, where $\varphi_\# = \sum_{i \in I} \varphi_i$.

This Chapter is organized as follows. We start by recalling some concepts. In Section 4.2 we present the basic properties of the intersections that we are interested in. Section 4.3 and Section 4.4 are dedicate to prove Theorem 4.0.1 and Theorem 4.0.2, respectively.

4.1 Preliminaries

Since the objects discussed in this subsection are used only in this Chapter, they were left aside of Chapter 2. First, we reformulate some equations from the last Chapter that will be important in the following sections. We also add an important formula that will be used extensively; see Lemma 4.1.1. Then, we describe the flat submanifolds that we will intersect.

Suppose that f and g are as in Section 3.1. Namely, $g : M^n \to \mathbb{R}_\nu^{n+p}$ is a non-degenerate deformation of a generic hypersurface $f : M^n \to \mathbb{R}^{n+1}$ of rank $(p+1) < n$. Then $g = g^{\varphi}$ is determined by a parallel section φ of the Sbrana bundle. Let X_i be the vectors determined by (3.4), η_i given by (3.6), and φ_i defined in (3.7). The Gauss equation (3.7) determines the shape operators A_{η_i} by

$$A_{\eta_i} X_j = \Big(1 + \frac{\delta_{ij}}{\varphi_i}\Big) A X_j. \tag{4.3}$$

Recall the normal connections 1-forms given by (3.13). Those 1-forms descend to the leaf space and, as shown in Lemma 3.1.16, they are given by

$$\phi_{ij}(\partial_k) = \delta_{ij} \frac{\partial_k(\varphi_i^{-1})}{2} + (1-\delta_{ij})\Big(\delta_{jk}\frac{\Gamma_{ji}^i}{\varphi_i} - \delta_{ik}\frac{\Gamma_{ij}^j}{\varphi_i}\Big), \tag{4.4}$$

which together with (3.14) determine the normal connection.

As seen above, the frame of directions defined by the coordinate vector fields of the conjugate chart of $h : L^{p+1} \to \mathbb{S}^n$ is unique and intrinsic to M^n, and the lifts of those vectors to $\Gamma_{\mathbb{C}}^{\perp}$ correspond to the eigenvectors of the splitting tensors of $\Delta_f = \Gamma$. However, the conjugate chart itself is not really unique, since two conjugate charts of h are related by $u_i = f_i(\hat{u}_i)$. In particular, the DMZ system Q itself depends on this choice of coordinates. On the other hand, the vanishment of the functions

$$m_{ij} = -\partial_i \Gamma^i_{ji} + \Gamma^i_{ji}\Gamma^j_{ij} - g_{ij},$$

$$m_{ijk} = \Gamma^i_{ji} - \Gamma^k_{jk},$$

called the (i, j) and $(i, j, k)-$*Laplace invariants of* Q, respectively, are invariant under this natural change of coordinates. More precisely, if $u_i = f_i(\hat{u}_i)$, then $\hat{m}_{ij} = f_i' f_j' m_{ij}$ and $\hat{m}_{ijk} = f_j' m_{ijk}$. Thus, conditions like $m_{ij} = 0$, $m_{ijk} = 0$, $m_{ij} = m_{ji}$, and so on, are intrinsic properties of M^n. Moreover, for any smooth function $\lambda \in \mathcal{C}^\infty(L^{p+1})$, the homothety λh is a solution to a DMZ system $\tilde{Q} = \tilde{Q}_{Q,\lambda}$ whose Laplace invariants coincide with those of Q. Laplace invariants were introduced in [33] as a generalization of the Laplace invariants of hyperbolic equations in the plane.

The Sbrana bundle and the species are also invariant under change of coordinates $u_i = f_i(\hat{u}_i)$. In this case, the Christoffel symbols and the metric are related by

$$\hat{\Gamma}^i_{ji} = f_j' \Gamma^i_{ji}, \quad \hat{g}_{ij} = f_i' f_j' g_{ij}.$$

Nevertheless, they are not invariant under homotheties. In this sense, the Sbrana bundle and the species are natural invariants of the net defined by the coordinate vectors.

We prove now an additional formula that will be useful later.

Lemma 4.1.1. *With the above notations, for $k \neq i$ we have that*

$$\langle AX_i, \nabla_{X_j} X_k \rangle + \delta_{jk} \langle AX_j, X_j \rangle \Gamma^j_{ij} = 0, \quad \forall i \neq j, \tag{4.5}$$

where ∇ is the Levi-Civita connection of M^n.

Proof. Notice that the Gauss map is given by $\rho = h \circ \pi$, where $\pi : M^n \to M^n/\Gamma = L^{p+1}$ is the quotient map. Hence, $\partial_i h = \rho_* X_i = -AX_i$ and $\partial^2_{ij} h = \overline{\nabla}_{X_j}(-AX_i)$, where $\overline{\nabla}$ is the connection of $\mathbb{R}^{n+1}$. Then we get that

$$0 = \langle Q_{ij}(h), X_k \rangle = \langle \overline{\nabla}_{X_j}(-AX_i) + \Gamma^i_{ji} AX_i + \Gamma^j_{ij} AX_j, X_k \rangle = \langle AX_i, \overline{\nabla}_{X_j} X_k \rangle + \Gamma^j_{ij} \langle AX_j, X_k \rangle,$$

and (4.5) follows. □

This description of non-degenerate deformations of a Euclidean hypersurface can be adapted to Riemannian hypersurfaces of the Lorentz space. Theorem 4.0.2 shows that those deformations are needed in order to describe our genuine deformations.

Let $f : M^n \to \mathbb{R}^{n+1}_\nu$ be a Riemannian hypersurface (hence $\nu \leq 1$) of rank $(p+1) < n$. Assume that f is generic and that it has non-degenerate deformations in $\mathbb{R}^{n+p}_\mu$ for some $\mu \leq p$. Call $\varepsilon := 1 - 2\nu \in \{\pm 1\}$, and set

$$\mathbb{Q}^n_\varepsilon = \begin{cases} \mathbb{S}^n & \text{the unit sphere for } \varepsilon = 1, \\ \mathbb{H}^n & \text{the hyperbolic space for } \varepsilon = -1. \end{cases}$$

In this case the, the Gauss map $h : M^n/\Gamma =: L^{p+1} \to \mathbb{Q}^n_\varepsilon \subseteq \mathbb{R}^{n+1}_\nu$ and the support function $\gamma = \langle h, f \rangle$ are solutions of the DMZ system of PDEs Q given by

$$Q_{ij} = \partial^2_{ij} - \Gamma^i_{ji}\partial_i - \Gamma^j_{ij}\partial_j + \varepsilon g_{ij}, \quad \forall i \neq j.$$

Observe that the normal vectors η_i defined by the same formula (3.6) satisfy

$$\langle \eta_i, \eta_j \rangle = \varepsilon \Big(1 + \frac{\delta_{ij}}{\varphi_i} \Big), \quad \forall i, j. \tag{4.6}$$

The Sbrana bundle, the concept of species, and the Laplace invariants naturally extend to the Lorentzian context.

4.1.1 Flat Euclidean submanifolds

In this work, we study deformations that arise as intersections. The first explicit example is an extension of the construction introduced in [14], where the authors intersect flat Euclidean hypersurfaces. However, for our purposes it is also necessary to consider flat submanifolds of the Lorentz space.

We describe here the flat submanifolds that are relevant for us.

Definition 4.1.2. We say that a flat submanifold $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^{n+p}_\nu$ is *non-degenerate* it there exists $X \in \mathrm{Re}(\alpha^F)$ such that $\alpha^F(X, TM) \subseteq T_F^\perp U$ is non-degenerate.

In particular, all flat submanifolds of $\mathbb{R}^{n+p}$ are non-degenerate. If $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^{n+p}_\nu$ is non-degenerate then the rank of F is at most p by Lemma 2.2.1. If it has maximal constant rank p, then the same lemma shows that F is *full*, that is, $\mathcal{S}(\alpha^F) = T_F^\perp U$. Codazzi equation implies that

$$\alpha^F(C_T X, Y) = \alpha^F(X, C_T Y), \quad \forall T \in \Delta_F, \forall X, Y \in \Delta_F^\perp. \tag{4.7}$$

When F is also generic, take $T_0 \in \Delta_F$ such that its splitting tensor C_{T_0} has distinct eigenvalues, and let $X_i \in \Delta_F^\perp$ be its eigenvectors. Then by (4.7) for $T = T_0$, we have that $\alpha^F(X_i, X_j) = 0$ for $i \neq j$. The flatness of α^F implies that the normal vectors $\{\alpha^F(X_i, X_i)\}_i$ are orthogonal, and as F is full then

$$\langle \alpha^F(X_i, X_i), \alpha^F(X_i, X_i) \rangle \neq 0.$$

Moreover, by (4.7) the X_i's are eigenvectors of all splitting tensors, so they define a smooth and uniquely determined (up to order and re-scaling) frame for $\Delta_F^\perp$. In addition, by Proposition A.3.4 there exists a local chart $(z_1, \dots z_s, x_{2s+1}, \dots, x_p) \in \mathbb{C}^s \times \mathbb{R}^{p-2s}$ for the leaf space $\pi : U \to L^p := U/\Delta_F$ such that, after re-scaling factors, we have

$$\pi_* X_i = \partial_i \circ \pi, \tag{4.8}$$

for $\partial_i = \partial_{u_i}$, where $(u_1, \dots, u_p) = (z_1, \overline{z_1}, \dots, z_s, \overline{z_s}, x_{2s+1}, \dots, x_p)$. We call $(u_1, \dots, u_p)$ *associated conjugate coordinates of* F.

4.2 Intersections of submanifolds with relative nullity

In this section, we provide a general technique to produce submanifolds with relative nullity by intersecting two others. We use this to construct examples of genuine deformations in the following sections. Due to the applications, again it is necessary to consider semi-Euclidean ambient spaces.

Consider two Riemannian submanifolds $F_i : M^{n+p_i} \to \mathbb{R}^{n+p_1+p_2}_\nu$, $i = 1, 2$. Assume that their relative nullities $\Delta_i = \Delta_{F_i} \subseteq TM_i$ are transversal, namely,

$$\Delta_1 + \Delta_2 := F_{1*}(\Delta_1) + F_{2*}(\Delta_2) = \mathbb{R}^{n+p_1+p_2}_\nu,$$

at any point in

$$M^n := F_1(M_1^{n+p_1}) \cap F_2(M_2^{n+p_2}).$$

In particular, M^n is a smooth manifold by transversality. Denote by $f_i : M^n \to M_i^{n+p_i}$ the respective inclusion and $f := F_1 \circ f_1 = F_2 \circ f_2$. Thus, we are in the situation described by the following diagram, where all the maps are isometric immersions:

$$\begin{tikzcd}
& M_2^{n+p_2} \arrow[dr, "F_2"] & \\
M^n \arrow[ur, "f_2"] \arrow[rr, "f"] \arrow[dr, "f_1"'] & & \mathbb{R}_\nu^{n+p_1+p_2} \\
& M_1^{n+p_1} \arrow[ur, "F_1"'] &
\end{tikzcd} \tag{4.9}$$

Notice that the second fundamental forms are related by

$$\alpha^f = \alpha^{f_1} + \gamma_1 = \alpha^{f_2} + \gamma_2, \tag{4.10}$$

where $\gamma_i = \alpha^{F_i}|_{TM\times TM}$. Furthermore, the nullity of γ_i is given by

$$D_{i+1} := \Delta_i \cap TM = \Delta_{\gamma_i} \subseteq TM, \tag{4.11}$$

where the index i is taken modulo 2. Indeed, the transversality of the relative nullities Δ_1 and Δ_2 implies that

$$TM_i = \Delta_i + (\Delta_{i+1} \cap TM_i) = \Delta_i + D_i. \tag{4.12}$$

So if $T_i \in \Delta_{\gamma_i}$ and $Y \in TM_i$, we decompose $Y = Y_i + \tilde{Y}$ for some $Y_i \in \Delta_i$ and $\tilde{Y} \in D_i \subseteq TM$ to obtain

$$\alpha^{F_i}(T_i, Y) = \alpha^{F_i}(T_i, \tilde{Y}) = \gamma_i(T_i, \tilde{Y}) = 0, \quad \forall Y \in TM,$$

which shows that $\Delta_{\gamma_i} \subseteq D_{i+1}$. This gives (4.11) since the other inclusion is obvious.

Notice that the vector bundles $W := T^\perp_{f_2}M \oplus T^\perp_{f_1}M$ and $\hat{W} := T^\perp_{F_1}M_1|_M \oplus T^\perp_{F_2}M_2|_M$ are naturally isometric. Indeed, for $\xi_i, \tilde{\xi}_i \in T^\perp_{f_i}M$ and $\eta_i, \tilde{\eta}_i \in T^\perp_{F_i}M_i|_M$ consider the inner products

$$\langle(\xi_2,\xi_1),(\tilde{\xi}_2,\tilde{\xi}_1)\rangle_W := \langle\xi_2,\tilde{\xi}_2\rangle_{T^\perp_{f_2}M} - \langle\xi_1,\tilde{\xi}_1\rangle_{T^\perp_{f_1}M}, \quad \langle(\eta_1,\eta_2),(\tilde{\eta}_1,\tilde{\eta}_2)\rangle_{\hat{W}} := \langle\eta_1,\tilde{\eta}_1\rangle_{T^\perp_{F_1}M_1} - \langle\eta_2,\tilde{\eta}_2\rangle_{T^\perp_{F_2}M_2}.$$

Then we have the natural isometry $I : W \to \hat{W}$ given by

$$I(\xi_1, \xi_2) = (\eta_2, \eta_1), \tag{4.13}$$

where (η_2, η_1) is uniquely determined by $\xi_1 + \eta_1 = \xi_2 + \eta_2 \in T^\perp_f M$.

Lemma 4.2.1. *Let $F_i : M_i^{n+p_i} \to \mathbb{R}^{n+p_1+p_2}_\nu$ be two Riemannian submanifolds with transversal relative nullities $\Delta_i := \Delta_{F_i}$, $i = 1, 2$. Then, with the notations of (4.9), we have*

$$\Delta := \Delta_f = \Delta_{f_1} \cap \Delta_{f_2} = \Delta_1 \cap \Delta_2 \subseteq TM. \tag{4.14}$$

Moreover, from the algebraic viewpoint the bilinear form $\beta := (\alpha^{f_2}, \alpha^{f_1}) : TM \times TM \to T^\perp_{f_2}M \oplus T^\perp_{f_1}M$ is equivalent (up to nullity) to $\alpha^{F_1} \oplus \alpha^{F_2}$.

Proof. The relation (4.10) shows that $\Delta \subseteq \Delta_{f_1} \cap \Delta_{f_2}$ and $\Delta \subseteq \Delta_{\gamma_1} \cap \Delta_{\gamma_2}$. Since $T^\perp_{f_1}M \cap T^\perp_{f_2}M = 0$ and $T^\perp_{F_1}M_1 \cap T^\perp_{F_2}M_2 = 0$, we conclude that

$$\Delta_{f_1} \cap \Delta_{f_2} = \Delta_{\gamma_1} \cap \Delta_{\gamma_2} = \Delta.$$

For the second part, consider the isometry $I : T^\perp_{f_2}M \oplus T^\perp_{f_1}M \to T^\perp_{F_1}M_1|_M \oplus T^\perp_{F_2}M_2|_M$ given by (4.13). As $D_1 + D_2 = TM$ by (4.12), define $T : TM/\Delta \to TM_1/\Delta_1 \oplus TM_2/\Delta_2$ by $T([v_1 + v_2]) = [v_2] + [v_1]$ for $v_i \in D_i \subseteq TM_{i+1}$, where the brackets denote the associated element in the corresponding quotient. Those maps define an isomorphism between $\beta = (\alpha^{f_2}, \alpha^{f_1})$ and $\alpha^{F_1} \oplus \alpha^{F_2}$. □

Remark 4.2.2. Notice that (4.10) and (4.12) show that $D_1 + D_2 = TM$, $\dim(D_i^\perp \cap TM) = \dim(\Delta^\perp_{i+1} \cap TM_{i+1})$ and $\alpha^f(D_1, D_2) = 0$. In particular, for $v_i, w_i \in D_i$ we have that $\alpha^f(v_i, w_i) = \alpha^{f_{i+1}}(v_i, w_i)$. Hence, in this sense, the second fundamental form of f is also the sum of the second fundamental forms of f_1 and f_2.

If F_1 and F_2 satisfy the hypotheses of Lemma 4.2.1 and they intersect non-orthogonally in M^n, that is

$$T^\perp_{F_{i+1}}M_{i+1} \cap F_{i*}(TM_i) = 0 \quad \text{for } i = 1, 2, \tag{4.15}$$

everywhere, then (4.14) becomes

$$\Delta = \Delta_{f_1} = \Delta_{f_2}. \tag{4.16}$$

Indeed, if $X \in \Delta_{f_i}$ then (4.10) gives for $Y \in D_i$ that

$$\alpha^{F_i}(X, Y) = \alpha^{f_{i+1}}(X, Y) \in T^\perp_{F_i}M \cap TM_{i+1}.$$

Thus the last expression is zero, which together with (4.12) proves that $X \in \Delta_i$. In this case (4.10) implies that $X \in \Delta_{\gamma_{i+1}}$, but since we showed that $\Delta_{\gamma_{i+1}} = \Delta_{i+1} \cap TM$, we get $X \in \Delta_1 \cap \Delta_2 = \Delta$, which proves (4.16).

Intersections can be used to produce examples of genuine deformations as we show next.

Example 4.2.3. Consider $F_i : U_i \subseteq \mathbb{R}^{n+p_1} \to \mathbb{R}_\nu^{n+p_1+p_2}$ two flat isometric immersions satisfying the hypotheses of Lemma 4.2.1. Assume that $\mathcal{S}(\alpha^{F_1} \oplus \alpha^{F_2})^\perp \subseteq T_{F_1}^\perp U_1 \oplus T_{F_2}^\perp U_2$ is a definite subspace (for example, when F_1 and F_2 are full). As discussed in the Preliminaries, if $\{f_1, f_2\}$ isometrically extends then $\mathcal{S}(\beta)^\perp$ has nontrivial null vectors, where $\beta = (\alpha^{f_2}, \alpha^{f_1})$. Hence, f_2 is a genuine deformation of f_1 since β is equivalent to $\alpha^{F_1} \oplus \alpha^{F_2}$.

More generally, we have:

Example 4.2.4. Take $G_i : M_i^{n+p_i} \to \mathbb{R}^{n+q_i}$ an isometric deformation of $F_i : M_i^{n+p_i} \to \mathbb{R}_\nu^{n+p_1+p_2}$ for $i = 1, 2$. Assume that F_1 and F_2 satisfy the conditions of Lemma 4.2.1. Call $h_i = G_i \circ f_i$ the isometric immersion of M^n induced by G_i. The following commutative diagram describes our situation:

$$\begin{tikzcd}
 & & \mathbb{R}^{n+q_2} \\
 & M_2^{n+p_2} \arrow[ur, "G_2"] \arrow[dr, "F_2"] & \\
M^n \arrow[ur, "f_2"] \arrow[dr, "f_1"'] \arrow[rr, "f"] \arrow[uurr, bend left, "h_2"] \arrow[ddrr, bend right, "h_1"'] & & \mathbb{R}_\nu^{n+p_1+p_2} \\
 & M_1^{n+p_1} \arrow[ur, "F_1"] \arrow[dr, "G_1"] & \\
 & & \mathbb{R}^{n+q_1}
\end{tikzcd} \tag{4.17}$$

Denote by $\beta_i = (\alpha^{G_i}, \alpha^{F_i}) : TM_i \times TM_i \to T_{G_i}^\perp M_i \oplus T_{F_i}^\perp M_i =: W_i$ and $\beta = (\alpha^{h_2}, \alpha^{h_1})$ the associated flat bilinear forms. If $\Delta_{\beta_1} + \Delta_{\beta_2} = \mathbb{R}_\nu^{n+q}$ then, just as in Lemma 4.2.1, we have β is equivalent to $\beta_2 \oplus \beta_1$ Assume that $\mathcal{S}(\beta_1 \oplus \beta_2)^\perp \subseteq W_1 \oplus W_2$ is a definite subspace (for example, when $\mathcal{S}(\beta_i) = W_i$ for $i = 1, 2$). Then h_2 is a genuine deformation of h_1 since $\mathcal{S}(\beta)^\perp$ is definite.

Remark 4.2.5. Under the spirit of genuine rigidity, it is not fair to consider h_1 and h_2 as above as a genuine pair, since they are produced by deformations of $M_1^{n+p_1}$ and $M_2^{n+p_2}$: $\{h_1, f\}$ and $\{f, h_2\}$ are not genuine pairs. This suggests to naturally extend the concept of genuine rigidity in order to make it a transitive one.

Lemma 4.2.1 deals with the linear algebra which involves our intersections. The next one analyzes the analytical properties of their distributions.

Lemma 4.2.6. *Let F_1 and F_2 as in Lemma 4.2.1. Then $\Delta \subseteq TM$ is totally geodesic, and $D_i \subseteq TM$ is integrable and satisfies $\nabla_\Delta D_i \subseteq D_i$ and $\nabla_{D_i} \Delta \subseteq D_i$, for $i = 1, 2$. Moreover, the map $f_1 \times f_2 : M^n \to M_1^{n+p_1} \times M_2^{n+p_2}$ naturally induces a local diffeomorphism*

$$M^n/\Delta = (M_1^{n+p_1} \cap M_2^{n+p_2})/\Delta_1 \cap \Delta_2 \cong (M_1^{n+p_1}/\Delta_1) \times (M_2^{n+p_2}/\Delta_2). \tag{4.18}$$

Proof. As discussed in the Preliminaries, the relative nullities $\Delta \subseteq TM$ and $\Delta_i \subseteq TM_i$ are totally geodesic for $i = 1, 2$. Hence $D_i \subseteq TM$ is integrable. Denote by ∇ and ∇^{M_i} the Levi-Civita connections of M^n and $M_i^{n+p_i}$ respectively for $i = 1, 2$. Given $X_i \in \mathfrak{X}(M)$ such that $X_i \in D_i$ pointwise and $T \in \Delta$, we have that

$$\nabla_T X_i = \nabla_T^{M_{i+1}} X_i \in \Delta_{i+1} \cap TM = D_i,$$

since $\Delta \subseteq \Delta_{f_i}$ and $\Delta_{i+1} \subseteq TM_{i+1}$ is totally geodesic. This shows that $\nabla_\Delta D_i \subseteq D_i$ and analogously $\nabla_{D_i} \Delta \subseteq D_i$.

Finally, denote by $\pi : M^n \to L^{r_1+r_2} := M^n/\Delta$ and $\pi_i : M_i^{n+p_i} \to L_i^{r_i} := M_i^{n+p_1}/\Delta_i$ the respective quotient maps. It is easy to verify that the map $\overline{f_1} \times \overline{f_2} : L^{r_1+r_2} \to L_1^{r_1} \times L_2^{r_2}$ given by $(\overline{f_1} \times \overline{f_2}) \circ \pi = (\pi_1 \circ f_1, \pi_2 \circ f_2)$ is well-defined and smooth. Since $TM_i = \Delta_i + D_i$ we have

$$(\overline{f_1} \times \overline{f_2})_* \pi_* D_1 = (\pi_{1*} f_{1*} D_1, \pi_{2*} f_{2*} D_1) = (TL_1, 0),$$

and analogously $(\overline{f_1} \times \overline{f_2})_* \pi_* D_2 = (0, TL_2)$. Thus $(\overline{f_1} \times \overline{f_2})_*$ is an isomorphism. $\square$

Remark 4.2.7. Set $\hat{D}_i := D_i \cap \Delta^\perp \subseteq TM$ and $p_i : TM_i \to \Delta_i^\perp$ the orthogonal projection onto $\Delta_i^\perp$. By (4.12) $\hat{p}_i := p_i|_{\hat{D}_i}$ is a bijection. Lemma 4.2.6 implies that, if C_T and C_T^i are the splitting tensors of $\Delta \subseteq TM$ and $\Delta_i \subseteq TM_i$ for $T \in \Delta \subseteq \Delta_i$, then

$$C_T^i(\hat{p}_i(X)) = -\left(\nabla^{M_i}_{\hat{p}_i(X)-X+X} T\right)_{\Delta_i^\perp} = \hat{p}_i(C_T(X)),$$

since $\nabla_{D_i} \Delta \subseteq \Delta$. Thus with respect to the decomposition $\Delta^\perp = \hat{D}_1 \oplus \hat{D}_2$ we have $C_T = (\hat{p}_1^{-1} C_T^1 \hat{p}_1) \oplus (\hat{p}_2^{-1} C_T^2 \hat{p}_2)$.

In the following sections we use the above results to describe examples of genuinelly deformable hypersurfaces. We are particularly interested in the following ones obtained as intersections.

Definition 4.2.8. For $i = 1, 2$, consider the Riemannian submanifolds $F_i : M^{n+p_i} \to N^{n+p_1+p_2}$. We say that they *intersect generically* if the following conditions are satisfied:

i) At each point of $M^n = F_1(M_1^{n+p_1}) \cap F_2(M_2^{n+p_2})$ we have $\Delta_1 + \Delta_2 = TN$;

ii) F_1 and F_2 intersect non-orthogonally in the sense of (4.15);

iii) There exists $T \in \Delta_{F_1} \cap \Delta_{F_2}$ such that the polynomial $\psi_{C_T^1} \cdot \psi_{C_T^2}$ has simple roots, where C_T^i is the splitting tensor associated with the relative nullity $\Delta_i := \Delta_{F_i} \subseteq TM_i$.

Remark 4.2.9. The last condition is to avoid the surface-like and ruled type of situation since the splitting tensor of $T \in \Delta \subseteq TM$ is $C_T^1 \oplus C_T^2$ up to conjugation. In this case, if $f : M^n \to N^{n+p_1+p_2}$ is the obvious inclusion, then F_1, F_2 and f are generic, and (4.16) holds.

Remark 4.2.10. The results of this section can be used for more general ambient spaces, that is, for submanifolds $F_i : M_i^{n+p_i} \to N^{n+p_1+p_2}$. However, in this situation, the relative nullity may not be totally geodesic. By Codazzi equation $\Delta_i \subseteq TM_i$ is totally geodesic if and only if

$$R_N(X, T)S \in TM_i, \quad \forall T, S \in \Delta_i, \quad \forall X \in TM_i,$$

where R_N is the curvature tensor of $N^{n+p_1+p_2}$. If the last condition holds for $i = 1, 2$, then all the properties discussed in this section can be used without any change. In particular, they hold if the ambient space has constant curvature.

4.3 Intersections of flat submanifolds

In this section, we intersect flat submanifolds in order to produce a hypersurface with a genuine deformation as the ones described in Section 4.1. We will see that certain Laplace invariants of the DMZ system vanish, a property that, in fact, characterizes analytically such geometric construction.

Consider two non-degenerate flat submanifolds $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_\nu^{n+p+1}$, $F_2 : U_2 \subseteq \mathbb{R}^{n+p} \to \mathbb{R}_\nu^{n+p+1}$ with $\nu \le 1$ (see Definition 4.1.2). Denote by $\Delta_1 = \Delta_{F_1}$ and $\Delta_2 = \Delta_{F_2}$ their respective relative nullities. Lemma 2.2.1 shows that $\dim(\Delta_2) - n \ge p - 1 \ge n - \dim(\Delta_1)$. Assume that

$$\dim(\Delta_2) - n = p - 1 = n - \dim(\Delta_1),$$

in particular, $\mathcal{S}(\alpha^{F_i}) = T_{F_i}^\perp U_i$. In addition, suppose that F_1 and F_2 intersect generically along $M^n := F_1(U_1) \cap F_2(U_2)$. Using the notations of (4.9), $f_1 : M^n \to \mathbb{R}^{n+1}$ has rank $(p+1)$ by (4.16), and its relative nullity coincides with the intrinsic nullity $\Gamma \subseteq TM$ for nowhere flat hypersurfaces. Hence

$$\Gamma = \Delta = \Delta_{f_1} = \Delta_{f_2} = \Delta_1 \cap \Delta_2. \tag{4.19}$$

Furthermore, Lemma 4.2.1 shows that $\beta = (\alpha^{f_2}, \alpha^{f_1}) \cong \alpha^{F_1} \oplus \alpha^{F_2}$, so f_2 is a genuine deformation of f_1 since $\mathcal{S}(\beta) = T_{f_2}^\perp M \oplus T_{f_1}^\perp M$. This already provides a large class of examples of hypersurfaces as the ones described in Theorem 3.0.1. In this section, we characterize such hypersurfaces f_1 as the ones for which the Laplace invariants m_{jp} of the associated DMZ system $Q = (Q_{ij})$ vanish for $j \neq p$.

Denote by $(u_0, \ldots, u_{p-1})$ and u_p the respective conjugate coordinates of $L^p := U_1/\Delta_1$ and $L^1 := U_2/\Delta_2$ as discussed in Section 4.1.1. Consider $\hat{X}_j$ a lift of ∂_j for each $j \le p$. Then for $j, k < p$ we have that

$$\alpha^{F_1}(\hat{X}_j, \hat{X}_k) = 0, \quad \forall j \neq k.$$

In M^n set $D_1 := \Delta_2 \cap TM$ and $D_2 := \Delta_1 \cap TM$, which are integrable distributions by Lemma 4.2.6. We can assume that along M^n we have $\hat{X}_j \in D_1$ (or $\hat{X}_j \in (D_1)_{\mathbb{C}}$ if u_j is a complex variable) for $j < p$ and $\hat{X}_p \in D_2$, by adding some vector field of the respective relative nullity if needed. Set

$$X_j = \hat{X}_j|_{M^n}.$$

We claim that the vector fields $\{X_j\}_{j=0}^p$ descend to the nullity leaf space $\pi : M^n \to M^n/\Gamma =: L^{p+1}$ as conjugate coordinates of the Gauss map $h : L^{p+1} \to \mathbb{S}^n$ of f_1.

To see this, first for $j, k < p$ we have that $[X_j, X_k] \in D_1$, since X_j and X_k belong to $D_1 \subseteq TM$ which is integrable. Moreover, $[\hat{X}_j, \hat{X}_k] \in \Delta_1$ since the vectors $\hat{X}_j$ and $\hat{X}_k$ descend as coordinate vector fields of $L^p = U_1/\Delta_1$, and thus $[X_j, X_k] \in D_1 \cap \Delta_1 = \Gamma$. Also, notice that $[X_j, \Gamma] \subseteq D_1$ for $j < p$ since D_1 is integrable, and $[X_j, \Gamma] \subseteq \Delta_1$ since $\hat{X}_j$ descends to the leaf space $L^p = U_2/\Delta_2$. Then $[X_j, \Gamma] \subseteq D_1 \cap \Delta_1 = \Gamma$. Analogously we get $[X_p, \Gamma] \subseteq \Gamma$. Since the vectors $\hat{X}_j$ and $\hat{X}_p$ descend to the respective leaf space, we have $[\Delta_2, \hat{X}_p] \subseteq \Delta_2$ and $[\hat{X}_j, \Delta_1] \subseteq \Delta_1$. This implies that $[\hat{X}_j, \hat{X}_p] \subseteq \Delta_1 \cap \Delta_2 = \Gamma$, and then $[X_j, X_p] \in \Gamma$. Therefore, by Proposition 10 of [15], the vector fields $\{X_j\}_{j=0}^p$ descend to the leaf space $\pi : M^n \to L^{p+1} = M^n/\Gamma$, as claimed.

Observe that the diffeomorphism of Lemma 4.2.6 is naturally given by putting together the conjugate coordinates of $L^p = U_1/\Delta_1$ and $L^1 = U_2/\Delta_2$, that is, $(u_0, \dots, u_p)$ are naturally the coordinates for L^{p+1}. With this identification, we have

$$\pi_* X_k = \partial_k \circ \pi = \partial_{u_k} \circ \pi \quad \text{ for } \ k \le p.$$

Furthermore, Lemma 4.2.1 and (4.10) show that

$$\beta(X_j, X_k) = 0, \quad \forall j \neq k. \tag{4.20}$$

Hence, the coordinates $(u_0, \dots, u_p)$ of L^{p+1} are conjugate coordinates of the Gauss map $h : L^{p+1} \to \mathbb{S}^n$.

For $i = 1, 2$ let $\Lambda_i : TM \times (TM \oplus T_{f_i}^{\perp} M) \to (T_{f_i}^{\perp} M)^{\perp} \subseteq T_f^{\perp} M$ be the flat tensor

$$\Lambda_i := \alpha^{F_i}|_{TM \times (TM \oplus T_{f_i}^{\perp} M)}.$$

We can rewrite (4.10) in terms of Λ_i as

$$\alpha^f(X, Y) = \alpha^{f_1}(X, Y) + \Lambda_1(X, Y) = \alpha^{f_2}(X, Y) + \Lambda_2(X, Y), \quad \forall X, Y \in TM. \tag{4.21}$$

Furthermore, the normal connection $\nabla^{\perp f}$ of f is related to the normal connections of f_1 and f_2 by

$$\nabla_X^{\perp f} \xi_i = \nabla_X^{\perp f_i} \xi_i + \Lambda_i(X, \xi_i), \quad \forall X \in TM, \quad \forall \xi_i \in \Gamma(T_{f_i}^{\perp} M). \tag{4.22}$$

The following result determines Λ_1 and Λ_2, which will be used to show that the (p, i)-Laplace invariants of $Q = (Q_{ij})$ are zero. Recall that $A = A_\rho$ is the shape operator of f_1 with respect to the Gauss map $\rho = h \circ \pi$ of f_1, where $\pi : M^n \to L^{p+1}$ is the quotient map and $h : L^{p+1} \to \mathbb{S}^n$ is the induced map.

Lemma 4.3.1. *The tensors Λ_i are determined by the normal vectors $\eta_i \in T_{f_2}^{\perp} M$ in (3.6) and the following conditions:*

i) *D_{i+1} is the left nullity of Λ_i, that is $D_{i+1} = \Delta_{\Lambda_i} := \{X \in TM : \Lambda_i(X, Y) = 0, \forall Y \in TM \oplus T_{f_i}^{\perp} M\}$;*

ii) *$\Lambda_1(X_j, X_k) = \delta_{jk} \langle AX_j, X_j \rangle (\eta_j - \rho), \quad \Lambda_2(X_p, X_p) = \langle AX_p, X_p \rangle (\rho - \eta_p), \quad \forall j, k < p;$*

iii) *$\Lambda_1(X_j, \rho) := \Gamma_{jp}^p (\eta_j - \rho), \quad \Lambda_2(X_p, \eta_k) := \Gamma_{pk}^k (\rho - \eta_p), \quad \forall k, j \neq p.$*

Moreover, the Christoffel symbols of the Gauss map $h : L^{p+1} \to \mathbb{S}^n$ of f_1 satisfy that

$$\partial_p \Gamma_{jp}^p - \Gamma_{jp}^p \Gamma_{pj}^j + g_{jp} = 0 \quad \forall j \neq p. \tag{4.23}$$

In particular, if $Q = (Q_{ij})$ is the associated DMZ-system, then the (p, j)-Laplace invariant m_{pj} is zero for all $0 \le j < p$.

Proof. Notice that (ii) is just (4.21). Also, as Λ_i is a restriction of α^{F_i}, we necessarily have $D_{i+1} = \Delta_i \cap TM \subseteq \Delta_{\Lambda_i}$. Then (i) is a consequence of (ii).

For $k \neq p$ set $\varepsilon_k = \frac{\alpha^{F_1}(X_k, X_k)}{\langle AX_k, X_k \rangle}$. Then $\eta_k = \rho + \varepsilon_k$ by (4.21). Thus, as $X_p \in D_2 \subseteq \Delta_1$,

$$\hat{\nabla}_{X_p} \eta_k = \hat{\nabla}_{X_p} \rho + \hat{\nabla}_{X_p} \varepsilon_k = \nabla_{X_p}^{\perp f_1} \varepsilon_k.$$

Using (4.5) and Codazzi equations for F_2 and A we get

$$\hat{\nabla}_{X_p} \eta_k = \frac{\langle AX_p, \nabla_{X_k} X_k \rangle}{\langle AX_k, X_k \rangle} \varepsilon_k = \Gamma_{pk}^k (\rho - \eta_k),$$

which together with (3.14) and (4.4) give us

$$\hat{\nabla}_{X_p} \eta_k = \Gamma_{pk}^k (\rho - \eta_k) = \nabla_{X_p}^{\perp f_2} \eta_k + \Lambda_2(X_p, \eta_k) = \Gamma_{pk}^k (\eta_p - \eta_k) + \Lambda_2(X_p, \eta_k).$$

This proves the second equality of (iii) for $k \neq p$.

Set $\varepsilon_p = \frac{\alpha^{F_2}(X_p, X_p)}{\langle AX_p, X_p \rangle}$. Hence $\rho = \eta_p + \varepsilon_p$ by (4.21). Thus, for $j \neq p$ we have that $X_j \in D_1 \subseteq \Delta_2$,

$$\hat{\nabla}_{X_j}\rho = \hat{\nabla}_{X_j}\eta_p + \hat{\nabla}_{X_j}\varepsilon_p = \nabla^{\perp f_2}_{X_j}\eta_p + \nabla^{\perp F_2}_{X_j}\varepsilon_p.$$

As before, by (4.5) and Codazzi equation for F_2 and A, we get

$$\hat{\nabla}_{X_j}\rho = \nabla^{\perp f_2}_{X_j}\eta_p + \frac{\langle AX_j, \nabla_{X_p}X_p \rangle}{\langle AX_p, X_p \rangle}\varepsilon_p = \nabla^{\perp f_2}_{X_j}\eta_p + \Gamma^p_{jp}(\eta_p - \rho).$$

Now using (3.14) and (4.4) we obtain

$$\Lambda_1(X_j, \rho) = \hat{\nabla}_{X_j}\rho = \Gamma^p_{jp}(\eta_j - \eta_p) + \Gamma^p_{jp}(\eta_p - \rho) = \Gamma^p_{jp}(\eta_j - \rho).$$

This proves the first equation of (iii).

We show now that (4.23) is equivalent to certain Ricci equations for f. Indeed, denote by $\hat{R}$ the normal curvature tensor of f. By (4.21) we see that A and A_{η_k} are the shape operators associated with $\rho, \eta_k \in T^\perp_f M$, respectively. Then, for $j \neq p$, Ricci equation says that

$$\langle \hat{R}(X_p, X_j)\rho, \eta_k \rangle = \langle [A, A_{\eta_k}]X_p, X_j \rangle = \Big(\frac{\delta_{kp}}{\varphi_k} - \frac{\delta_{kj}}{\varphi_j}\Big)\langle AX_p, AX_j \rangle = \Big(\frac{\delta_{kp}}{\varphi_k} - \frac{\delta_{kj}}{\varphi_j}\Big)g_{jp}, \tag{4.24}$$

where we used (4.3) and $\partial_i h = -AX_i$. On the other hand, using (4.22) and the definition of Λ_i, we have

$$\begin{aligned}\hat{R}(X_p, X_j)\rho &= \hat{\nabla}_{X_p}(-\Gamma^p_{jp}(\rho - \eta_j)) = -\partial_p\Gamma^p_{jp}(\rho - \eta_j) - \Gamma^p_{jp}\hat{\nabla}_{X_p}(\rho - \eta_j) \\ &= -\partial_p\Gamma^p_{jp}(\rho - \eta_j) + \Gamma^p_{jp}(\nabla^{\perp, f_2}_{X_p}\eta_j + \Gamma^j_{pj}(\rho - \eta_p)).\end{aligned}$$

Using (3.14) and (4.4) we have for $j \neq p$ that

$$\hat{R}(X_p, X_j)\rho = -\partial_p\Gamma^p_{jp}(\rho - \eta_j) + \Gamma^p_{jp}(\Gamma^j_{pj}\eta_p - \Gamma^j_{pj}\eta_j + \Gamma^j_{pj}(\rho - \eta_p)) = -(\partial_p\Gamma^p_{jp} - \Gamma^p_{jp}\Gamma^j_{pj})(\rho - \eta_j).$$

Therefore, as $\eta_j = \rho + \varepsilon_k$ and $\rho = \eta_p + \varepsilon_p$, we have that $\langle \rho, \eta_k \rangle = 1 + \frac{\delta_{kp}}{\varphi_p}$, and so

$$\langle \hat{R}(X_p, X_j)\rho, \eta_k \rangle = -(\partial_p\Gamma^p_{jp} - \Gamma^p_{jp}\Gamma^j_{pj})\Big(\frac{\delta_{kp}}{\varphi_k} - \frac{\delta_{kj}}{\varphi_j}\Big), \quad \forall k, \forall j \neq p. \tag{4.25}$$

Combining this with (4.24), we conclude that these Ricci equations hold if and only if (4.23) is satisfied. □

Remark 4.3.2. The tensors Λ_i are Codazzi with respect to the connection $\overline{\nabla}$ induced by the one of the ambient space $\tilde{\nabla}$. Indeed, by definition

$$(\overline{\nabla}_X\Lambda_i)(Y, v) := (\tilde{\nabla}_X(\Lambda_i(Y, v))_{L_i} - \Lambda_i(\nabla_X Y, v) - \Lambda_i(Y, (\tilde{\nabla}_X v)_{TM \oplus L_i}),$$

where the subindex denotes the orthogonal projection to the respective subspace. Then

$$\overline{\nabla}_X\Lambda_i(Y, v) - \overline{\nabla}_Y\Lambda_i(X, v) = (\tilde{R}(X, Y)v)_{L_i} = 0,$$

since the ambient space has curvature zero.

We have all the ingredients to prove Theorem 4.0.1.

Proof of Theorem 4.0.1. We have already proved the direct statement. For the converse, since this is a local result we can assume that M^n is simply connected. We will construct a semi-Riemannian rank $(p+1)$ vector bundle $E \to M^n$, and a symmetric bilinear form $\hat{\alpha} : TM \times TM \to E$. Then we will endow E with a compatible connection $\hat{\nabla}$ such that $(E, \hat{\nabla}, \hat{\alpha})$ satisfies Gauss, Codazzi, and Ricci equations. By the Fundamental Theorem of submanifolds, we will obtain an isometric immersion $f : M^n \to \mathbb{R}^{n+p+1}_\nu$ with this data as normal bundle and second fundamental form. We end the proof by showing that f is a composition of both f_1 and f_2, giving us the maps F_1 and F_2.

As discussed in Chapter 3, $f_2 : M^n \to \mathbb{R}^{n+p}$ is associated with some parallel section φ of the Sbrana bundle, that is $f_2 = f_2^\varphi$. Let $E := T^\perp_{f_2}M \oplus T^\perp_{f_1}M \to M^n$ be the Whitney sum. Consider on E the unique inner product such that the trivial inclusions $T^\perp_{f_i}M \to E$ are isometries and

$$\langle \rho, \eta_i \rangle := 1 + \frac{\delta_{ip}}{\varphi_p}, \quad \forall i. \tag{4.26}$$

This is well-defined by (3.9). As $\langle \rho - \eta_p, \eta_i\rangle = 0$ for any i, $(\rho - \eta_p)$ is orthogonal to $T^\perp_{f_2}M$ in E and

$$\langle \rho - \eta_p, \rho - \eta_p\rangle = -\varphi_p^{-1}.$$

So the index of this inner product is $\nu := (\operatorname{sign}(\varphi_p) + 1)/2 \in \{0, 1\}$.

For $i = 1, 2$, let $\Lambda_i : TM \times (TM \oplus T^\perp_{f_i}M) \to (T^\perp_{f_i}M)^\perp \subseteq E$ be the bilinear tensor defined in Lemma 4.3.1. A straightforward computation shows that Λ_i is flat. Define the symmetric bilinear tensor $\hat\alpha : TM \times TM \to E$ by

$$\hat\alpha(X, Y) := \alpha^{f_1}(X, Y) + \Lambda_1(X, Y) = \alpha^{f_2}(X, Y) + \Lambda_2(X, Y), \quad \forall X, Y \in TM,$$

as in (4.21). The flatness of Λ_i implies that $\hat\alpha$ satisfies Gauss equation, since α^{f_1} and α^{f_2} also do. In addition, the shape operator in the direction $\xi_i \in T^\perp_{f_i}M$ associated to $\hat\alpha$ coincides with the one associated to α^{f_i}, since

$$\langle \hat A_{\xi_i} X, Y\rangle := \langle \hat\alpha(X, Y), \xi_i\rangle = \langle \alpha^{f_i}(X, Y) + \Lambda_i(X, Y), \xi_i\rangle = \langle \alpha^{f_i}(X, Y), \xi_i\rangle =: \langle A_{\xi_i} X, Y\rangle, \quad \forall X, Y \in TM.$$

Consider on E the connection $\hat\nabla$ given by

$$\hat\nabla_X \xi_i := \nabla^{\perp f_i}_X \xi_i + \Lambda_i(X, \xi_i), \quad \forall X \in TM, \quad \forall \xi_i \in \Gamma(T^\perp_{f_i}M),$$

as in (4.22). We prove next that $(E, \hat\nabla, \hat\alpha)$ satisfies the hypothese of the Fundamental Theorem of submanifolds through a series of claims.

Claim 1. *The connection $\hat\nabla$ is compatible with the inner product.*

Proof. Since the connection $\nabla^{\perp f_i}$ is compatible with the inner product of $T^\perp_{f_i}M$, we have for $\xi, \zeta \in \Gamma(T^\perp_{f_i}M)$ that

$$\langle \hat\nabla_X \xi, \zeta\rangle + \langle \xi, \hat\nabla_X \zeta\rangle = \langle \nabla^{\perp f_i}_X \xi, \zeta\rangle + \langle \xi, \nabla^{\perp f_i}_X \zeta\rangle = X\langle \xi, \zeta\rangle.$$

Furthermore, using (3.9) and (3.14) we get

$$\begin{aligned}
\langle \hat\nabla_X \eta_i, \rho\rangle + \langle \eta_i, \hat\nabla_X \rho\rangle &= \langle \nabla^{\perp f_2}_X \eta_i + \Lambda_2(X, \eta_i), \rho\rangle + \langle \eta_i, \Lambda_1(X, \rho)\rangle \\
&= \sum_j \phi_{ij}(X)\varphi_j\langle \eta_j, \rho\rangle + \langle \Lambda_2(X, \eta_i), \rho\rangle + \langle \eta_i, \Lambda_1(X, \rho)\rangle \\
&= \sum_j \phi_{ij}(X)\varphi_j + \phi_{ip}(X) + \langle \Lambda_2(X, \eta_i), \rho\rangle + \langle \eta_i, \Lambda_1(X, \rho)\rangle \\
&= \phi_{ip}(X) + \langle \Lambda_2(X, \eta_i), \rho\rangle + \langle \eta_i, \Lambda_1(X, \rho)\rangle.
\end{aligned}$$

Hence, the compatibility is clear for $X \in \Gamma$. Using (4.4) and analyzing first for $X = X_j$ with $p \neq j \neq i$, then for $X = X_i$ with $i \neq p$, and finally for $X = X_p$, we see that the last computation, in any case, gives $X(\frac{\delta_{ip}}{\varphi_p}) = X\langle \eta_i, \rho\rangle$. This proves the first claim by (4.26). □

Now, the covariant derivative of Λ_i is

$$(\hat\nabla_X \Lambda_i)(Y, v) := \hat\nabla_X(\Lambda_i(Y, v)) - \Lambda_i(\nabla_X Y, v) - \Lambda_i(Y, (\tilde\nabla_X v)_{TM \oplus T^\perp_{f_i}M}), \quad X, Y \in TM, \quad v \in TM \oplus T^\perp_{f_i}M,$$

where $\tilde\nabla$ is the connection on $TM \oplus E$ defined by

$$\tilde\nabla_X Y = \nabla_X Y + \hat\alpha(X, Y) \quad \text{and} \quad \tilde\nabla_X \xi = \hat\nabla_X \xi - \hat A_\xi X, \quad \forall X, Y \in TM, \quad \xi \in E. \tag{4.27}$$

Claim 2. *$(E, \hat\nabla, \hat\alpha)$ satisfies Codazzi and Ricci equations if and only if Λ_i is a Codazzi tensor for $i = 1$ or 2, namely,*

$$(\hat\nabla_X \Lambda_i)(Y, v) = (\hat\nabla_Y \Lambda_i)(X, v), \quad \forall X, Y \in TM, \quad \forall v \in T^\perp_{f_i}M. \tag{4.28}$$

Proof. First, notice that

$$\begin{aligned}
(\hat\nabla_X \hat\alpha)(Y, Z) &= \hat\nabla_X(\alpha^{f_i}(Y, Z) + \Lambda_i(Y, Z)) - (\alpha^{f_i}(\nabla_X Y, Z) + \Lambda_i(\nabla_X Y, Z)) - (\alpha^{f_i}(Y, \nabla_X Z) + \Lambda_i(Y, \nabla_X Z)) \\
&= \nabla^{\perp f_i}_X(\alpha^{f_i}(Y, Z)) + \Lambda_i(X, \alpha^{f_i}(Y, Z)) + \hat\nabla_X(\Lambda_i(Y, Z)) - (\alpha^{f_i}(\nabla_X Y, Z) + \Lambda_i(\nabla_X Y, Z)) \\
&\quad - (\alpha^{f_i}(Y, \nabla_X Z) + \Lambda_i(Y, \nabla_X Z)) \\
&= (\nabla^{\perp f_i}_X \alpha^{f_i})(Y, Z) + (\hat\nabla_X \Lambda_i)(Y, Z) + \Lambda_i(X, \alpha^{f_i}(Y, Z)) + \Lambda_i(Y, \alpha^{f_i}(X, Z)).
\end{aligned}$$

Hence, $\hat{\alpha}$ satisfies Codazzi if and only if (4.28) holds for some $i = 1, 2$ and for $v = Z \in TM$ (and consequently for both $i = 1, 2$). Furthermore, for $\xi \in T_{f_i}^{\perp}M$ we have

$$\begin{aligned}
\hat{R}(X,Y)\xi_i &= \hat{\nabla}_X\hat{\nabla}_Y\xi_i - \hat{\nabla}_Y\hat{\nabla}_X\xi_i - \hat{\nabla}_{[X,Y]}\xi \\
&= \hat{\nabla}_X(\nabla_Y^{\perp f_i}\xi_i + \Lambda_i(Y,\xi_i)) - \hat{\nabla}_Y(\nabla_X^{\perp f_i}\xi_i + \Lambda_i(X,\xi_i)) - \nabla_{[X,Y]}^{\perp f_i}\xi_i - \Lambda_i([X,Y],\xi_i) \\
&= R^{\perp f_i}(X,Y)\xi_i + \Lambda_i(X,\nabla_Y^{\perp f_i}\xi_i) + \hat{\nabla}_X(\Lambda_i(Y,\xi_i)) - \Lambda_i(Y,\nabla_X^{\perp f_i}\xi_i) - \hat{\nabla}_Y(\Lambda_i(X,\xi_i)) - \Lambda_i([X,Y],\xi_i) \\
&= R^{\perp f_i}(X,Y)\xi_i + (\hat{\nabla}_X\Lambda_i)(Y,\xi_i) + \Lambda_i(Y,-A_{\xi_i}X) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i) - \Lambda_i(X,-A_{\xi_i}Y) \\
&= \alpha^{f_i}(X,A_{\xi_i}Y) - \alpha^{f_i}(Y,A_{\xi_i}X) + (\hat{\nabla}_X\Lambda_i)(Y,\xi_i) + \Lambda_i(Y,-A_{\xi_i}X) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i) - \Lambda_i(X,-A_{\xi_i}Y),
\end{aligned}$$

where in the last equality we used Ricci equation for f_i. So

$$\hat{R}(X,Y)\xi_i = (\alpha^{f_i}(X,A_{\xi_i}Y) + \Lambda_i(X,A_{\xi_i}Y)) - (\alpha^{f_i}(Y,A_{\xi_i}X) + \Lambda_i(Y,A_{\xi_i}X)) + (\hat{\nabla}_X\Lambda_i)(Y,\xi_i) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i).$$

In other words,

$$\hat{R}(X,Y)\xi_i = \hat{\alpha}(X,A_{\xi_i}Y) - \hat{\alpha}(Y,A_{\xi}X) + (\hat{\nabla}_X\Lambda_i)(Y,\xi_i) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i). \tag{4.29}$$

Then $(E,\hat{\nabla},\hat{\alpha})$ satisfies Ricci equation if and only if (4.28) holds for $v = \xi_i \in T_{f_i}^{\perp}M$, $i = 1, 2$. Moreover, the latter for $i = 1$ is equivalent to the one for $i = 2$. Indeed, for $\xi_i, \tilde{\xi}_i \in T_{f_i}^{\perp}M$, the compatibility of the connection with the inner product and the fact that Λ_i is flat give us

$$\begin{aligned}
\langle(\hat{\nabla}_X\Lambda_i)(Y,\xi_i) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i),\tilde{\xi}_i\rangle &= \langle\hat{\nabla}_X(\Lambda_i(Y,\xi_i)),\tilde{\xi}_i\rangle - \langle\hat{\nabla}_Y(\Lambda_i(X,\xi_i)),\tilde{\xi}_i\rangle \\
&= -\langle\Lambda_i(Y,\xi_i),\Lambda_i(X,\hat{\xi}_i)\rangle + \langle\Lambda_i(X,\xi_i),\Lambda_i(Y,\hat{\xi}_i)\rangle = 0.
\end{aligned} \tag{4.30}$$

On the other hand, for $\xi_{i+1} \in T_{f_{i+1}}^{\perp}M$, (4.29) shows that

$$\begin{aligned}
\langle(\hat{\nabla}_X\Lambda_i)(Y,\xi_i) - (\hat{\nabla}_Y\Lambda_i)(X,\xi_i),\xi_{i+1}\rangle &= \langle\hat{R}(X,Y)\xi_i,\xi_{i+1}\rangle + \langle[A_{\xi_{i+1}},A_{\xi_i}]X,Y\rangle \\
&= \langle-\hat{R}(X,Y)\xi_{i+1} + \hat{\alpha}(X,A_{\xi_{i+1}}Y) - \hat{\alpha}(Y,A_{\xi_{i+1}}X),\xi_i\rangle \\
&= \langle-(\hat{\nabla}_X\Lambda_{i+1})(Y,\xi_{i+1}) + (\hat{\nabla}_Y\Lambda_{i+1})(X,\xi_{i+1}),\xi_i\rangle.
\end{aligned}$$

Then Λ_i is Codazzi if and only if Λ_{i+1} is. This proves the second claim. □

Claim 3. *Λ_2 is a Codazzi tensor.*

Proof. As the frame $\{X_i\}_i \in \Gamma_{\mathbb{C}}^{\perp}$ descends to $L^{p+1} = M^n/\Gamma$ as coordinate vectors, we have

$$[\Gamma,\Gamma] \subseteq \Gamma, \quad [\Gamma,X_i] \subseteq \Gamma, \quad [X_i,X_j] \in \Gamma, \quad \forall i \neq j. \tag{4.31}$$

Hence $D_1 \subseteq TM$ and $D_2 \subseteq TM$ are integrable distributions. To simplify notations, set

$$\begin{aligned}
P(X,Y,v) &:= (\hat{\nabla}_X\Lambda_2)(Y,v) - (\hat{\nabla}_Y\Lambda_2)(X,v) \\
&= \hat{\nabla}_X(\Lambda_2(Y,v)) - \hat{\nabla}_Y(\Lambda_2(X,v)) - \Lambda_2([X,Y],v) - \Lambda_2(Y,(\tilde{\nabla}_Xv)_{TM\oplus T_{f_2}^{\perp}M}) + \Lambda_2(X,(\tilde{\nabla}_Yv)_{TM\oplus T_{f_2}^{\perp}M}).
\end{aligned}$$

In particular, if $X, Y \in D_1 = \Delta_{\Lambda_2}$ then $P(X,Y,v) = 0$, since $D_1 \subseteq TM$ is integrable. Hence, to prove Codazzi for Λ_2 is enough to consider the case $X \in D_1$ and $Y = X_p$, in which case the last expression reduces to

$$P(X,X_p,v) = \hat{\nabla}_X(\Lambda_2(X_p,v)) - \Lambda_2([X,X_p],v) - \Lambda_2(X_p,(\tilde{\nabla}_Xv)_{TM\oplus T_{f_2}^{\perp}M}). \tag{4.32}$$

We prove that the right hand side of (4.32) is zero by analyzing first the case $v \in D_1$, then $v = X_p$, and finally $v = \eta_k$.

We show first that $P(D_1,X_p,D_1) = 0$. This holds if $X, v \in \Gamma$ since Γ is totally geodesic. If $X = X_i$ and $v \in \Gamma$ for $i \neq p$, then

$$P(X_i,X_p,v) = -\Lambda_2(X_p,\nabla_{X_i}v) = \Lambda_2(X_p,C_vX_i) = 0,$$

since X_i is an eigenvalue of all the splitting tensors. If $v = X_i$ and $X \in \Gamma$ with $i \neq p$, then by (4.31) we have

$$P(X,X_p,X_i) = -\Lambda_2(X_p,\nabla_XX_i) = -\Lambda_2(X_p,[X,X_i] + \nabla_{X_i}X) = \Lambda_2(X_p,C_XX_i) = 0,$$

as before. Now if $X = X_i$ and $v = X_j$ for $i, j \neq p$, then

$$P(X_i, X_p, X_j) = -\Lambda_2(X_p, \nabla_{X_i} X_j + \alpha^{f_2}(X_i, X_j)) = -\langle AX_p, \nabla_{X_i} X_j\rangle(\rho - \eta_p) + \delta_{ij}\Gamma^j_{pj}\langle AX_j, X_j\rangle(\rho - \eta_p),$$

which is zero by (4.5). This concludes the proof that $P(D_1, X_p, D_1) = 0$.

Now, let us show that $P(D_1, X_p, X_p) = 0$. Using Codazzi equation for A in (4.32), we have

$$\begin{aligned} P(X, X_p, X_p) &= \hat{\nabla}_X(\Lambda_2(X_p, X_p)) - \Lambda_2([X, X_p], X_p) - \Lambda_2(X_p, (\nabla_X X_p)) \\ &= X(\langle AX_p, X_p\rangle)(\rho - \eta_p) + \langle AX_p, X_p\rangle\hat{\nabla}_X(\rho - \eta_p) - (\langle A[X, X_p], X_p\rangle + \langle AX_p, \nabla_X X_p\rangle)(\rho - \eta_p) \\ &= -\langle AX, \nabla_{X_p} X_p\rangle(\rho - \eta_p) + \langle AX_p, X_p\rangle(\Lambda_1(X, \rho) - \nabla^{\perp f_2}_X \eta_p). \end{aligned}$$

If $X \in \Gamma$ the last expression is zero. If $X = X_i$ then (3.14) and (4.4) give us

$$P(X_i, X_p, X_p) = -\langle AX_i, \nabla_{X_p} X_p\rangle(\rho - \eta_p) + \langle AX_p, X_p\rangle(\Gamma^p_{ip}(\eta_i - \rho) - \Gamma^p_{ip}(\eta_i - \eta_p)),$$

which is zero by (4.5).

Finally, take $v = \eta_k$. By (4.30) is enough to prove that

$$\langle P(X, X_p, \eta_k), \rho\rangle = 0.$$

Equations (4.31) and (4.32) prove this for $X \in \Gamma$. If $X = X_i$ for $i \neq p$, then (4.29) and the same computations of Lemma 4.3.1 (in particular (4.24) and (4.25)) give us

$$\langle P(X_i, X_p, \eta_k), \rho\rangle = \langle \hat{R}(X_i, X_p)\eta_k, \rho\rangle = -(\partial_p\Gamma^p_{ip} - \Gamma^p_{ip}\Gamma^i_{pi} + g_{ip})\Big(\frac{\delta_{kp}}{\varphi_k} - \frac{\delta_{kj}}{\varphi_j}\Big),$$

which is zero by hypothesis. This concludes the proof. □

The three claims above show that $(E, \hat{\nabla}, \hat{\alpha})$ satisfies Gauss, Codazzi, and Ricci equations. By the Fundamental Theorem of submanifolds, locally (on a simply connected neighborhood) there exists an isometric immersion $f : M^n \to \mathbb{R}^{n+p+1}_\nu$ whose normal bundle is given by $(E = T^\perp_{f_2}M \oplus T^\perp_{f_1}M, \langle\cdot,\cdot\rangle, \hat{\nabla})$, and its second fundamental form is $\hat{\alpha}$, up to a parallel isometry of vector bundles.

The following claim concludes the proof.

Claim 4. *The map f is a composition of f_1 and f_2. Namely, there exist isometric immersions $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+p+1}_\nu$ and $F_2 : U_2 \subseteq \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+1}_\nu$ such that locally $f = F_1 \circ f_1 = F_2 \circ f_2$. Furthermore, F_1 and F_2 intersect generically in M^n.*

Proof. We prove that f is a composition of f_1 and f_2 by using the techniques of [28], in particular, Propositions 17 and 18 therein. Let $L_i := T^\perp_{f_i}M \subseteq E$ and the tensors $\phi_{L_i^\perp} : TM \times (TM \oplus L_i) \to L_i^\perp \subseteq E$ be given by

$$\phi_{L_i^\perp}(Z, v) = (\tilde{\nabla}_Z v)_{L_i^\perp},$$

where $\tilde{\nabla}$ is the connection of $\mathbb{R}^{n+p+1}_\nu$. Then (4.21) and (4.22) show that $\phi_{L_i^\perp} = \Lambda_i$, and for $X = \sum_j X_j$ we have $\Lambda_i(X, TM) = L_i^\perp$. Proposition 18 of [28] (which is also valid for semi-Euclidean ambient space) shows that $f = F_1 \circ f_1 = F_2 \circ f_2$ for some isometric immersions $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+p+1}_\nu$ and $F_2 : U_2 \subseteq \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+1}_\nu$.

Clearly F_1 and F_2 intersect transversally in M^n, and (4.26) implies that F_1 and F_2 intersect non-orthogonally. For $X = \sum_j X_j$ we see that $\alpha^{F_i}(X, TM) = \Lambda_i(X, TM) = T^\perp_{F_i}U_i$, so F_1 and F_2 are full. By Lemma 2.2.1 we have that the rank of F_1 and F_2 are at least p and 1, respectively. Moreover, if Z belongs to Δ_1 and Δ_2, the respective nullities of F_1 and F_2, then $Z \in TM$. By (4.21) we have that $Z \in \Gamma$, and thus

$$\dim(\Delta_1 + \Delta_2) = \dim(\Delta_1) + \dim(\Delta_2) - \dim(\Delta_1 \cap \Delta_2) \geq n + 1 - p + n + p - 1 - (n - p - 1) = n + p + 1.$$

So Δ_1 and Δ_2 intersect transversally in M^n, and necessarily the ranks of F_1 and F_2 are p and 1, respectively. Finally, (iii) of Definition 4.2.8 is trivially satisfied since it is equivalent to $\Gamma \subseteq TM$ being generic. This shows that F_1 and F_2 intersect generically in a neighborhood of M^n. □

Remark 4.3.3. The same proof characterizes all non-degenerate deformations $f_2 : M^n \to \mathbb{R}^{n+p}_\mu$ of a generic hypersurface $f_1 : M^n \to \mathbb{R}^{n+1}$ of rank $(p+1) < n$ satisfying (4.23). They are locally obtained as an intersection of non-degenerate flat submanifolds in $\mathbb{R}^{n+p+1}_\nu$ with $\nu = \mu + (\mathrm{sign}(\varphi_p) + 1)/2$.

4.4 Intersecting a flat submanifold with a deformable hypersurface

In this section we intersect a flat submanifold with a hypersurface as in Theorem 3.0.1, which provides another deformable hypersurface. In addition, we characterize analytically such geometric construction.

Consider a Riemannian hypersurface $F : \hat{M}^{n+q} \to \mathbb{R}^{n+q+1}_{\nu}$, and $G : \hat{M}^{n+q} \to \mathbb{R}^{n+q+p}$ a non-degenerate deformation of F as discussed in Section 4.1. In this case, let $\hat{\Gamma} = \Delta_F = \Delta_G \subseteq T\hat{M}$ be the nullity of $\hat{M}^{n+q}$, and $\hat{\pi} : \hat{M}^{n+q} \to \hat{M}^{n+q}/\hat{\Gamma} =: L^{p+1}$ be the quotient map to the associated leaf space. Also, let $\hat{h} : L^{p+1} \to \mathbb{Q}^{n+q}_{\varepsilon}$ be the Gauss map of F and $\hat{\gamma} \circ \hat{\pi} = \langle \hat{h} \circ \hat{\pi}, F\rangle$ its support function. Fix a conjugate chart $\{u_i\}_i$ associated to F, and call $\hat{Q}$ the associated DMZ system such that $\hat{Q}(\hat{h}) = 0 = \hat{Q}(\hat{\gamma})$. Finally, call $\hat{\varphi}$ the parallel section of the Sbrana bundle such that $G = G^{\hat{\varphi}}$.

Let $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+q+1}_{\nu}$ be a flat submanifold as the ones described in Section 4.1.1. That is, F_1 is full, it has rank q, and is generic. Fix conjugate coordinates $\{u_\alpha\}_\alpha$ associated to F_1. To avoid confusion, through this section we use Latin letters for the indices of the conjugate coordinates of F, and Greek letters for the ones of F_1.

Assume that F intersects generically with F_1. We use the notations of (4.9) for $F_2 := F$. Set $g = G \circ f_2$. Our situation is then described by diagram (4.2). We denote by Δ_1 and Δ_2 the respective relative nullities of F_1 and F_2. As discussed before, $\Delta_2 = \Delta_G = \hat{\Gamma} \subseteq T\hat{M}$ is the intrinsic nullity of $\hat{M}^{n+q}$ and coincides with the relative nullity of G.

Notice that g is a non-degenerate deformation of the generic hypersurface $f_1 : M^n \to \mathbb{R}^{n+1}$ of rank $(p + q + 1) < n$. Indeed, let $\beta = (\alpha^g, \alpha^f)$ and $\hat{\beta} = (\alpha^G, \alpha^{F_2})$ be the associated flat bilinear forms. Example 4.2.4 shows that $\beta \cong \hat{\beta} \oplus \alpha^{F_1}$, so g is a non-degenerate deformation of f_1 since G is a non-degenerate deformation of F_2 and F_1 is a non-degenerate flat submanifold. Thus, g is a deformation of f_1 as in Theorem 3.0.1. In particular, the intrinsic nullity $\Gamma \subseteq TM$ coincides with the relative nullities, namely,

$$\Gamma = \Delta_g = \Delta_{f_1} = \Delta_1 \cap \Delta_2.$$

Let $\pi : M^n \to M^n/\Gamma =: L^{p+q+1}$ be the quotient map of the associated leaf space.

Let $\hat{X}_j \in \mathfrak{X}(\hat{M})$ and $\hat{X}_\alpha \in \mathfrak{X}(U)$ be lifts of $\partial_j := \partial_{u_j}$ and $\partial_\alpha := \partial_{u_\alpha}$, respectively. In M^n set $D_1 := \Delta_2 \cap TM$ and $D_2 := \Delta_1 \cap TM$. We can assume that along M^n we have $\hat{X}_j \in D_2$ and $\hat{X}_\alpha \in D_1$. Let $X_j = \hat{X}_j|_{M^n}$ and $X_\alpha = \hat{X}_\alpha|_{M^n}$ be the associated vector fields in M^n. Then we can identify $\{u_i, u_\alpha\}_{i,\alpha}$ with the conjugate coordinates associated to f_1 as in Section 4.3. In particular, the Gauss map $h : L^{p+q+1} \to \mathbb{S}^n$ of f_1 and its support function $\gamma \circ \pi := \langle f_1, h \circ \pi\rangle$ satisfy $Q(h) = 0 = Q(\gamma)$, where Q is the DMZ system associated to this conjugate chart. Also, let φ be the parallel section of the Sbrana bundle such that $g = g^{\varphi}$.

Set $\rho = h \circ \pi$ and $\hat{\rho} = \hat{h} \circ \hat{\pi}$. Denote by $A = A_\rho$ and $\hat{A} = A_{\hat{\rho}}$ their respective shape operators. Recall the normal vectors $\{\eta_i, \eta_\alpha\}_{i,\alpha} \subseteq T^{\perp}_g M$ and $\{\hat{\eta}_i\}_i \subseteq T^{\perp}_G \hat{M}$ defined by (3.6), and let $\{\varphi_i, \varphi_\alpha\}_{i,\alpha}$ and $\{\hat{\varphi}_\alpha\}_\alpha$ the associated functions given by (4.6). Notice that $\operatorname{span}\{\eta_\alpha\}_\alpha = T^{\perp}_{f_2} M$, so define

$$\varphi_{\#} := -\Big(1 + \sum_\alpha \varphi_\alpha\Big) = \sum_i \varphi_i \quad \text{and} \quad \eta_{\#} := -\frac{1}{\varphi_{\#}}\Big(\sum_\alpha \varphi_\alpha \eta_\alpha\Big) \in T^{\perp}_{f_2} M. \tag{4.33}$$

Straightforward computations show that

$$\langle \eta_{\#}, \eta_{\#}\rangle = 1 + \frac{1}{\varphi_{\#}} \quad \text{and} \quad \langle \eta_{\#}, \eta_i\rangle = \langle \eta_{\#}, \eta_\alpha\rangle = 1, \quad \forall i, \alpha.$$

Hence, $\langle \eta_i - \eta_{\#}, \eta_\alpha\rangle = 0$ for all i, α, so $\eta_{\#}$ is the orthogonal projection of η_i onto $T^{\perp}_{f_2} M$ for any i. Thus

$$\eta_{\#} = \frac{\alpha^{f_2}(X_i, X_i)}{\langle AX_i, X_i\rangle}, \quad \forall i. \tag{4.34}$$

Using (4.10) we have

$$\rho = \frac{\alpha^{f_1}(X_i, X_i)}{\langle AX_i, X_i\rangle} = \frac{\alpha^{f}(X_i, X_i)}{\langle AX_i, X_i\rangle} = \frac{\alpha^{F_2}(X_i, X_i)}{\langle AX_i, X_i\rangle} + \frac{\alpha^{f_2}(X_i, X_i)}{\langle AX_i, X_i\rangle} = \kappa_i \varepsilon \hat{\rho} + \eta_{\#}, \quad \forall i,$$

where $\kappa_i = \frac{\langle \hat{A}X_i, X_i\rangle}{\langle AX_i, X_i\rangle}$ and $\varepsilon = \langle \hat{\rho}, \hat{\rho}\rangle = 1 - 2\nu$. Take inner products between these expressions to obtain

$$\varepsilon \kappa_i \kappa_j = -\varphi_{\#}^{-1}, \quad \forall i, j.$$

In particular $\kappa := \kappa_i$ does not depend on i and

$$\varepsilon \kappa^2 = -\varphi_{\#}^{-1}. \tag{4.35}$$

Moreover, the last equation determines the index of the ambient space inner product via

$$\varepsilon = -\operatorname{sign}(\varphi_{\#}) = 1 - 2\nu. \tag{4.36}$$

For $i = 1, 2$ set $L_i := T^{\perp}_{f_i}M$ and $\Lambda_i : TM \times (TM \oplus L_i) \to L_i^{\perp} \subseteq T^{\perp}_f M$ by

$$\Lambda_i := \alpha^{F_i}|_{TM\times(TM\oplus L_i)}.$$

Then Λ_1 is flat since $U \subseteq \mathbb{R}^{n+1}$ is flat, yet Λ_2 is not since $\hat{M}^{n+q}$ is nowhere flat. We can rewrite (4.10) in terms of the tensors Λ_i and the second fundamental forms of g and f as

$$\alpha^f(X,Y) = \alpha^{f_1}(X,Y) + \Lambda_1(X,Y) = (\alpha^g(X,Y))_{L_2} + \Lambda_2(X,Y), \quad \forall X, Y \in TM, \tag{4.37}$$

where the subindex L_2 denotes the orthogonal projection on L_2. Furthermore, the normal connection $\nabla^{\perp f}$ of f is related to the normal connections of f_1 and g by

$$\begin{aligned} \nabla^{\perp f}_X \xi_1 &= \nabla^{\perp f_1}_X \xi_1 + \Lambda_1(X, \xi_1), \quad \forall X \in TM, \quad \forall \xi_1 \in \Gamma(L_1), \\ \nabla^{\perp f}_X \xi_2 &= (\nabla^{\perp g}_X \xi_2)_{L_2} + \Lambda_2(X, \xi_2), \quad \forall X \in TM, \quad \forall \xi_2 \in \Gamma(L_2). \end{aligned} \tag{4.38}$$

As in the last section, we proceed to determine the tensors Λ_i for $i = 1, 2$ and using these tensors we will show that certain Laplace invariants of Q vanish.

Lemma 4.4.1. *The Christoffel symbols of the Gauss map $h : L^{p+1} \to \mathbb{S}^n$ of f_1 satisfy that*

$$\Gamma_\alpha := \Gamma^i_{\alpha i} = \Gamma^j_{\alpha j}, \quad \forall i, j, \alpha, \tag{4.39}$$

$$\partial_j \Gamma_\alpha - \Gamma_\alpha \Gamma^\alpha_{j\alpha} + g_{j\alpha} = 0, \quad \forall j, \alpha. \tag{4.40}$$

Furthermore, the tensors Λ_i are uniquely determined by the following conditions:

i) D_{i+1} is the left nullity of Λ_i, that is $D_{i+1} = \Delta_{\Lambda_i} = \{X \in TM : \Lambda_i(X,Y) = 0, \quad \forall Y \in TM \oplus T^{\perp}_{f_i}M\}$;

ii) $\Lambda_1(X_\alpha, X_\beta) = \delta_{\alpha\beta}\langle AX_\alpha, X_\alpha\rangle(\eta_\alpha - \rho), \quad \Lambda_2(X_i, X_j) = \delta_{ij}\langle AX_i, X_j\rangle(\rho - \eta_{\#}), \quad \forall i, j, \alpha, \beta$;

iii) $\Lambda_1(X_\alpha, \rho) = \Gamma_\alpha(\eta_\alpha - \rho), \quad \Lambda_2(X_i, \eta_\alpha) = \Gamma^\alpha_{i\alpha}(\rho - \eta_{\#}), \quad \forall i, \alpha.$

Proof. We prove first (4.39). In $M^n \subseteq \mathbb{R}^{n+q+p}$ we have the orthogonal decomposition $T^{\perp}_g M = T^{\perp}_G \hat{M} \oplus T^{\perp}_{f_2} M$, so

$$\eta_i = \frac{\alpha^g(X_i, X_i)}{\langle AX_i, X_i\rangle} = \frac{\alpha^G(X_i, X_i)}{\langle AX_i, X_i\rangle} + \frac{\alpha^{f_2}(X_i, X_i)}{\langle AX_i, X_i\rangle} = \kappa\hat{\eta}_i + \eta_{\#}.$$

Using (4.6) and $\varepsilon\kappa^2 = -\varphi_{\#}^{-1}$, we get

$$\varepsilon\kappa^2 = \frac{\hat{\varphi}_i}{\varphi_i}, \quad \forall i.$$

From (3.22) and the fact that $X_\alpha \in D_2 \subseteq \hat{\Gamma}$, we obtain

$$\partial_\alpha \kappa = -\Gamma^i_{\alpha i}\kappa, \quad \forall \alpha, \ \forall i. \tag{4.41}$$

This proves (4.39) since $\kappa \neq 0$.

We now determine the tensors Λ_1 and Λ_2. Notice that (ii) is just (4.34) and (4.38). Also, as Λ_i is a restriction of α^{F_i}, we necessarily get $D_{i+1} = \Delta_i \cap TM \subseteq \Delta_{\Lambda_i}$. Then we conclude (i) from (ii). On the other hand, by (3.14) and (4.4) we have

$$\nabla^{\perp g}_{X_\alpha} \eta_i = \Gamma_\alpha(\eta_\alpha - \eta_i).$$

Then using (4.34) and that $X_\alpha \in \Delta_2 = \Delta_G$, we have

$$\hat{\nabla}_{X_\alpha}\eta_{\#} = \nabla^{\perp f_2}_{X_\alpha}\eta_{\#} = \nabla^{\perp g}_{X_\alpha}\eta_{\#} = \nabla^{\perp g}_{X_\alpha}\big((\eta_{\#} - \eta_i) + \eta_i\big) = (\nabla^{\perp g}_{X_\alpha}\eta_i)_{T^{\perp}_{f_2}M} = \Gamma_\alpha(\eta_\alpha - \eta_{\#}).$$

Since $\rho = \varepsilon\kappa\hat{\rho} + \eta_{\#}$, from (4.41) we get

$$\hat{\nabla}_{X_\alpha}\rho = \hat{\nabla}_{X_\alpha}(\varepsilon\kappa\hat{\rho} + \eta_{\#}) = \varepsilon\partial_\alpha\kappa\hat{\rho} + \Gamma_\alpha(\eta_\alpha - \eta_{\#}) = \Gamma_\alpha(-\varepsilon\kappa\hat{\rho} + \eta_\alpha - \eta_{\#}) = \Gamma_\alpha(\eta_\alpha - \rho),$$

which proves the first equality of *(iii)*. Finally, by (3.14) and (4.4) we obtain

$$\nabla^{\perp g}_{X_i}\eta_\alpha = \Gamma^\alpha_{i\alpha}(\eta_i - \eta_\alpha).$$

Then

$$\nabla^{\perp f_2}_{X_i}\eta_\alpha = \Gamma^\alpha_{i\alpha}(\eta_\# - \eta_\alpha) \quad \text{and} \quad \alpha^G(X_i,\eta_\alpha) = \Gamma^\alpha_{i\alpha}(\eta_i - \eta_\#) = \Gamma^\alpha_{i\alpha}\kappa\hat{\eta}_i. \tag{4.42}$$

Since $\alpha^G(X_i,\eta_\alpha) = \langle \hat{A}X_i,\eta_\alpha\rangle\hat{\eta}_i$ by definition of $\hat{\eta}_i$, we obtain $\langle \hat{A}X_i,\eta_\alpha\rangle = \Gamma^\alpha_{i\alpha}\kappa$. Thus,

$$\begin{aligned}\hat{\nabla}_{X_i}\eta_\alpha &= \nabla^{\perp f_2}_{X_i}\eta_\alpha + \alpha^{F_2}(X_i,\eta_\alpha) = \Gamma^\alpha_{i\alpha}(\eta_\# - \eta_\alpha) + \langle \hat{A}X_i,\eta_\alpha\rangle\varepsilon\hat{\rho}\\ &= \Gamma^\alpha_{i\alpha}(\eta_\# - \eta_\alpha) + \Gamma^\alpha_{i\alpha}\kappa\varepsilon\hat{\rho} = \Gamma^\alpha_{i\alpha}(\eta_\# - \eta_\alpha) + \Gamma^\alpha_{i\alpha}(\rho - \eta_\#),\end{aligned}$$

which proves the second equality of *(iii)*.

We prove now that (4.40) is equivalent to some Ricci equations of f. Let $\hat{R}$ be the normal curvature tensor of f. Then

$$\begin{aligned}\hat{R}(X_j,X_\alpha)\rho = \hat{\nabla}_{X_j}\hat{\nabla}_{X_\alpha}\rho &= \hat{\nabla}_{X_j}(\Lambda_1(X_\alpha,\rho)) = \hat{\nabla}_{X_j}(\Gamma_\alpha(\eta_\alpha - \rho))\\ &= \partial_j\Gamma_\alpha(\eta_\alpha - \rho) + \Gamma_\alpha((\nabla^{\perp g}_{X_j}\eta_\alpha)_{L_2} + \Lambda_2(X_j,\eta_\alpha)\\ &= \partial_j\Gamma_\alpha(\eta_\alpha - \rho) + \Gamma_\alpha\big(\Gamma^\alpha_{j\alpha}(\eta_j - \eta_\alpha)_{L_2} + \Gamma^\alpha_{j\alpha}(\rho - \eta_\#)\big)\\ &= (\partial_j\Gamma_\alpha - \Gamma_\alpha\Gamma^\alpha_{j\alpha})(\eta_\alpha - \rho).\end{aligned}$$

Moreover, as $\eta_\beta = \rho + \frac{\alpha^{F_1}(X_\beta,X_\beta)}{\langle AX_\beta,X_\beta\rangle}$, we have that $\langle\eta_\alpha,\rho\rangle = 1$ and hence

$$\langle\hat{R}(X_j,X_\alpha)\rho,\eta_\beta\rangle = (\partial_j\Gamma_\alpha - \Gamma_\alpha\Gamma^\alpha_{j\alpha})\frac{\delta_{\alpha\beta}}{\varphi_\alpha}. \tag{4.43}$$

Furthermore, using (4.3) and Ricci equation of f, the last expression is also given by

$$\langle[A,A_{\eta_\beta}]X_j,X_\alpha\rangle = -\frac{\delta_{\alpha\beta}}{\varphi_\alpha}\langle AX_j,AX_\beta\rangle = -\frac{\delta_{\alpha\beta}}{\varphi_\alpha}g_{j\beta}. \tag{4.44}$$

This proves (4.40) and ends the proof. □

We have all that is needed to prove Theorem 4.0.2.

Proof of Theorem 4.0.2. The proof is similar to the one for Theorem 4.0.1. We have already proved the direct statement. For the converse, we will first construct $f : M^n \to \mathbb{R}^{n+q+1}_\nu$ as in (4.2) using the Fundamental Theorem of submanifolds, then we will verify that f is a composition of f_1, and prove that $\{g,f\}$ isometrically extends to $\hat{M}^{n+q}$.

As discussed in Chapter 3, $g = g^\varphi : M^n \to \mathbb{R}^{n+q+p}$ is associated with some parallel section φ of the Sbrana bundle. Set $L_1 = T^\perp_f M$ and $L_2 = \operatorname{span}\{\eta_\alpha\}_\alpha \subseteq T^\perp_g M$. In the Whitney sum $E := L_1 \oplus L_2 \to M^n$ consider the inner product such that the trivial inclusions $L_i \to E$ are isometries and

$$\langle\rho,\eta_\alpha\rangle := 1, \quad \forall\alpha. \tag{4.45}$$

Define $\varphi_\#$ and $\eta_\#$ as in (4.33). Then $\langle\rho - \eta_\#,\eta_\alpha\rangle = 0$ for any α, so $(\rho - \eta_\#)$ is orthogonal to L_2 in E, and

$$\langle\rho - \eta_\#,\rho - \eta_\#\rangle = -\varphi_\#^{-1}.$$

Therefore, this inner product in E has index $\nu := (\operatorname{sign}(\varphi_\#)+1)/2 \in \{0,1\}$. Moreover, we have $\langle\eta_i - \eta_\#,\eta_\alpha\rangle = 0$ for all i,α, so $\eta_\#$ is the orthogonal projection of η_i onto L_2.

Consider for $i = 1,2$ the symmetric bilinear tensor $\Lambda_i : TM \times (TM \oplus L_i) \to L_i^\perp \subseteq E$ by the same formulas of Lemma 4.4.1. Straightforward computations show that Λ_1 is flat. Define the symmetric bilinear tensor $\hat{\alpha} : TM \times TM \to E$ by

$$\hat{\alpha}(X,Y) = \alpha^{f_1}(X,Y) + \Lambda_1(X,Y) = (\alpha^g(X,Y))_{L_2} + \Lambda_2(X,Y), \quad \forall X,Y \in TM,$$

as in (4.37). The flatness of Λ_1 shows that $\hat{\alpha}$ satisfies Gauss equation, since α^{f_1} does.

Endow E with the connection $\hat{\nabla}$ in (4.38). As in Theorem 4.0.2, we verify that $(E,\hat{\nabla},\hat{\alpha})$ satisfies the hypothese of the Fundamental Theorem of submanifolds through a series of claims.

Claim 1. *The connection $\hat{\nabla}$ is compatible with the inner product.*

Proof. Since the connection $\nabla^{\perp g}$ is compatible with the inner product of $T_g^{\perp}M$, we have for $\xi, \zeta \in \Gamma(L_2)$ that

$$\langle \hat{\nabla}_X \xi, \zeta \rangle + \langle \xi, \hat{\nabla}_X \zeta \rangle = \langle \nabla_X^{\perp g} \xi, \zeta \rangle + \langle \xi, \nabla_X^{\perp g} \zeta \rangle = X \langle \xi, \zeta \rangle,$$

and $2\langle \hat{\nabla}_X \rho, \rho \rangle = 0 = X(\langle \rho, \rho \rangle)$. Also, using (3.14) and that $\eta_\#$ is the orthogonal projection of η_i into L_2 we get

$$(\nabla_X^{\perp, g} \eta_\alpha)_{L_2} = \Big(\sum_j \phi_{\alpha j}(X) \varphi_j \Big) \eta_\# + \sum_\beta \phi_{\alpha\beta}(X) \varphi_\beta \eta_\beta.$$

Hence

$$\langle \hat{\nabla}_X \eta_\alpha, \rho \rangle + \langle \eta_\alpha, \hat{\nabla}_X \rho \rangle = \Big(\sum_j \phi_{\alpha j}(X) \varphi_j \Big) \langle \eta_\#, \rho \rangle + \sum_\beta \phi_{\alpha\beta}(X) \varphi_\beta + \langle \Lambda_2(X, \eta_\alpha), \rho \rangle + \langle \eta_\alpha, \Lambda_1(X, \rho) \rangle. \tag{4.46}$$

To prove the claim it remains to show that the right hand side of the last equation is equal to $X\langle \eta_\alpha, \rho \rangle$ for all α and X. This is clear if $X \in \Gamma$. If $X = X_i$ for some i, then using (4.4) we get that (4.46) is equal to

$$\langle \hat{\nabla}_{X_i} \eta_\alpha, \rho \rangle + \langle \eta_\alpha, \hat{\nabla}_{X_i} \rho \rangle = \Gamma_{i\alpha}^\alpha \langle \eta_\#, \rho \rangle - \Gamma_{i\alpha}^\alpha + \langle \Gamma_{i\alpha}^\alpha (\rho - \eta_\#), \rho \rangle = 0 = X_i \langle \eta_\alpha, \rho \rangle.$$

Analogously for $X = X_\gamma$ we complete the proof of the claim using (3.9) and (3.13) since

$$\begin{aligned}
\langle \hat{\nabla}_{X_\gamma} \eta_\alpha, \rho \rangle + \langle \eta_\alpha, \hat{\nabla}_{X_\gamma} \rho \rangle &= \Big(\sum_j \phi_{\alpha j}(X_\gamma) \varphi_j \Big) \Big(1 + \frac{1}{\varphi_\#} \Big) + \sum_\beta \phi_{\alpha\beta}(X_\gamma) \varphi_\beta + \langle \eta_\alpha, \Lambda_1(X_\gamma, \rho) \rangle \\
&= \Big(\sum_j \phi_{\alpha j}(X_\gamma) \varphi_j + \sum_\beta \phi_{\alpha\beta}(X_\gamma) \varphi_\beta \Big) + \Big(\sum_j \phi_{\alpha j}(X_\gamma) \varphi_j \Big) \varphi_\#^{-1} + \langle \eta_\alpha, \Gamma_\gamma (\eta_\gamma - \rho) \rangle \\
&= \Big\langle \nabla_{X_\gamma}^{\perp g} \eta_\alpha, \sum_j \varphi_j \eta_j + \sum_\beta \varphi_\beta \eta_\beta \Big\rangle - \frac{\delta_{\gamma\alpha} \Gamma_\alpha}{\varphi_\alpha} \sum_j \varphi_j \varphi_\#^{-1} + \delta_{\gamma\alpha} \Gamma_\alpha \varphi_\alpha^{-1} \\
&= \delta_{\gamma\alpha} \Gamma_\alpha \varphi_\alpha^{-1} \Big(- \sum_j \varphi_j \varphi_\#^{-1} + 1 \Big) = 0 = X_\alpha \langle \eta_\alpha, \rho \rangle.
\end{aligned}$$

□

Now, we argue that $(E, \hat{\nabla}, \hat{\alpha})$ satisfies Codazzi and Ricci equations. As in the proof of Theorem 4.0.1, this is equivalent to Λ_1 being Codazzi, or Λ_2 and ϕ_2 being Codazzi, where $\phi_2 : TM \times (TM \oplus L_2) \to L_2^\perp \subseteq T_g^\perp M$ is the tensor

$$\phi_2(X, v) = (\overline{\nabla}_X v)_{L_2^\perp},$$

and $\overline{\nabla}$ is the connection of the ambient space $\mathbb{R}^{n+q+p}$.

Claim 2. Λ_1 *is Codazzi.*

Proof. To simplify notations, set

$$\begin{aligned}
P(X, Y, v) &:= (\hat{\nabla}_X \Lambda_1)(Y, v) - (\hat{\nabla}_Y \Lambda_1)(X, v) \\
&= \hat{\nabla}_X (\Lambda_1(Y, v)) - \hat{\nabla}_Y (\Lambda_1(X, v)) - \Lambda_1([X, Y], v) - \Lambda_1(Y, (\tilde{\nabla}_X v)_{TM \oplus L_1}) + \Lambda_1(X, (\tilde{\nabla}_Y v)_{TM \oplus L_1}),
\end{aligned}$$

where $\tilde{\nabla}$ is the connection on $TM \oplus E$ defined by (4.27). Notice that for $X, Y \in D_2 = \Delta_{\Lambda_1}$ we have $P(X, Y, v) = 0$ since $D_2 \subseteq TM$ is integrable. Hence, to prove that Λ_1 is Codazzi, is enough to show that $P(D_2, X_\beta, v) = 0$ and $P(X_\alpha, X_\beta, v) = 0$ for any v and $\alpha \neq \beta$.

First, the case $X \in D_2$ and $Y = X_\beta$ is analogous to the proof of Theorem 4.0.1 for $X \in D_1$ and $Y = X_p$. Indeed, those computations are similar to the ones of Lemma 4.4.1, which in turn are equivalent to (4.40).

Finally, we have that $[X_\alpha, X_\beta] \in \Gamma$ for $\alpha \neq \beta$ since they project as coordinate vectors. So

$$P(X_\alpha, X_\beta, v) = \hat{\nabla}_{X_\alpha} (\Lambda_1(X_\beta, v)) - \hat{\nabla}_{X_\beta} (\Lambda_1(X_\alpha, v)) - \Lambda_1(X_\beta, (\tilde{\nabla}_{X_\alpha} v)_{TM \oplus L_1}) + \Lambda_1(X_\alpha, (\tilde{\nabla}_{X_\beta} v)_{TM \oplus L_1}). \tag{4.47}$$

Assuming first that $v \in D_2 = \Delta_{\Lambda_1}$, (4.47) reduces to

$$P(X_\alpha, X_\beta, v) = \Lambda_1(X_\alpha, \nabla_{X_\beta} v) - \Lambda_1(X_\beta, \nabla_{X_\alpha} v) = \langle AX_\alpha, \nabla_{X_\beta} v \rangle (\eta_\alpha - \rho) - \langle AX_\beta, \nabla_{X_\alpha} v \rangle (\eta_\beta - \rho).$$

This vanishes if $v \in \Gamma$ since X_α and X_β are eigenvectors of the splitting tensor C_v and $\langle AX_\alpha, X_\beta \rangle = 0$. Moreover, if $v = X_i$ then (4.5) implies that the last expression is zero. This proves that $P(X_\alpha, X_\beta, D_1) = 0$.

Assume now that $v = X_\gamma$ and γ is distinct to α and β. In this case (4.47) reduces to

$$P(X_\alpha, X_\beta, X_\gamma) = \Lambda_1(X_\alpha, \nabla_{X_\beta} X_\gamma) - \Lambda_1(X_\beta, \nabla_{X_\alpha} X_\gamma) = \langle AX_\alpha, \nabla_{X_\beta} X_\gamma\rangle(\eta_\alpha - \rho) - \langle AX_\beta, \nabla_{X_\alpha} X_\gamma\rangle(\eta_\beta - \rho),$$

which is zero by (4.5).

Assume now that $v = X_\beta$ (the case $v = X_\alpha$ is symmetric). Then using Codazzi for A, (4.47) gives

$$\begin{aligned} P(X_\alpha, X_\beta, X_\beta) &= \hat\nabla_{X_\alpha}\big(\langle AX_\beta, X_\beta\rangle(\eta_\beta - \rho)\big) - \Lambda_1(X_\beta, \nabla_{X_\alpha} X_\beta) + \Lambda_1(X_\alpha, \nabla_{X_\beta} X_\beta + \langle AX_\beta, X_\beta\rangle\rho) \\ &= \big(\langle A\nabla_{X_\alpha} X_\beta, X_\beta\rangle - \langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle\big)(\eta_\beta - \rho) + \langle AX_\beta, X_\beta\rangle(\nabla^{\perp g}_{X_\alpha}\eta_\beta - \Gamma_\alpha(\eta_\alpha - \rho)) \\ &\quad - \langle AX_\beta, \nabla_{X_\alpha} X_\beta\rangle(\eta_\beta - \rho) + \langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle(\eta_\alpha - \rho) + \langle AX_\beta, X_\beta\rangle\Gamma_\alpha(\eta_\alpha - \rho). \end{aligned}$$

Hence from (3.14) and (4.4) we get

$$\begin{aligned} P(X_\alpha, X_\beta, X_\beta) &= (\langle A\nabla_{X_\alpha} X_\beta, X_\beta\rangle - \langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle)(\eta_\beta - \rho) + \langle AX_\beta, X_\beta\rangle(\Gamma^\beta_{\alpha\beta}(\eta_\alpha - \eta_\beta) - \Gamma_\alpha(\eta_\alpha - \rho)) \\ &\quad - \langle AX_\beta, \nabla_{X_\alpha} X_\beta\rangle(\eta_\beta - \rho) + \langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle(\eta_\alpha - \rho) + \langle AX_\beta, X_\beta\rangle\Gamma_\alpha(\eta_\alpha - \rho) \\ &= ((\langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle + \langle AX_\beta, X_\beta\rangle\Gamma^\beta_{\alpha\beta})(\eta_\alpha - \eta_\beta) \\ &= ((\langle AX_\alpha, \nabla_{X_\beta} X_\beta\rangle + \langle AX_\beta, X_\beta\rangle\Gamma^\beta_{\alpha\beta})(\eta_\alpha - \eta_\beta), \end{aligned}$$

which is zero by (4.5).

Finally, for $v = \rho$ we have

$$\begin{aligned} P(X_\alpha, X_\beta, \rho) &= \hat\nabla_{X_\alpha}(\Gamma_\beta(\eta_\beta - \rho)) - \hat\nabla_{X_\beta}(\Gamma_\alpha(\eta_\alpha - \rho)) - \Lambda_1(X_\beta, -AX_\alpha) + \Lambda_1(X_\alpha, -AX_\beta) \\ &= \partial_\alpha\Gamma_\beta(\eta_\beta - \rho) + \Gamma_\beta(\nabla^{\perp g}_{X_\alpha}\eta_\beta - \Gamma_\alpha(\eta_\alpha - \rho)) - \partial_\beta\Gamma_\alpha(\eta_\alpha - \rho) - \Gamma_\alpha(\nabla^{\perp g}_{X_\beta}\eta_\alpha - \Gamma_\beta(\eta_\beta - \rho)) \\ &\quad + \langle AX_\beta, AX_\alpha\rangle(\eta_\beta - \rho) - \langle AX_\alpha, AX_\beta\rangle(\eta_\alpha - \rho). \end{aligned}$$

Thus using (3.14) and (4.4) we get

$$\begin{aligned} P(X_\alpha, X_\beta, \rho) &= (\partial_\beta\Gamma_\alpha - \partial_\alpha\Gamma_\beta)\rho + (\partial_\alpha\Gamma_\beta + \Gamma_\alpha\Gamma_\beta - \Gamma_\alpha\Gamma^\alpha_{\beta\alpha} - \Gamma_\beta\Gamma^\beta_{\alpha\beta} + g_{\alpha\beta})\eta_\beta \\ &\quad - (\partial_\beta\Gamma_\alpha + \Gamma_\beta\Gamma_\alpha - \Gamma_\beta\Gamma^\beta_{\alpha\beta} - \Gamma_\alpha\Gamma^\alpha_{\beta\alpha} + g_{\beta\alpha})\eta_\alpha. \end{aligned} \tag{4.48}$$

Notice that the right hand side is zero since they are the integrability conditions of the DMZ system. In fact, they are also obtained as Gauss equation for $h : L^{p+q+1} \to \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ as

$$\begin{aligned} g_{i\alpha}\partial_\beta - g_{\alpha\beta}\partial_i = R(\partial_\beta, \partial_i)\partial_\alpha &= (\partial_\beta\Gamma^\alpha_{i\alpha} - \partial_i\Gamma^\alpha_{\beta\alpha})\partial_\alpha + (\partial_\beta\Gamma_\alpha + \Gamma_\alpha\Gamma_\beta - \Gamma_\alpha\Gamma^\alpha_{\beta\alpha} - \Gamma_\beta\Gamma^\beta_{\alpha\beta})\partial_i \\ &\quad - (\partial_i\Gamma^\beta_{\alpha\beta} + \Gamma^\beta_{\alpha\beta}\Gamma^\beta_{i\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{i\alpha} - \Gamma^\beta_{i\beta}\Gamma_\alpha)\partial_\beta. \end{aligned}$$

This proves that one of the terms of the right hand side of (4.48) is zero. By the analogous equations for $R(\partial_i, \partial_\alpha)\partial_\beta$ and $R(\partial_\alpha, \partial_\beta)\partial_i$ we prove that the remaining terms are also zero, and the claim is proved. $\square$

Therefore, by the Fundamental Theorem of submanifolds, locally (on a simply connected neighborhood) there exists an isometric immersion $f : M^n \to \mathbb{R}^{n+q+1}_\nu$ whose normal bundle is $(E = L_1 \oplus L_2, \langle\cdot,\cdot\rangle, \hat\nabla)$ and whose second fundamental form is $\hat\alpha$, up to a parallel isometry of vector bundles.

The following claim concludes the proof.

Claim 3. *The map f is a composition of f_1 and $\{g, f\}$ isometrically extends. Namely, there exist a Riemannian manifold $\hat M^{n+q}$ and isometric immersions $F_1 : U_1 \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^{n+q+1}_\nu$, $f_2 : M^n \to \hat M^{n+q}$, $G : \hat M^{n+q} \to \mathbb{R}^{n+q+p}$ and $F_2 : \hat M^{n+q} \to \mathbb{R}^{n+q+1}_\nu$ such that locally $f = F_1 \circ f_1 = F_2 \circ f_2$ and $g = G \circ f_2$. Furthermore, F_1 and F_2 intersect generically in M^n and G is a genuine deformation of the rank $(p+1)$ generic hypersurface F_2.*

Proof. We prove now that locally f is a composition $f = F_1 \circ f_1$ in the same way as we did it for Theorem 4.0.1. Define the tensor $\phi : TM \times (TM \oplus L_2) \to L_2^\perp \oplus L_2^\perp \subseteq T_g^\perp M \oplus T_f^\perp M$ as

$$\phi(X, v) := \Big(\big(\overline\nabla_X v\big)_{L_2^\perp}, \big(\tilde\nabla_X v\big)_{L_2^\perp}\Big) = (\phi_2(X, v), \Lambda_2(X, v)), \quad \forall X \in TM,\ \forall v \in TM \oplus L_2,$$

where $\overline\nabla$ and $\tilde\nabla$ are the ambient space connections of $\mathbb{R}^{n+q+p}$ and $\mathbb{R}^{n+q+1}_\nu$ respectively. We use Proposition 12 of [28] to verify that $\{g, f\}$ extends isometrically. Straightforward computations show that the identity map $\tau : L_2 \subseteq T_g^\perp M \to L_2 \subseteq T_f^\perp M$ preserves the respective components of the normal connections and the second fundamental forms.

We show next that $\phi(X, TM \oplus L_2) = 0$ for all $X \in D_2$. This is clear for $X \in \Gamma$, so take $X = X_\alpha$. We clearly have $\phi(X_\alpha, TM) = 0$, and by (3.14) and (4.4) we get

$$\phi(X_\alpha, \eta_\beta) = ((\nabla^{\perp g}_{X_\alpha}\eta_\beta)_{L_2^\perp}, 0) = \Big(\sum_i \phi_{\beta i}(\partial_\alpha)(\varphi_i\eta_i)_{L_2^\perp}, 0\Big) = \Big(-\delta_{\alpha\beta}\Gamma_\beta\varphi_\beta^{-1}\Big(\sum_i \varphi_i\eta_i\Big)_{L_2^\perp}, 0\Big), \quad \forall \alpha, \beta.$$

But since the last expression is zero by (3.9), we obtain that $\phi(D_2, TM \oplus L_2) = 0$.

Notice that

$$\phi(X_i, X_i) = (\alpha^g(X_i, X_i)_{L_2^\perp}, \Lambda_2(X_i, X_i)) = \langle AX_i, X_i\rangle\big((\eta_i)_{L_2^\perp}, \rho - \eta_\#\big). \tag{4.49}$$

Using (3.14) and (4.4) we get

$$\phi(X_i, \eta_\alpha) = \big((\nabla^{\perp g}_{X_i}\eta_\alpha)_{L_2^\perp}, \Lambda_2(X_i, \eta_\alpha)\big) = \Gamma^\alpha_{i\alpha}\langle AX_i, X_i\rangle^{-1}\phi(X_i, X_i).$$

Define

$$\lambda_\alpha := \eta_\alpha - \sum_j \Gamma^\alpha_{j\alpha}\langle AX_j, X_j\rangle^{-1}X_j \in TM \oplus L_2, \quad \forall \alpha.$$

We argue that $\phi(X, \lambda_\alpha) = 0$ for all $X \in TM$. Indeed, this is clear since $\phi(D_2, \lambda_\alpha) = 0$ and

$$\phi(X_i, \lambda_\alpha) = \phi(X_i, \eta_\alpha) - \Gamma^\alpha_{i\alpha}\langle AX_i, X_i\rangle^{-1}\phi(X_i, X_i) = 0.$$

Define the rank q bundle over M^n given by $\Lambda = \operatorname{span}\{\lambda_\alpha\}_\alpha \subseteq TM \oplus L_2$, which as a manifold has dimension $(n+q)$. Then $\phi(TM, \Lambda) = 0$, and so $\{f, g\}$ isometrically extends by Proposition 12 of [28]. That is, there is an open subset $\hat{M}^{n+q} \subseteq \Lambda$ of the zero section $f_2 : M^n \to \Lambda$, and there are immersions $G : \hat{M}^{n+q} \to \mathbb{R}^{n+q+p}$ and $F_2 : \hat{M}^{n+q} \to \mathbb{R}^{n+q+1}_\nu$ which induce the same metric on $\hat{M}^{n+q}$, with $g = G \circ j$, $f = F_2 \circ j$.

We prove next that F_1 and F_2 intersect generically in a similar way as we did for Theorem 4.0.1. Let $\hat{\beta} = (\alpha^G, \alpha^{F_2})$ be the induced flat bilinear form. Notice that $\beta(X, TM) = T^\perp_G M \oplus T^\perp_{F_2} M$ for $X = \sum_i X_i$ by (4.49) since $\beta = \phi|_{TM \times T\hat{M}}$. Then G is a non-degenerate deformation of F_2 and by Lemma 2.2.1 we get

$$\dim(\Delta_\beta) = \dim(\Delta_G \cap \Delta_{F_2}) \geq n + q - (p+1).$$

As for the proof of Theorem 4.0.1 we have that F_1 is full and $\dim(\Delta_{F_1}) \geq n + 1 - q$. Notice that, if $Z \in \Delta_\beta \cap \Delta_{F_1}$ then $\hat{\alpha}(Z, X) = \alpha^{f_1}(Z, X) = \alpha^g(Z, X)_{L_2}$ for all $X \in TM$, and so $Z \in \Delta_{f_1} = \Gamma$. Therefore,

$$\dim(\Delta_\beta + \Delta_{F_1}) = \dim(\Delta_\beta) + \dim(\Delta_{F_1}) - \dim(\Delta_\beta \cap \Delta_{F_1}) \geq n + q - (p+1) + n + 1 - q - (n - p - q - 1) = n + q + 1.$$

We conclude that $\Delta_\beta + \Delta_{F_1} = \mathbb{R}^{n+q+1}_\nu$, $\dim(\Delta_\beta) = n + q - p - 1$ and $\dim(\Delta_{F_1}) = n + 1 - q$. Furthermore, as $\alpha^{F_1}(X_i, X_i) = \Lambda_1(X_i, X_i) \neq 0$, by dimension reasons we have $\Delta_\beta = \Delta_{F_2} = \hat{\Gamma}$. This proves that F_1 and F_2 intersect generically and G is a non-degenerate deformation of the rank $(p+1)$ generic hypersurface F_2. In particular, G is a genuine deformation of F_2. □

This procedure gives us new examples of genuine deformations from old ones, and the new DMZ system is closely related to the old one.

Proposition 4.4.2. *With the above notations set $\lambda = \kappa^{-1}$. Then*

$$Q_{ij}(\lambda\hat{h}) = 0, \quad \forall i \neq j. \tag{4.50}$$

Moreover,

$$\hat{Q}_{ij}(\hat{h}) = \partial^2_{ij}\hat{h} - \hat{\Gamma}^i_{ji}\partial_i\hat{h} - \hat{\Gamma}^j_{ij}\partial_j\hat{h} + \hat{g}_{ij}\hat{h} = 0, \quad \forall i \neq j,$$

and the coefficients of Q and $\hat{Q}$ are related by the following formulas

$$\hat{\Gamma}^i_{ji} = \Gamma^i_{ji} - \frac{\partial_j\lambda}{\lambda}, \quad \hat{g}_{ij} = \frac{Q_{ij}(\lambda)}{\lambda}, \quad \forall i \neq j.$$

Furthermore, the Laplace invariants of $\hat{Q}$ coincide with the Laplace invariants of $\tilde{Q} = (Q_{ij})$.

Proof. We only prove (4.50), the remaining being straightforward computations. Let $h : L^{p+q+1} \to \mathbb{S}^n$ and $\hat{h} : L^{p+1} \to \mathbb{Q}_\varepsilon^{n+q}$ be the Gauss maps of f_1 and F_1 respectively. Namely, let $\pi : M^n \to L^{p+p+1}$ and $\hat{\pi} : \hat{M}^{n+q} \to L^{q+1}$ be the quotient maps and $\rho = h \circ \pi$ and $\hat{\rho} = \hat{h} \circ \hat{\pi}$ unit normal vector fields. Call A and $\hat{A}$ the shape operators of ρ and $\hat{\rho}$ respectively. Then $-AX_i = \partial_i h$ and $-\hat{A}X_i = \partial_i \hat{h}$. Since we showed before that $\rho = \kappa\varepsilon\hat{\rho} + \eta_\#$ as vectors on $\mathbb{R}_\nu^{n+q+1}$, we have that

$$h = \eta_\# + \varepsilon\kappa\hat{h}. \tag{4.51}$$

Differentiate the last equation and use that $X_i \in \Delta_{F_1}$ to get

$$\partial_i h = \tilde{\nabla}_{X_i}\eta_\# + \varepsilon\partial_i(\kappa\hat{h}), \tag{4.52}$$

where $\tilde{\nabla}$ the connection of $\mathbb{R}_\nu^{n+q+1}$. Using the definition of $\eta_\#$, (4.3) and (3.22) we obtain

$$0 = \tilde{\nabla}_{X_i}\Big(\varphi_\#\eta_\# + \sum_\alpha \varphi_\alpha\eta_\alpha\Big) = \partial_i\varphi_\#\eta_\# + \varphi_\#(\tilde{\nabla}_{X_i}\eta_\#) + \sum_\alpha \big(2\Gamma_{i\alpha}^\alpha\eta_\alpha - AX_i + \nabla_{X_i}^{\perp f}\eta_\alpha\big)\varphi_\alpha.$$

From (3.14), (4.4) and Lemma 4.4.1 we have

$$\begin{aligned}
\tilde{\nabla}_{X_i}\eta_\# &= -\varphi_\#^{-1}(\partial_i\varphi_\#)\eta_\# - \varphi_\#^{-1}\sum_\alpha \big(2\Gamma_{i\alpha}^\alpha\eta_\alpha + \partial_i h + (\nabla_{X_i}^{\perp g}\eta_\alpha)_{T_{f_2}^\perp M} + \Lambda_2(X_i,\eta_\alpha)\big)\varphi_\alpha\\
&= -\varphi_\#^{-1}(\partial_i\varphi_\#)\eta_\# - \varphi_\#^{-1}\sum_\alpha \big(2\Gamma_{i\alpha}^\alpha\eta_\alpha + \partial_i h + (\Gamma_{i\alpha}^\alpha(\eta_i - \eta_\alpha)_{T_{f_2}^\perp M} + \Gamma_{i\alpha}^\alpha(\rho - \eta_\#)\big)\varphi_\alpha.
\end{aligned}$$

Hence, since $\eta_\#$ is the orthogonal projection of η_i onto $T_{f_2}^\perp M$ we conclude that

$$\tilde{\nabla}_{X_i}\eta_\# = -\varphi_\#^{-1}(\partial_i\varphi_\#)\eta_\# - \varphi_\#^{-1}\sum_\alpha \big(\partial_i h + \Gamma_{i\alpha}^\alpha(\rho + \eta_\alpha)\big)\varphi_\alpha = \Big(1 + \frac{1}{\varphi_\#}\Big)\partial_i h - \varphi_\#^{-1}(\partial_i\varphi_\#)\eta_\# - \varphi_\#^{-1}\sum_\alpha \Gamma_{i\alpha}^\alpha(\rho + \eta_\alpha)\varphi_\alpha.$$

This equation in (4.52) gives

$$\begin{aligned}
\partial_i h &= \partial_i\varphi_\#\eta_\# + \sum_\alpha \Gamma_{i\alpha}^\alpha(h + \eta_\alpha)\varphi_\alpha - \varepsilon\varphi_\#\partial_i(\kappa\hat{h})\\
&= \partial_i\varphi_\#\eta_\# + \Big(2\sum_\alpha \Gamma_{i\alpha}^\alpha\varphi_\alpha\Big)h + \sum_\alpha \Gamma_{i\alpha}^\alpha(\eta_\alpha - h)\varphi_\alpha - \varepsilon\varphi_\#\partial_i(\kappa\hat{h})\\
&= \partial_i\varphi_\#(\eta_\# - h) + \sum_\alpha \Gamma_{i\alpha}^\alpha(\eta_\alpha - h)\varphi_\alpha + \varepsilon\varphi_\#\partial_i(\kappa\hat{h}).
\end{aligned}$$

Thus by (4.51) we get

$$\partial_i h = \sum_\alpha \varphi_\alpha\Gamma_{i\alpha}^\alpha(\eta_\alpha - h) - \varepsilon\partial_i(\varphi_\#\kappa\hat{h}). \tag{4.53}$$

Differentiating the last equation with respect to X_j for $j \neq i$, and computing the ambient space derivative as before we obtain

$$\begin{aligned}
\partial_{ij}^2 h &= \sum_\alpha \varphi_\alpha\big((\partial_j\Gamma_{i\alpha}^\alpha + 2\Gamma_{i\alpha}^\alpha\Gamma_{j\alpha}^\alpha)(\eta_\alpha - h) + \Gamma_{i\alpha}^\alpha\tilde{\nabla}_{X_j}(\eta_\alpha - h)\big) - \varepsilon\partial_{ij}^2(\varphi_\#\kappa\hat{h})\\
&= \sum_\alpha \varphi_\alpha\big((\partial_j\Gamma_{i\alpha}^\alpha + 2\Gamma_{i\alpha}^\alpha\Gamma_{j\alpha}^\alpha)(\eta_\alpha - h) + \Gamma_{i\alpha}^\alpha\Gamma_{j\alpha}^\alpha(h - \eta_\alpha)\big) - \varepsilon\partial_{ij}^2(\varphi_\#\kappa\hat{h})\\
&= \sum_\alpha \varphi_\alpha(\partial_j\Gamma_{i\alpha}^\alpha + \Gamma_{i\alpha}^\alpha\Gamma_{j\alpha}^\alpha)(\eta_\alpha - h) - \varepsilon\partial_{ij}^2(\varphi_\#\kappa\hat{h}).
\end{aligned}$$

Now, combine the last equation with (4.53) to get

$$0 = Q_{ij}(h) = \sum_\alpha \varphi_\alpha(\partial_j\Gamma_{i\alpha}^\alpha + \Gamma_{i\alpha}^\alpha\Gamma_{j\alpha}^\alpha - \Gamma_{i\alpha}^\alpha\Gamma_{ji}^i - \Gamma_{j\alpha}^\alpha\Gamma_{ij}^j)(\eta_\alpha - h) - \varepsilon(Q_{ij}(\varphi_\#\kappa\hat{h}) - g_{ij}\varphi_\#\kappa\hat{h}) + g_{ij}h.$$

The term involving the Christoffel symbols in the last equation is equal to $-g_{ij}$. Indeed, this is an integrability condition of the DMZ system Q and can be obtained as a Gauss equation for h as we did in the proof of Theorem 4.0.2. Then the last equation is equivalent to

$$\begin{aligned}
0 &= -g_{ij}\sum_\alpha \big(\varphi_\alpha\eta_\alpha - \varphi_\alpha h\big) - \varepsilon(Q_{ij}(\varphi_\#\kappa\hat{h}) - g_{ij}\varphi_\#\kappa\hat{h}) + g_{ij}h\\
&= g_{ij}\varphi_\#\eta_\# - g_{ij}(1 + \varphi_\#)h + g_{ij}h - \varepsilon(Q_{ij}(\varphi_\#\kappa\hat{h}) - g_{ij}\varphi_\#\kappa\hat{h}) = g_{ij}\varphi_\#(\eta_\# - h + \varepsilon\kappa\hat{h}) - \varepsilon Q_{ij}(\varphi_\#\kappa\hat{h}).
\end{aligned}$$

Now (4.50) is a consequence of (4.35) and (4.51). □

Remark 4.4.3. In some contexts Q and $\tilde{Q}$ can be identified since they only differ by the factor λ; see for example [34].

CHAPTER 5

Chern-Kuiper's inequalities and compositions

So far we have worked with two naturally associated distributions to a submanifold $g : M^n \to \mathbb{R}^{n+p}$, the nullity $\Gamma \subseteq TM$ of the curvature tensor and the relative nullity $\Delta_g \subseteq TM$, i.e., the nullity of the second fundamental form α of g. Gauss equation implies that $\Delta_g \subseteq \Gamma$. The relative nullity plays a fundamental role in many aspects of submanifold theory; see for example [9], [14], [19], [27], and [28]. In many of them, this distribution coincides with the nullity, turning the distribution into an intrinsic one; besides the ones already cited, see [13], [29], [31]. Chern and Kuiper proved in [5] that the ranks $\mu := \dim(\Gamma)$ and $\nu_g := \dim(\Delta_g)$ are related by the inequalities

$$\nu_g \leq \mu \leq \nu_g + p.$$

There are two natural families of submanifolds with $\nu_g \neq \mu$. Firstly, if M^n is flat and g is not (an open subset of) an affine subspace then $\nu_g < \mu = n$. Secondly, we have the compositions, that is, if $\hat{g} : M^n \to \mathbb{R}^{n+q}$ has nontrivial nullity and $h : U \subseteq \mathbb{R}^{n+q} \to \mathbb{R}^{n+p}$ is a flat submanifold with $\hat{g}(M^n) \subseteq U$ then generically $g = h \circ \hat{g} : M^n \to \mathbb{R}^{n+p}$ has less relative nullity, in particular $\Delta_g \neq \Gamma$. Theorem 1 of [30] is an example of this phenomenon.

Straightforward computations show that if $\nu_g = \mu - p$ then M^n is flat and $\nu_g = \mu - p = n - p$. Proposition 7 of [13] analyses the next case of the Chern-Kuiper's inequalities in a restricted situation. It shows that if $g : M^n \to \mathbb{R}^{n+2}$ has $\nu_g = \mu - 1 = n - 3$ then g is a composition. However, the authors' approach seems difficult to generalize. The first result of this Chapter extends that proposition, and the generalization is in two directions. We allow higher codimensions and do not impose a particular rank for the nullity.

Theorem 5.0.1. *Let $g : M^n \to \mathbb{R}^{n+p}$ be a submanifold with $p \geq 2$ and*

$$\nu_g = \mu - p + 1 \leq n - p - 1.$$

Then $g = j \circ \hat{g}$ is a composition, where $j : N^{n+1} \to \mathbb{R}^{n+p}$ is a flat submanifold and $\hat{g} : M^n \to N^{n+1}$ is an isometric embedding with $\Delta_{\hat{g}} = \Gamma$.

In particular, we are able to characterize locally the submanifolds $g : M^n \to \mathbb{R}^{n+2}$ with $\mu \neq \nu_g$. Observe that the inequality condition in the last result is equivalent to M^n being nowhere flat.

Using our technique, we analyze the next case of Chern-Kuiper's inequalities. We show that if $p \geq 3$ and $\nu_g = \mu - p + 2 \leq n - p - 1$ then they are also compositions in an extended sense. Namely, we use the concept of *singular isometric extensions* introduced in [28].

Theorem 5.0.2. *Let $g : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion with $p \geq 3$ and*

$$\nu_g = \mu - p + 2 \leq n - p - 1.$$

Let U be a connected component of an open dense subset of M^n where $(p - k) := \dim(\mathcal{S}(\alpha|_{TM \times \Gamma}))$ is constant. Then $k \in \{1, 2\}$ and there is an isometric immersion $\hat{g} : U \subseteq M^n \to \mathbb{R}^{n+k}$ such that $\Delta_{\hat{g}} = \Gamma$ and

i) g is locally a composition of $\hat{g}$ if $k = 2$;

ii) $\{g, \hat{g}\}$ *singularly isometrically extends for* $k = 1$.

In the isometric rigidity theory, Beez-Killing Theorem states that if $f, g : M^n \to \mathbb{R}^{n+1}$ are two non-congruent isometric immersions then $\dim(\Delta_f \cap \Delta_g) \geq n - 2$. Allendoerfer in [1] proved an extension for higher codimensions with an algebraic assumption on the second fundamental form. Theorem 1 of [12] extends these classical results using genuine rigidity. Another extension is Theorem 1 in [28] by allowing some natural singularities. When $f : M^n \to \mathbb{R}^{n+q}$ and $g : M^n \to \mathbb{R}^{n+p}$ are genuine deformation, we can compose f or g with flat submanifolds to generically obtain new genuine deformations. The notion of honest deformations was introduced in [27] to exclude these compositions. Namely, we say that f and g are *strongly honest deformations* (of each other) if they are genuine deformations in the singular sense and none of them is locally a singular composition. Using the results proved in this work and the theorems described before, we obtain the following Beez-Killing type theorem that generalizes Theorem 1 of [12] for low codimensions.

Theorem 5.0.3. *Let* $f : M^n \to \mathbb{R}^{n+q}$ *and* $g : M^n \to \mathbb{R}^{n+p}$ *be strongly honest deformations with* $(p+q) < \min\{6, n\}$. *Then*

$$\dim(\Delta_f \cap \Delta_g) \geq n - p - q. \tag{5.1}$$

5.1 Chern-Kuiper's inequalities

In this section, we characterize locally submanifolds $g : M^n \to \mathbb{R}^{n+p}$ whose relative nullity Δ_g does not coincide with the intrinsic nullity Γ. In particular, we describe all of them in codimension $p = 2$.

Let $g : M^n \to \mathbb{R}^{n+p}$ be a submanifold with non-trivial intrinsic nullity $\Gamma \neq 0$. Call α its second fundamental form and Δ_g its relative nullity. Gauss equation implies that $\Delta_g \subseteq \Gamma$ and the bilinear tensor $\beta := \alpha|_{TM \times \Gamma}$ is flat. Let Δ_β be the (left) nullity of β. By Gauss equation, we have

$$\alpha(Y, X) \in \mathcal{S}(\beta)^\perp, \quad \forall Y \in \Delta_\beta, \forall X \in TM. \tag{5.2}$$

So in particular

$$\alpha(Y, X) \in \mathcal{S}(\beta) \cap \mathcal{S}(\beta)^\perp = 0, \quad \forall Y \in \Delta_\beta \cap \Gamma, \forall X \in TM,$$

which shows that $\Delta_g = \Delta_\beta \cap \Gamma$. Then, we have the following relation

$$\nu_g + \dim(\Delta_\beta + \Gamma) = \dim(\Delta_\beta) + \mu. \tag{5.3}$$

Notice that $\Delta_\beta \subseteq TM$ is an integrable distribution. Indeed, Codazzi equation for $T_1, T_2 \in \Delta_\beta$ gives

$$\alpha([T_1, T_2], Z) = \alpha(T_1, \nabla_{T_2} Z) - \alpha(T_2, \nabla_{T_1} Z), \quad \forall Z \in \Gamma,$$

but the left hand side belongs to $\mathcal{S}(\beta)$ and the right hand side to $\mathcal{S}(\beta)^\perp$ by (5.2). So $[T_1, T_2] \in \Delta_\beta$.

Let us recall Chern-Kuiper's inequalities, and provide a quick proof.

Proposition 5.1.1 (Chern-Kuiper's inequalities [5])**.** *Let* $g : M^n \to \mathbb{R}^{n+p}$ *be a submanifold, then*

$$\nu_g \leq \mu \leq \nu_g + p. \tag{5.4}$$

Proof. As $\Delta_g \subseteq \Gamma$ then $\nu_g \leq \mu$. Take $Z \in \mathrm{Re}(\beta) \subseteq \Gamma$ a (right) regular element of β, then by Lemma 2.2.1 and (5.3) we get that

$$\nu_g + n \geq \nu_g + \dim(\Delta_\beta + \Gamma) = \dim(\Delta_\beta) + \mu = n - \dim(\mathrm{Im}(\beta^Z)) + \mu \geq n - p + \mu. \tag{5.5}$$

which proves the second inequality of (5.4). ◻

We are interested in submanifolds $g : M^n \to \mathbb{R}^{n+p}$ with $\Delta_g \neq \Gamma$. For example, if M^n is flat and g is not totally geodesic then $\nu_g \neq \mu = n$.

5.1.1 The case $\nu_g = \mu - p$

In this subsection we analyze the maximal case of Proposition 5.1.1.

The next result shows that only flat submanifolds attain equality in the second inequality of (5.4).

Proposition 5.1.2. *Let $g : M^n \to \mathbb{R}^{n+p}$ be a submanifold with*

$$\nu_g = \mu - p. \tag{5.6}$$

Then M^n is flat, in particular $\nu_g = n - p$.

Proof. In this case we must have equalities in (5.5). Hence $\Delta_\beta + \Gamma = TM$ and $\mathrm{Im}(\beta^Z) = \mathcal{S}(\beta) = T_g^\perp M$. Then (5.2) implies that $\Delta_\beta = \Delta_g \subseteq \Gamma$, and thus $\Gamma = \Delta_\beta + \Gamma = TM$. □

Remark 5.1.3. There are natural parametrizations for flat submanifolds attaining (5.6); see [16] for $p = 1$ and [27] for $p = 2$. This has been recently generalized in [32] for any $p \leq n$. Namely, if $\nu_g = n - p$ then g can be locally described in terms of a flat normal submanifold $h : L^p \to \mathbb{R}^{n+p}$ and a solution $\gamma \in \mathcal{C}^\infty(L^p)$ to a certain system of PDEs.

Chern-Kuiper's inequalities and Proposition 5.1.2 characterize the hypersurfaces with $\Delta_g \neq \Gamma$. They are flat and described by means of the Gauss parametrization. Hence, we assume from now on that $p \geq 2$.

There is a natural way to produce submanifolds $g : M^n \to \mathbb{R}^{n+p}$ with $\Delta_g \neq \Gamma$ using compositions. Consider a submanifold $\hat{g} : M^n \to \mathbb{R}^{n+q}$ with $D := \Delta_{\hat{g}} \neq 0$, $q < p$, and let $j : U \subseteq \mathbb{R}^{n+q} \to \mathbb{R}^{n+p}$ be an isometric immersion of an open subset U of $\mathbb{R}^{n+q}$ with $\hat{g}(M^n) \subseteq U$. Then $g := j \circ \hat{g}$ generically has less nullity that $\hat{g}$, so $\Delta_g \neq \Gamma$. Conversely, we will use the following strategy to prove that such a g must be a composition. We may not know who $\hat{g}$ would be, yet suppose that we have a candidate $D \supseteq \Delta_g$ to be its relative nullity, possibly $D = \Gamma$. Consider the flat bilinear form $\beta = \beta^D := \alpha|_{TM \times D}$. Naively, $\mathcal{S}(\beta)$ should be $T_j^\perp U$, and so $L := \mathcal{S}(\beta)^\perp \subseteq T_g^\perp M$ has to be $T_{\hat{g}}^\perp M$. Hence, we project α and the normal connection of g onto L. If that data satisfies Gauss, Codazzi, and Ricci equations then we have $\hat{g}$ locally by the Fundamental Theorem of submanifolds. Finally, we need to verify that g is a composition of $\hat{g}$.

In [28] this idea of projecting the data onto a subbundle $L \subseteq T_g^\perp M$ is discussed in order to prove that g is a composition. Following this plan, define the tensor $\phi = \phi_L : TM \times (TM \oplus L) \to L^\perp \subseteq T_g^\perp M$ by

$$\phi(X, v) = (\tilde{\nabla}_X v)_{L^\perp}, \tag{5.7}$$

where the subindex denotes the orthogonal projection to $L^\perp$ and $\tilde{\nabla}$ is the connection of $\mathbb{R}^{n+p}$. We will analyze the properties of ϕ to prove that g is a composition. Proposition 17 of [28] guarantees the local existence of $\hat{g}$ as long as ϕ is flat.

5.1.2 The case $\nu_g = \mu - p + 1$

Proposition 5.1.2 characterizes the first case of inequality in the Chern-Kuiper's inequalities. We prove now Theorem 5.0.1 which characterizes the second case for nowhere-flat submanifolds, and it is an extension of Proposition 7 of [13].

Proof of Theorem 5.0.1. Notice that $\mu = (n-1)$ is not possible by the skew symmetries of the curvature tensor, so $\mu \leq n-2$ since M^n is nowhere flat. We claim that $\mathcal{S}(\beta) \neq T_g^\perp M$ everywhere. Indeed, if that is not the case then (5.2) shows that $\Delta_\beta = \Delta_g$. However, by Lemma 2.2.1 we have that

$$n - p \leq \dim(\Delta_\beta) = \nu_g = \mu - p + 1 \leq n - 2 - p + 1,$$

which is a contradiction.

Take $Z \in \mathrm{Re}(\beta) \subseteq \Gamma$ a regular element of β. Lemma 2.2.1 shows that

$$\dim(\Delta_\beta) \geq n - \dim(\mathrm{Im}(\beta^Z)) \geq n - \dim \mathcal{S}(\beta) \geq n - p + 1,$$

which together with (5.3) imply that $\dim(\Delta_\beta + \Gamma) \geq n$. Hence, all these inequalities must be equalities. In particular, $\mathcal{S}(\beta)$ has rank $(p-1)$. We conclude the proof using the next lemma for $D^d = \Gamma$. □

Lemma 5.1.4. *Let $g : M^n \to \mathbb{R}^{n+p+k}$ be a submanifold with a totally geodesic distribution $D \subseteq TM$ such that $\Delta_g \subseteq D^d \subseteq \Gamma$. Let $\beta := \alpha|_{TM \times D}$, where α is the second fundamental form of g, and suppose that $\mathcal{S}(\beta)$ has rank $d - \nu_g$. Then $g = j \circ \hat{g}$ is a composition where $j : N^{n+p} \to \mathbb{R}^{n+p+k}$ is a flat submanifold and $\hat{g} : M^n \to N^{n+p+k}$ an isometric embedding with $\Delta_{\hat{g}} \supseteq D^d$. In particular, $\Delta_{\hat{g}} = \Gamma$ if $D^d = \Gamma$.*

Proof. Set $L := \mathcal{S}(\beta)^\perp \subseteq T_g^\perp M$ and consider the tensor $\phi : TM \times (TM \oplus L) \to L^\perp$ given by (5.7). First, we will use Propositions 17 and 18 of [28] to prove that g is locally such a composition by showing that ϕ is flat and $\alpha(TM, Z) = L^\perp$ for some $Z \in D$. Finally, we prove that g is a composition globally.

Let $\Delta_\beta \subseteq TM$ be the left nullity of the flat bilinear form β. Notice that for $T \in \Delta_\beta \cap D$ we have by flatness of β that

$$\alpha(T, TM) \in \mathcal{S}(\beta)^\perp.$$

Since $\alpha(T, TM) \in \mathcal{S}(\beta)$, so $\Delta_\beta \cap D = \Delta_g$. For a regular value $Z \in \mathrm{Re}(\beta) \subseteq D$ of β, Lemma 2.2.1 implies that

$$\dim(\Delta_\beta + D) = \dim(\Delta_\beta) + \dim(D) - \dim(\Delta_g) = n - \dim(\mathrm{Im}(\beta^Z)) + d - \nu_g \geq n.$$

This shows that $\alpha(Z, TM) = L^\perp$ for some $Z \in D$ and $\Delta_\beta + D = TM$. Thus $\mathcal{S}(\beta) = \mathcal{S}(\alpha|_{D\times D})$.

We show next that ϕ is flat. If $Y \in \Delta_\beta$ then by flatness of β we have $\alpha(Y, X) \in L$, and so $\phi(Y, TM) = 0$. Moreover, Codazzi equation for $\xi \in L$ and $Z_1, Z_2 \in D$ gives us that

$$\langle \phi(Y, \xi), \alpha(Z_1, Z_2)\rangle = \langle \nabla_Y^\perp \xi, \alpha(Z_1, Z_2)\rangle = -\langle \xi, (\nabla_Y^\perp \alpha)(Z_1, Z_2)\rangle = \langle \xi, \alpha(Y, \nabla_{Z_1} Z_2)\rangle = 0, \quad \forall Z_1, Z_2 \in D,$$

since $D \subseteq TM$ is totally geodesic. Then $\phi(Y, \xi) = 0$ since $\mathcal{S}(\alpha|_{\Gamma\times\Gamma}) = \mathcal{S}(\beta) = L^\perp$, and so $\phi(\Delta_\beta, TM \oplus L) = 0$. Therefore, as $TM = \Delta_\beta + D$, to prove that ϕ is flat it is enough to show that $\phi|_{D\times(D\oplus L)}$ is flat.

First, $\phi|_{D\times D} = \alpha|_{D\times D}$ is flat. If $Z_1, Z_2, Z_3 \in D$ and $\xi \in L$ then

$$\langle \phi(Z_1, \xi), \phi(Z_2, Z_3)\rangle = \langle \nabla_{Z_1}^\perp \xi, \alpha(Z_2, Z_3)\rangle = -\langle \xi, (\nabla_{Z_1}^\perp \alpha)(Z_2, Z_3)\rangle,$$

which is symmetric in Z_1 and Z_2 by Codazzi equation. Notice that the nullity of $\alpha|_{D\times D}$ is $\Delta_\beta \cap D = \Delta_g$. Thus, $\alpha|_{D\times D}$ is completely described by Theorem 2 of [36]. Namely, there are vectors $Z_1, \ldots, Z_{d-\nu_g} \in D \cap \Delta_g^\perp$ such that $\alpha(Z_i, Z_j) = 0$ for $i \neq j$ and the set $\{\rho_i := \alpha(Z_i, Z_i)\}_{i=1}^{d-\nu_g}$ is an orthonormal basis of $L^\perp$. Given $\xi \in L$ then Codazzi equation implies that

$$\langle \phi(Z_i, \xi), \rho_j\rangle = -\langle \xi, (\nabla_{Z_i}^\perp \alpha)(Z_j, Z_j)\rangle = \langle \xi, \nabla_{Z_j}^\perp(\alpha(Z_i, Z_j)) - \alpha(\nabla_{Z_j} Z_i, Z_j) - \alpha(Z_i, \nabla_{Z_j} Z_j)\rangle = 0, \quad \forall i \neq j.$$

Then $\phi(Z_i, \xi) = \lambda_i(\xi)\rho_i$ for some linear maps $\lambda_i : L \to \mathbb{R}$. We conclude that $\phi|_{D\times(D\oplus L)}$ is flat since

$$\langle \phi(Z_i, \xi_1), \phi(Z_j, \xi_2)\rangle = \delta_{ij}\lambda_i(\xi_1)\lambda_j(\xi_2), \quad \forall i, j,\ \forall \xi_1, \xi_2 \in L.$$

Therefore, Proposition 17 and 18 of [28] tell us that $g = j \circ \hat{g}$ is a composition locally. Moreover, the second fundamental form of $\hat{g}$ is the orthogonal projection of α onto L. As $\alpha(D, TM) = L^\perp$ then $D \subseteq \Delta_{\hat{g}}$. Observe that $D = \Gamma$ then $\Delta_{\hat{g}} = \Gamma$ by Gauss equation.

Finally, consider the rank p subbundle $N = \ker(\phi^Z) \cap \Delta_\beta^\perp \subseteq TM \oplus L$. Observe that $N \cap TM = 0$ and $\phi(TM, N) = 0$ by Lemma 2.2.1. Therefore, by Proposition 12 of [28] g is a global composition. □

We completely describe locally the submanifolds $g : M^n \to \mathbb{R}^{n+2}$ with $\Delta_g \neq \Gamma$.

Proposition 5.1.5. *Let $g : M^n \to \mathbb{R}^{n+2}$ be a submanifold with $\Gamma \neq \Delta_g$. Denote by $\mu = \dim(\Gamma)$ and $\nu_g = \dim(\Delta_g)$. Then, on each connected component U of an open dense subset of M^n, we have one of the following possibilities:*

i) $\mu = \nu_g + 1$ and $g|_U = j \circ \hat{g}$ is a composition where $\hat{g} : U \to V \subseteq \mathbb{R}^{n+1}$ and $j : V \to \mathbb{R}^{n+2}$ are isometric immersions with $\Gamma = \Delta_{\hat{g}}$;

ii) $\mu = \nu_g + 2$ and U is flat.

Proof of Theorem 5.1.5. By Proposition 5.1.1, Proposition 5.1.2, and Theorem 5.0.1, it remains to analyze the case $\mu = n = \nu_g + 1$. Consider the line bundle $L := \mathcal{S}(\alpha)^\perp \subseteq T_g^\perp M$. Proposition 18 of [28] for L proves that g is such a composition. Furthermore, $\hat{g}$ is totally geodesic since its second fundamental form is the orthogonal projection of α onto L. □

Remark 5.1.6. Each case of Theorem 5.1.5 is naturally parametrizable. For (i) we use the Gauss parametrization described in [16], and Corollary 18 of [27] describes the second case.

5.1.3 The case $\nu_g = \mu - p + 2$

We now characterize the next case of Chern-Kuiper's inequalities. We will show that they are also compositions in an extended sense for $\mu \leq n-3$. However, the composition may be singular in the sense of [28].

Let us recall the definition of a singular isometric extension given in [28].

Definition 5.1.7. We say that a pair of isometric immersions $g : M^n \to \mathbb{R}^{n+p}$ and $\hat{g} : M^n \to \mathbb{R}^{n+q}$ *singularly extend isometrically* when there are an embedding $j : M^n \hookrightarrow N^{n+s}$ and isometric maps $G : N^{n+s} \to \mathbb{R}^{n+p}$ and $\hat{G} : N^{n+s} \to \mathbb{R}^{n+q}$ with $g = G \circ j$ and $\hat{g} = \hat{G} \circ j$, with the set of points where G and $\hat{G}$ fail to be an immersion (that may be empty) is contained in $j(M)$.

Remark 5.1.8. If the set of singularities of G and $\hat{G}$ is empty then we recover the notions of isometric extensions and local compositions for $s = q \leq p$.

We are ready to prove Theorem 5.0.2.

Proof of Theorem 5.0.2. Denote by $\beta = \alpha|_{TM\times\Gamma}$ the associated flat bilinear form. Since β is smooth, k is constant on each connected component U of an open dense subset of M^n. As the problem is local, we assume that U is simply connected.

We prove first that $\mathcal{S}(\beta) \neq T_g^\perp M$ just as we did for Theorem 5.0.1, but using $\mu \leq n-3$ instead. Lemma 2.2.1 and (5.3) imply that

$$n + \nu_g \geq \dim(\Delta_\beta + \Gamma) + \dim(\Delta_g) = \dim(\Delta_\beta) + \dim(\Gamma) \geq n - \dim(\mathcal{S}(\beta)) + \nu_g + p - 2. \tag{5.8}$$

This proves that $k \in \{1, 2\}$ since $p \geq 3$.

If $k = 2$, then Lemma 5.1.4 for $D = \Gamma$ proves that $g = j \circ \hat{g}$ is a composition of a flat submanifold $j : N^{n+2} \to \mathbb{R}^{n+p}$ and $\hat{g} : U \to N^{n+2}$.

Assume now that $k = 1$. Then by (5.8) we get

$$\dim(\Delta_\beta + \Gamma) \geq (n-1). \tag{5.9}$$

Consider ϕ as in (5.7), where $L = \mathcal{S}(\beta)^\perp \subseteq T_g^\perp M$ is again a line bundle. Notice first that $\phi|_{TM\times\Gamma} = \beta$ and $\phi(\Delta_\beta, TM) = 0$ by (5.2), so $\phi|_{(\Gamma+\Delta_\beta)\times TM}$ is flat. Hence, $\phi|_{TM\times TM}$ is also flat by (5.9). Thus, if $\rho \in L$ is a unit vector field (unique up to sign) then its shape operator $\hat{A} := A_\rho$ satisfies Gauss equation.

We verify now that $\hat{A}$ satisfies Codazzi equation, that is

$$(D\hat{A})(X,Y) := (\nabla_X \hat{A})Y - (\nabla_Y \hat{A})X = 0. \tag{5.10}$$

Codazzi and Gauss equations for $\hat{A}$ are equivalent to ϕ being flat. Thomas's Theorem states that (5.10) is a consequence of Gauss equation if the rank of $\hat{A}$ is at least 4. As $\hat{A}$ may have rank 3, we cannot directly apply this result. However, we can adapt Thomas's proof to our situation.

Fix a point $q \in M^n$. Consider a geodesic frame $e_1, \dots, e_n$ around q, that is, they are an orthonormal and

$$(\nabla_{e_i} e_j)(q) = 0, \quad \forall i, j.$$

After a linear transformation, we can assume that the e_i's are eigenvectors of $\hat{A}_q$, namely

$$\hat{A}_q(e_i(q)) = \lambda_i e_i(q).$$

The second Bianchi identity at q gives

$$\begin{aligned} 0 &= (\nabla_{e_i} R)(e_j, e_k, e_l, e_m) + (\nabla_{e_l} R)(e_j, e_k, e_m, e_i) + (\nabla_{e_m} R)(e_j, e_k, e_i, e_l) = \sum (\nabla_{e_i} R)(e_j, e_k, e_l, e_m) \\ &= \sum \nabla_{e_i} \left[\langle \hat{A}e_j, e_m \rangle \langle \hat{A}e_k, e_l \rangle - \langle \hat{A}e_j, e_l \rangle \langle \hat{A}e_k, e_m \rangle \right] \\ &= \sum \left[\langle (\nabla_{e_i} \hat{A}) e_j, e_m \rangle \delta_{kl} \lambda_k + \delta_{jm} \lambda_j \langle (\nabla_{e_i}) \hat{A} e_k, e_l \rangle - \langle (\nabla_{e_i} \hat{A}) e_j, e_l \rangle \delta_{km} \lambda_k - \delta_{jl} \lambda_j \langle (\nabla_{e_i}) \hat{A} e_k, e_m \rangle \right], \end{aligned}$$

where the sum is over the cyclic permutations of the indices i, l, and m. After a rearrangement of factors, the last equation is equivalent to

$$0 = \sum \left[\langle (D\hat{A})(e_i, e_m), e_j \rangle \delta_{kl} \lambda_k + \langle (D\hat{A})(e_i, e_l), e_k \rangle \delta_{jm} \lambda_j \right]. \tag{5.11}$$

We will prove (5.10) first for $X = e_i$ with $\lambda_i = 0$ and $Y = e_m$. We can assume that $m \neq i$. Take in (5.11) three distinct indices j, m and $k = l$ with $\lambda_k \neq 0$ to get
$$0 = \langle (D\hat{A})(e_i, e_m), e_j \rangle \lambda_k.$$
Hence, we can write $(D\hat{A})(e_i, e_m) = a_{im} e_m + a_{ik} e_k$. Nevertheless, as $\operatorname{rank}(\hat{A}) \geq 3$ we can vary e_k, so we necessarily have that $(D\hat{A})(e_i, e_m) = a_{im} e_m$. If $\lambda_m = 0$ then a symmetry argument gives
$$(D\hat{A})(e_i, e_m) = -(D\hat{A})(e_m, e_i) = -a_{mi} e_i.$$
Thus $(D\hat{A})(e_i, e_m) = 0$ since the vectors e_i and e_m are linearly independent. If $\lambda_m \neq 0$, then (5.11) for $k \neq m$ with $\lambda_k \neq 0$ and $j = m$ implies that
$$0 = a_{im}\lambda_k + a_{ik}\lambda_m.$$
However, as $\operatorname{rank}(\hat{A}) \geq 3$, there is an $r \neq m, k$ with $\lambda_r \neq 0$. Then, by symmetry we have
$$0 = a_{ik}\lambda_r + a_{ir}\lambda_k,$$
$$0 = a_{ir}\lambda_m + a_{im}\lambda_r.$$
The last three equations imply that $a_{im} = 0$, and then $(D\hat{A})(e_i, e_m) = 0$. This proves that $(D\hat{A})(\Gamma, TM) = 0$.

As in Theorem 5.0.1 we have $\ker(\hat{A}) = \Gamma$. For $Y \in \Delta_\beta$, $Z \in \Gamma$ and $X \in TM$, Codazzi equation for α and $(D\hat{A})(\Gamma, TM) = 0$ gives
$$\begin{aligned}\langle \nabla^\perp_Y \rho, \alpha(Z, X)\rangle &= Y\langle \hat{A}Z, X\rangle - \langle \rho, \nabla^\perp_Y(\alpha(Z, X))\rangle = \langle (\nabla_Y \hat{A})Z, X\rangle - \langle \rho, (\nabla^\perp_Y \alpha)(Z, X)\rangle \\ &= \langle (\nabla_Z \hat{A})Y, X\rangle - \langle \rho, (\nabla^\perp_Z \alpha)(Y, X)\rangle = Z\langle \hat{A}Y, X\rangle - \langle \rho, \nabla^\perp_Z(\alpha(X, Y))\rangle \\ &= \langle \nabla^\perp_Z \rho, \alpha(X, Y)\rangle.\end{aligned}$$
Since $\nabla^\perp_Z \rho \in L^\perp$ and $\alpha(Y, X) \in L$ by (5.2), the last expression vanishes for any $X \in TM$. Hence, $\nabla^\perp_Y \rho \in L^\perp \cap L = 0$ for any $Y \in \Delta_\beta$.

We prove now that $(D\hat{A})(\Delta_\beta, TM) = 0$. For $Y \in \Delta_\beta$, $X \in TM$, Codazzi equation gives
$$\begin{aligned}0 &= (\nabla_Y A)(X, \rho) - (\nabla_X A)(Y, \rho) = \nabla_Y(\hat{A}X) - \hat{A}(\nabla_Y X) - A_{\nabla^\perp_Y \rho} X - \nabla_X(\hat{A}Y) + \hat{A}(\nabla_X Y) + A_{\nabla^\perp_X \rho} Y \\ &= (D\hat{A})(Y, X) + A_{\nabla^\perp_X \rho} Y.\end{aligned}$$
On the other hand, for any $W \in TM$ we have by (5.2) that
$$\langle A_{\nabla^\perp_X \rho} Y, W\rangle = \langle \nabla^\perp_X \rho, \alpha(Y, W)\rangle = 0, \quad \forall W \in TM.$$
Hence $(D\hat{A})(Y, X) = 0$. This proves that $(D\hat{A})(\Delta_\beta, TM) = 0$.

Thus, as $(D\hat{A})(\Gamma, TM) = (D\hat{A})(\Delta_\beta, TM) = 0$, we have $(D\hat{A})(\Delta_\beta + \Gamma, TM) = 0$. We conclude from (5.9) that $\hat{A}$ satisfies Codazzi equation.

As $\hat{A}$ satisfies Gauss and Codazzi equations, by the Fundamental Theorem of submanifolds, there exists an isometric immersion of $\hat{g} : U \to \mathbb{R}^{n+1}$ whose shape operator is $\hat{A}$. Since $\phi(TM, \Delta_\beta) = 0$, Propositions 12 and 13 of [28] for $\Lambda = \Delta_\beta$ says that g is a (possible singular) composition of $\hat{g}$ unless g and $\hat{g}$ are mutually Δ_β-ruled. But in the latter case, $\langle \hat{A}\Delta_\beta, \Delta_\beta\rangle = 0$, and so $\langle \hat{A}(\Gamma + \Delta_\beta), (\Gamma + \Delta_\beta)\rangle = 0$. However, this and (5.9) would imply that $\operatorname{rank}(\hat{A}) \leq 2$, which is a contradiction since $\nu_{\hat{g}} = \mu \leq n - 3$. $\square$

5.2 A Beez-Killing type theorem

In this section we apply the results of this chapter to obtain a Beez-Killing type theorem.

Following [28], we say that two isometric immersions $f : M^n \to \mathbb{R}^{n+q}$ and $g : M^n \to \mathbb{R}^{n+p}$ are *strongly honest deformations* (of each other) if they are strongly genuine deformations and none of them is a singular composition.

Proof of Theorem 5.0.3. We may assume that $q \leq p$. Along the subset of flat points of M^n (5.1) follows by Lemma 2.2.2 since

$$\dim(\Delta_f \cap \Delta_g) = \dim(\Delta_f) + \dim(\Delta_g) - \dim(\Delta_f + \Delta_g) \geq n - p + n - q - n = n - p - q.$$

Suppose then that M^n is nowhere flat. Theorem 1 of [12] says that either (5.1) holds, or f and g are simultaneously $D = D^d$-ruled for $d \geq (n - p - q + 3)$ and the normal subbundle

$$L_D^f = \operatorname{span}\{\alpha^f(Z, X) : Z \in D, X \in TM\} \subseteq T_f^{\perp} M,$$

has rank 1, the analogous bundle $L_D^g \subseteq T_g^{\perp} M$ is naturally isometric to L_D^f. This is also valid for singular rigidity, see Remark 16 of [28]. Hence, we only need to analyze the second case for $(p + q) \geq 4$.

Suppose first that $q = 1$. Lemma 2.3.2 shows that

$$n - \mu = n - \nu_f \leq 2p - 4.$$

If $p = 3$ then the rank of M^n is 2, and by the results discussed in this section we get

$$\dim(\Delta_g) \geq (n - 3) > (n - 4),$$

which proves (5.1). If $p = 4$, then the rank of M^n is at most 4. We easily prove (5.1) by the same argument as for $p = 3$.

For $p = q = 2$ take $X \in TM$ being orthogonal to the ruling $D = D^{n-1} \subseteq TM$, and consider

$$N_f = \ker(Y \to \alpha^f(X, Y)) \cap D.$$

Notice that $N_f \subseteq \Delta_f$ and $\dim(N_f) \geq n - 2$ since $\dim(L_D) = 1$. Then $\Delta_f = N_f = \Gamma$ since M^n is nowhere flat and $\dim(\Gamma) = n - 2$. Similarly, $\Delta_g = \Gamma$.

Finally, assume that $p = 3$ and $q = 2$. Consider X_1, X_2 an orthogonal basis of D^{n-2} (the case $D = D^{n-1}$ is similar to $p = q = 2$). Define

$$K_i := \ker(Y \to \alpha^f(Y, X_i)).$$

Then $D \cap K_1 \cap K_2 \subseteq \Delta_f$. As L_D has rank 1, then $\dim(D \cap K_1) \geq n - 3$. Thus

$$n - 3 + n - 2 \leq \dim(D \cap K_1) + \dim(K_2) = \dim(\Delta_f) + \dim((D \cap K_1) + K_2) \leq \nu_f + n.$$

Therefore $n - 5 \leq \nu_f = \mu$. We conclude this case in a similar way as before. $\square$

The following Corollary is a consequence of the last result and the discussion before.

Corollary 5.2.1. *Under the hypothesis of Theorem 5.0.3, if $\mu \leq (n - \max\{p, q\})$ then*

$$\Gamma = \Delta_f = \Delta_g. \tag{5.12}$$

Proof. By Theorem 5.0.1 and Theorem 5.0.2, we need to show (5.12) only for $(q, p) = (1, 4)$ with $\mu = n - 4$. Remark 3.1.2 shows that either $\Delta_g = \Gamma = \Delta_f$, or $\mathcal{S}(\beta)$ is non-degenerate. In the second case, if $\Delta_g \neq \Gamma$, then by Lemma 2.2.2 we get

$$n - 5 = \dim(\Delta_g) = \dim(\Delta_\beta) \geq n - \dim \mathcal{S}(\beta),$$

and this is a contradiction since Lemma B.0.1 for $\gamma = \alpha^f$ proves that g is a composition. Hence, $\Gamma = \Delta_g = \Delta_f$. $\square$

Remark 5.2.2. The discussion in this section suggests that if $g : M^n \to \mathbb{R}^{n+p}$ is a submanifold such that $\nu_g < \mu \leq (n - p)$, then g is not a honest immersion in some sense. It also indicates that the hardest case to deal with is when $\nu_g + 1 = \mu$.

APPENDIX A

Appendix of Chapter 3

In this section, we prove some minor technical results used in Chapter 3.

A.1 Description of an admissible φ and its index

Here we characterize the property of a tuple $\varphi = (\varphi_i)_{i=0}^p$ to be admissible (with respect to a basis $\{e_i\}_{i=0}^p$ of $\mathbb{W}_{\mathbb{C}}$ and a conjugation of indices $\overline{e_i} = e_{\bar{i}}$); see Definition 3.1.10. This description relates φ with a non-degenerate inner product and the index of φ coincides with the index of such product. We assume that the first $2s$ coordinates are complex conjugate and the remaining are real.

Consider a tuple $\varphi = (\varphi_i)_{i=0}^p$ such that $\overline{\varphi_i} = \varphi_{\bar{i}} \neq 0$ for all i. Let $D_\varphi : \mathbb{W}_{\mathbb{C}} \to \mathbb{W}_{\mathbb{C}}$ the linear map defined by $D_\varphi(e_i) = \sum_j d_{ij} e_j$, where $d_{ij} = 1 + \frac{\delta_{ij}}{\varphi_i}$. Since $\overline{D_\varphi(e_i)} = D_\varphi(e_{\bar{i}})$, this linear map can be considered as a real one $D_\varphi : \mathbb{W} \to \mathbb{W}$. A simple induction process proves the following.

Lemma A.1.1. *If $\varphi = (\varphi_i)_{i=0}^p$ is such that $\varphi_i \neq 0$ for all i, then*

$$\det(D_\varphi) = \frac{1 + \sum_i \varphi_i}{\Pi_i \varphi_i}.$$

If $\det(D_\varphi) \neq 0$, then D_φ^{-1} is given by

$$(D_\varphi^{-1})_{ij} = \delta_{ij}\varphi_i - \frac{\varphi_i^2}{1 + \sum_k \varphi_k}.$$

When φ is admissible the above lemma implies that D_φ has a kernel of dimension exactly 1. Indeed, in this case the determinant of the minor of D_φ obtained by deleting the i^{th} row and column is $\frac{-\varphi_i}{\Pi_{j\neq i}\varphi_j} \neq 0$. Moreover, we can verify that

$$\ker(D_\varphi) = \operatorname{span}\left\{ \sum_j \varphi_j e_j \right\}. \tag{A.1}$$

Therefore, when φ is admissible, D_φ induces a non-degenerate inner product on the p-dimmensional real vector space $\mathbb{W}/\ker(D_\varphi)$ by the formula

$$\langle [e_i], [e_j] \rangle = d_{ij} = 1 + \frac{\delta_{ij}}{\varphi_i} \quad \forall i, j \in I, \tag{A.2}$$

where $[e] := e + \ker(D_\varphi)$.

Proposition A.1.2. *If φ is admissible then the index of φ is precisely the index of the non-degenerate inner product given in (A.2).*

Proof. Denote by μ the index of the product given by (A.2). Consider on $\mathbb{W}_{\mathbb{C}}$ the bilinear product $\langle e_i, e_j\rangle = \frac{\delta_{ij}}{\varphi_i}$. This defines an inner product on $\mathbb{W} = \mathrm{Re}_C(\mathbb{W}_{\mathbb{C}})$, where C denotes the conjugation given by the conjugation of indices. We identify the signature of this product in two ways. Set

$$\xi_{2j} = \frac{1}{\sqrt{2}}\Big(\omega_j e_{2j} + \overline{(\omega_j e_{2j})}\Big) \quad \text{and} \quad \xi_{2j+1} = \frac{1}{\sqrt{2}}\Big(i\omega_j e_{2j} + \overline{(i\omega_j e_{2j})}\Big) \quad \text{for } 0 \le j < s,$$

where ω_j is any of the two complex roots of φ_{2j}. For $j \geq 2s$ define

$$\xi_j = \omega_j e_j,$$

where ω_j is the positive root of $|\varphi_j|$. Then $\{\xi_j\}_{j=0}^{p}$ is an orthonormal basis of $\mathbb{W}$ of index $p+1-(s+P)$.

Setting $\xi = \sum \varphi_j e_j$ and $v_j = e_j + \xi$, then $\langle \xi, \xi\rangle = -1$, $\langle v_j, v_j\rangle = 1 + \frac{\delta_{ij}}{\varphi_i} = d_{ij}$ and $\langle \xi, v_j\rangle = 0$. This gives us the orthogonal decomposition $\mathbb{W} = \mathrm{Re}(\mathrm{span}\{v_j\}) \oplus \mathrm{span}\{\xi\}$, and then the product has index $\mu + 1$. Thus $\mu = p - (s+P)$. ☐

A.2 The shared dimension of two curves

In this subsection, we extend the concept of shared dimension of two curves, which was introduced in [27] for the Euclidean ambient space, to the semi-Euclidean case.

Given two curves $\alpha_i : I_i \subseteq \mathbb{R} \to \mathbb{R}^N_\mu$ $(i = 0, 1)$ in a semi-Euclidean ambient space, we define the index $\overline{I}(\alpha_1, \alpha_2)$ as the minimum integer k such that $\langle \alpha_1'(u), \alpha'(v)\rangle$ can be written as a sum $\sum_{j=1}^{k} a_j(u) b_j(v)$ for some smooth functions a_j, b_j, $1 \le j \le k$. Define

$$I(\alpha_1, \alpha_2)(u, v) = \lim_{\varepsilon \to 0} \overline{I}(\alpha_1|_{(u-\varepsilon, u+\varepsilon)}, \alpha_2|_{(v-\varepsilon, v+\varepsilon)}),$$

which is semicontinuous and constant along connected components of an open dense subset of the parameters (u, v). Following [27], we call this integer the *shared dimension* between α_1 and α_2. For Euclidean ambient spaces, this agrees (locally) with the dimension of $\mathrm{span}(\alpha_1) \cap \mathrm{span}(\alpha_2)$, where $\mathrm{span}(\alpha_i)$ is the smallest subspace which contains the image of the curve α_i. Yet, this is not true for semi-Euclidean ambient spaces. If $\mathrm{span}(\alpha_1)$, $\mathrm{span}(\alpha_2)$ and $\mathrm{span}(\alpha_1) \cap \mathrm{span}(\alpha_2)$ are non-degenerate subspaces of $\mathbb{R}^N_\nu$, then clearly

$$\dim\Big(\mathrm{span}(\alpha_1) \cap \mathrm{span}(\alpha_2)\Big) \geq I(\alpha_1, \alpha_2).$$

The following lemma allows us to decompose the ambient space in relation to the shared dimension. The proof is similar to the one of Lemma 10 in [27].

Lemma A.2.1. *Let α_1, α_2 be curves in $\mathbb{R}^N_\nu$ such that $\big(span(\alpha_i)\big)^\perp \subseteq \mathbb{R}^N_\nu$ is a definite subspace for $i = 1, 2$ and*

$$\mathbb{U} := span(\alpha_1) + span(\alpha_2) \subseteq \mathbb{R}^N_\nu,$$

is non-degenerate. Then there exists an orthogonal decomposition $\mathbb{R}^N_\nu = \mathbb{V}_1 \oplus \mathbb{V}^l \oplus \mathbb{V}_2$ such that $l \le I(\alpha_1, \alpha_2)$, and $span(\alpha_i) \subseteq \mathbb{V}_i \oplus \mathbb{V}^l$, $i = 1, 2$. In particular, $\dim(span(\alpha_1) \cap span(\alpha_2)) \le I(\alpha_1, \alpha_2)$.

Proof. Clearly, we can assume that $\mathbb{U} = \mathbb{R}^N_\nu$. Write $\langle \alpha_1'(u), \alpha_2'(v)\rangle = \sum_{i=1}^{k} a_i(u) b_i(v)$, and set

$$\hat{\alpha}_1(u) = \Big(\alpha_1(u), -\int_0^u a_1(s)ds, \ldots, -\int_0^u a_k(s)ds\Big), \quad \text{and} \quad \hat{\alpha}_2(v) = \Big(\alpha_2(v), \int_0^v b_1(s)ds, \ldots, \int_0^v b_k(s)ds\Big),$$

as orthogonal curves in $\mathbb{R}^{N+k}_\nu = \mathbb{R}^N_\nu \oplus \mathbb{R}^k_0$. Consider $\mathcal{E} = \mathrm{span}(\hat{\alpha}_1) \cap \mathrm{span}(\hat{\alpha}_2) \subseteq \mathbb{R}^{N+k}_\nu$ which is a null subspace. Then using a pseudo-orthogonal basis we can express $\mathbb{R}^{N+k}_\nu = \hat{\mathbb{V}}_1^{n_1} + \hat{\mathbb{V}}_2^{n_2}$, with $\hat{\mathbb{V}}_1^{n_1}$, $\hat{\mathbb{V}}_2^{n_2}$ orthogonal and $\hat{\mathbb{V}}_1^{n_1} \cap \hat{\mathbb{V}}_1^{n_1} = \mathcal{E}$. Define for $i = 1, 2$ the subspaces $\mathbb{V}_i = \hat{\mathbb{V}}_i \cap (\mathbb{R}^N_\nu \times 0) \subseteq \mathbb{R}^N_\nu$. Notice that $\mathbb{V}_i \subseteq \mathrm{span}(\alpha_{i+1})^\perp$ (index modulo 2). Hence $\mathbb{V}_1$ and $\mathbb{V}_2$ are orthogonal definite subspaces. Define then $\mathbb{V}^l := (\mathbb{V}_1 \oplus \mathbb{V}_2)^\perp$. Thus, $\mathrm{span}(\alpha_i) \subseteq \mathbb{V}_i \oplus \mathbb{V}^l$ and

$$l = \dim(\mathbb{V}^l) = N - \dim(\mathbb{V}_1) - \dim(\mathbb{V}_2) \le N - (n_1 - k) - (n_2 - k) = N + 2k - (N + k - \dim(\mathcal{E})) \le k.$$

☐

A.3 Diagonalizable Codazzi tensors

The main goal of this subsection will be to prove Proposition A.3.4 which gives conditions for diagonalizing directions of a Codazzi tensor to descend as coordinate vectors to the leaf space of the nullity distribution of such tensor. This result is presented in a general context since it has independent interest. It was present in several works in the literature when the leaf space has dimension 2, see for example [27], [14], [7] and [15].

Definition A.3.1. Consider a real vector bundle $F \to M^n$ with a connection $\nabla = \nabla^F$. We say that a bilinear symmetric tensor $\beta : TM \times TM \to F$ *satisfies Codazzi equation* if

$$(\nabla_X \beta)(Y, Z) = (\nabla_Y \beta)(X, Z), \quad \forall X, Y, Z \in TM. \tag{A.3}$$

We denote $\Delta = \Delta_\beta$ the nullity of β.

Remark A.3.2. Codazzi equation implies that the nullity is in fact a totally geodesic distribution on an open dense subset of M^n, along any open subset where Δ has constant dimension. We assume that this is the case and that $L^l = M^n/\Delta$ is smooth.

We define the splitting tensor of Δ in the same way as in Definition 3.1.7, but for Δ instead of Γ. Here we also denote X^h for the projection of $X \in TM$ on $\Delta^\perp$. In this context, equation (3.3) is also valid for the splitting tensor of Δ since β satisfies Codazzi equation.

Definition A.3.3. Suppose that $\beta : TM \times TM \to F$ is a bilinear tensor with $l = \dim(\Delta^\perp)$ and that it is diagonalizable by the smooth frame $X_1, X_2, \ldots, X_l \in \Gamma(\Delta^\perp_{\mathbb{C}})$ with $\overline{X_{2j-1}} = X_{2j}$ for $j \leq s$ and $\overline{X_j} = X_j$ for $j > 2s$ for some s. We say that it diagonalizes *strongly* if for every non-empty subset $S \subseteq \{1, \ldots, l\}$ with $\#S \leq 3$, the set $\{\beta(X_i, X_i)\}_{i \in S}$ is pointwise $\mathbb{C}$-linearly independent.

As before, we will denote by $\bar{j}$ the index associated to j such that $\overline{X_j} = X_{\bar{j}}$.

Proposition A.3.4. *Let β be a bilinear tensor satisfying Codazzi equation and*

$$(R(X,T)S)^h = 0 \quad \forall T, S \in \Delta,\ X \in TM. \tag{A.4}$$

Assume that β strongly diagonalizes by $X_1, \ldots, X_l$. Then there exist $f_i : M^n \to \mathbb{C}$ satisfying $\overline{f_j} = f_{\bar{j}}$ for all j and (local) coordinates $(z_1, \ldots, z_s, w_{2s+1}, \ldots, w_l) \in \mathbb{C}^s \times \mathbb{R}^r$ for L^l such that, for $Z_i = f_i X_i$, we have $\partial_{u_i} \circ \pi = \pi_* \circ Z_i$, where $(u_0, \ldots, u_l) = (z_1, \overline{z_1}, \ldots, z_s, \overline{z_s}, w_{2s+1}, \ldots, w_l)$ and $\pi : M^n \to L^l$ is the canonical projection.

Proof. Given any vector Y, we write Y^i for the component of Y^h with respect to X_i, that is, $Y^h = \sum_i Y^i X_i$. By (3.3) for $X = X_i$ and $Y = X_j$ with $i \neq j$, we have that

$$(C_T X_i)^j \beta(X_j, X_j) = (C_T X_j)^i \beta(X_i, X_i).$$

Since β diagonalizes strongly, the last equation implies that there exist 1-forms $\lambda_i : \Delta_{\mathbb{C}} \to \mathbb{C}$ such that $C_T X_i = \lambda_i(T) X_i$. Using Codazzi equation (A.3) for $X = T$, $Y = X_i$ and $Z = X_j$, we get

$$(\nabla_T X_i)^j \beta(X_j, X_j) + (\nabla_T X_j)^i \beta(X_i, X_i) = 0.$$

Hence, $\nabla_T X_i = a_i(T) X_i$ for some 1-forms a_i.

First, we claim that we can assume that $a_i = 0$ to simplify computations. Equation (A.4) can be expressed in terms of the splitting tensor as

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S}.$$

Thus

$$0 = (\nabla_T C_S(X_i) - C_S C_T(X_i) - C_{\nabla_T S}(X_i)) - (\nabla_S C_T(X_i) - C_T C_S(X_i) - C_{\nabla_S T}(X_i)) = d\lambda_i(T, S) X_i. \tag{A.5}$$

Using Jacobi identity for T, S and X_i, and analyzing the vertical component, we get that

$$da_i(T, S) + d\lambda_i(T, S) = 0.$$

We get from (A.5) that $da_i(T, S) = 0$. We integrate the 1-forms a_i along the nullity leaves giving arbitrary values along a transversal submanifold. This defines functions r_i such that $dr_i(T) = a_i(T)$. Notice that we can do this in a way that $\overline{r_i} = r_{\bar{i}}$. By replacing X_i with $e^{-r_i} X_i$, we can assume that $\nabla_T X_i = 0$ for all $T \in \Delta$, as we claimed.

Codazzi equation (A.3) for $X = X_i$, $Y = X_j$ and $Z = X_k$ with $i \neq j \neq k \neq i$ gives

$$-([X_i, X_j])^k \beta(X_k, X_k) = -(\nabla_{X_i} X_k)^j \beta(X_j, X_j) - (\nabla_{X_j} X_k)^i \beta(X_i, X_i).$$

As β diagonalizes strongly, we get that $(\nabla_{X_i} X_j)^k = 0$ for all distinct indices. Then, there exists $a_i^j, b_j^i, r_i^j : M^n \to \mathbb{C}$, $1 \leq i \neq j \leq l$, such that

$$\nabla_{X_i} X_j + a_i^j X_i - b_j^i X_j \in \Delta_{\mathbb{C}} \quad \text{and} \quad [X_i, X_j] + r_i^j X_i - r_j^i X_j \in \Delta_{\mathbb{C}}. \tag{A.6}$$

Clearly, $r_i^j = a_i^j + b_i^j$.

As in Proposition 10 of [15], to project Z_i to L^l we need that $[Z_i, T] \in \Delta_{\mathbb{C}}$ for all $T \in \Delta_{\mathbb{C}}$, and to be a local coordinate system we also need that $[Z_i, Z_j] \in \Delta_{\mathbb{C}}$ for any i, j. Write $f_i = e^{g_i}$. The first condition is equivalent to $T(g_i) = -\lambda_i(T)$, while the second one is equivalent to $X_i(g_j) = -r_j^i$ and $X_j(g_i) = -r_i^j$. To find such functions, consider the $\mathbb{C}$-linear 1-form $\hat{\sigma}_i : \mathrm{span}_{\mathbb{C}}\{\Delta, X_j\}_{j \neq i} \to \mathbb{C}$, given by

$$\hat{\sigma}_i(T) = -\lambda_i(T), \quad \hat{\sigma}_i(X_j) = -r_i^j.$$

Let's prove that σ_i is closed. We have already proved that $d\hat{\sigma}_i|_{\Delta \times \Delta} = 0$ in (A.5). Now, we need to prove that

$$d\hat{\sigma}_i(T, X_j) = -T(r_i^j) + X_j(\lambda_i(T)) - \lambda_i(\nabla^v_{X_j} T) + \lambda_j(T) r_i^j = 0, \quad \forall j \neq i. \tag{A.7}$$

By Jacobi identity for $i \neq j$,

$$\begin{aligned}
0 &= [T, [X_i, X_j]]^h + [X_j, [T, X_i]]^h - [X_i, [T, X_j]]^h \\
&= (\nabla_T [X_i, X_j]^h + C_T([X_i, X_j]^h) + [X_i, -\nabla^v_{X_j} T + \lambda_j(T) X_j]^h - [X_j, -\nabla^v_{X_i} T + \lambda_i(T) X_j]^h \\
&= -T(r_i^j) X_i + T(r_j^i) X_j - \lambda_i(T) r_i^j X_i + \lambda_j(T) r_j^i X_j + \lambda_i(\nabla^v_{X_j} T) X_i + X_i(\lambda_j(T)) X_j \\
&\quad + \lambda_j(T)(-r_i^j X_i - r_j^i X_j) - \lambda_j(\nabla^v_{X_i} T) X_j + X_j(\lambda_i(T)) X_i - \lambda_i(T)(-r_j^i X_j - r_i^j X_i) \\
&= d\hat{\sigma}_i(T, X_j) X_i - d\hat{\sigma}_j(T, X_i) X_j,
\end{aligned}$$

which proves (A.7). Also by Jacobi identity, we have for three distinct indices that

$$\begin{aligned}
0 &= \sum [X_i, [X_j, X_k]]^h = \sum \Big(-\lambda_i([X_j, X_k]^v) X_i + \nabla^h_{X_i}(-r_j^k X_j + r_k^j X_k) - \nabla^h_{-r_j^k X_j + r_k^j X_k} X_i \Big) \\
&= \sum \Big(-\lambda_i([X_j, X_k]^v) X_i - X_i(r_j^k) X_j - r_j^k \nabla^h_{X_i} X_j + X_i(r_k^j) X_k + r_k^j \nabla^h_{X_i} X_k + r_j^k \nabla^h_{X_j} X_i - r_k^j \nabla^h_{X_k} X_i \Big) \\
&= \sum \Big(-\lambda_i([X_j, X_k]^v) X_i - X_i(r_j^k) X_j - r_j^k(-a_i^j X_i + b_j^i X_j) + X_i(r_k^j) X_k + r_k^j(-a_i^k X_i + b_k^i X_k) \\
&\quad + r_j^k(-a_j^i X_j + b_i^j X_i) - r_k^j(-a_k^i X_k + b_i^k X_i) \Big) \\
&= \sum \Big(-\lambda_i([X_j, X_k]^v) X_i + r_j^k r_i^j X_i - r_k^j r_i^k X_i - X_i(r_j^k) X_j - r_j^k r_j^i X_j + X_i(r_k^j) X_k + r_k^j r_k^i X_k \Big) \\
&= \sum \Big(-\lambda_i([X_j, X_k]^v) X_i + r_j^k r_i^j X_i - r_k^j r_i^k X_i - X_k(r_i^j) X_i - r_i^j r_i^k X_i + X_j(r_i^k) X_i + r_i^k r_i^j X_i \Big) \\
&= \sum \Big(\hat{\sigma}_i([X_j, X_k]) + X_k(\hat{\sigma}_i(X_j)) - X_j(\hat{\sigma}_i(X_k)) \Big) X_i = -\sum d\hat{\sigma}_i(X_j, X_k) X_i.
\end{aligned}$$

This shows that $d\hat{\sigma}_i(X_j, X_k) = 0$ and proves the exactness of $\hat{\sigma}_i$ in a simply connected neighborhood.

For $1 \leq i \leq l$, consider

$$\hat{\Omega}_i = \mathrm{span}_{\mathbb{C}}\{\Delta, X_j\}_{j \neq i, \bar{i}}.$$

As the X_i's are the eigenvectors of the splitting tensors, by (A.6) $\hat{\Omega}_i$ is involutive, namely, it is closed with respect to the Lie bracket extended by $\mathbb{C}-$bilinearity. Since $\hat{\Omega}_i$ is closed with respect to conjugation of indices, this implies that $\Omega_i = \mathrm{Re}(\hat{\Omega}_i) \subseteq TM$ is integrable in the Frobenius sense. Consider $\sigma_i = \hat{\sigma}_i|_{\Omega_i}$ which is a closed 1-form since $\hat{\sigma}_i$ is closed. Therefore, we can integrate σ_i on M^n by defining arbitrary values along a transversal submanifold to Ω_i. Thus, there exists g_i's such that $dg_i|_{\Omega_i} = \sigma_i$. This can be done in a way that $\overline{g_i} = g_{\bar{i}}$.

Consider then $Y_i = e^{g_i} X_i$. Those vectors satisfy that $[Y_i, T] \in \Delta_{\mathbb{C}}$ and $[Y_i, Y_j] \in \Delta_{\mathbb{C}}$ for any $T \in \Delta$ and $i \neq \bar{j}$. Using Proposition 10 of [15], let $A_i \in (TL)_{\mathbb{C}}$ be the local frame such that $A_i \circ \pi = \pi_* Y_i$. They satisfy that $[A_i, A_j] = 0$ for any $i \neq \bar{j}$. If there are no complex indices, we are done. Thus, suppose that this is not the case.

Write $A_{2j} = U_j + iV_j$ for $j \leq s$. By (A.6), there exist $a_j, b_j : L^l \to \mathbb{R}$ such that

$$[A_{2j-1}, A_{2j}] + (a_j + ib_j) A_{2j-1} - (a_j - ib_j) A_{2j} = 0,$$

which in terms of the U_j's and V_j's can be expressed as

$$[U_j, V_j] + b_j U_j - a_j V_j = 0.$$

For $k \neq 2j, 2j-1$, from Jacobi identity, using the last condition we get that

$$A_k(a_j) = A_k(b_j) = 0. \tag{A.8}$$

Thus, there are (local) functions $\hat{a}_j, \hat{b}_j : L^l \to \mathbb{R}$ such that the frame $\{e^{\hat{a}_1}U_1, e^{\hat{b}_1}V_1, \ldots, e^{\hat{a}_s}U_s, e^{\hat{b}_s}V_s, A_{2s+1}, \ldots, A_l\}$ is commutative. Then there is a local chart $(x_1, y_1, \ldots, x_s, y_s, w_{2s+1}, \ldots, w_l)$ such that the canonical vectors are this frame (locally) and $\hat{a}_j = \hat{a}_j(x_j, y_j)$ $\hat{b}_j = \hat{b}_j(x_j, y_j)$ by (A.8).

To conclude, consider on the plane (x_j, y_j) the metric $g(\partial_{x_j}, \partial_{x_j}) = e^{2\hat{a}_j}$, $g(\partial_{y_j}, \partial_{y_j}) = e^{2\hat{b}_j}$ and $g(\partial_{x_j}, \partial_{y_j}) = 0$. Since all the surfaces possess isothermal charts, there are functions $p_j = p_j(x_j, y_j)$ and $q_j = q_j(x_j, y_j)$ with $(p_j, q_j) \neq (0,0)$ such that $[p_iU_i - q_iV_i, p_iV_i + q_iV_i] = 0$. Thus, there is a local chart $(\hat{x}_1, \hat{y}_1, \ldots, \hat{x}_s, \hat{y}_s, w_{2s+1}, \ldots, w_l)$ such that $\partial_{\hat{x}_i} = p_iU_i - q_iV_i$, $\partial_{\hat{y}_i} = q_iU_i + p_iV_i$. This chart is the chart we are looking for. Define $z_j = \hat{x}_j + i\hat{y}_j$, $f_{2j} = e^{g_{2j}}(p_j + iq_j)$, $f_{2j-1} = \overline{f_{2j}}$ for $j \leq s$ and $f_k = e^{g_k}$ for $k > 2s$. □

APPENDIX B

Appendix of Chapter 5

In this appendix, we prove a technical lemma which extends Proposition 7 of [13]. We present it in a general context. This result is used in Chapter 5, but mainly in a future paper.

Lemma B.0.1. *Consider $g : M^n \to \mathbb{R}^{n+p+\ell}$ a submanifold with $\Gamma \neq \Delta_g$. Suppose that $\gamma : TM \times TM \to (E, \langle\cdot,\cdot\rangle)$ is a bilinear form which satisfies Gauss equation for some semi-Riemannian vector bundle E. Let $\beta = (\alpha^g, \gamma)$ be the associated flat bilinear form, and assume that $\Delta_\beta = \Delta_g \subset \Delta_\gamma$ and that $\Delta_\gamma \subseteq TM$ is a totally geodesic distribution. Suppose in addition that $\mathcal{S}(\beta) \subseteq T_g^\perp M \oplus E$ is a non-degenerate subspace whose index is at most 1, with*

$$\dim(\Delta_g^\perp) = \dim(\mathcal{S}(\beta)) = \dim(\Delta_\gamma^\perp) + \ell.$$

Then, locally, $g = h \circ \hat{g}$ is a composition, where $\hat{g} : M^n \to \mathbb{R}^{n+p}$ and $h : U \subseteq \mathbb{R}^{n+p} \to \mathbb{R}^{n+p+\ell}$ are isometric immersions with $\Delta_{\hat{g}} = \Delta_\gamma$.

Proof. To simplify notations, set $\Delta = \Delta_g$, $\alpha = \alpha^g$, $W := \mathcal{S}(\beta)$, and for $Y \in \Delta^\perp$ call β^Y the linear map $\beta^Y = \beta(Y, \cdot) : \Delta^\perp \to W$. We are going to use Moore's techniques in the proof of Theorem 2 in [36]. Corollaries 1 and 2 therein show that there is $X \in \Delta^\perp$ such that β^X is an isomorphism. Consider the automorphism

$$B(Y) = \beta^Y \circ (\beta^X)^{-1} : W \to W, \quad Y \in \Delta^\perp.$$

Moore proved that those maps are commutative and symmetric with respect to the inner product in W. Furthermore, there exists a decomposition $W_{\mathbb{C}} = \bigoplus_{i \in I} W_i$ such that

$$B(Y)|_{W_i} = \lambda_i(Y)\mathrm{Id}_{W_i} + N_i(Y), \quad \forall i \in I \quad \forall Y \in \Delta^\perp,$$

where $\lambda_i : \Delta^\perp \to \mathbb{C}$ are distinct complex 1-forms, and $N_i(Y) : W_i \to W_i$ form a commutative family of nilpotent operators. Furthermore, there is a conjugation of indices such that $\overline{W_i} = W_{\bar{i}}$, $\overline{\lambda_i(Y)} = \lambda_{\bar{i}}(Y)$ and $\overline{N_i(Y)} = N_{\bar{i}}(Y)$. Since $B(X) = \mathrm{Id}$, $\lambda_i \neq 0$ for all i.

Set $V_i := (\beta^X)^{-1}(W_i) \subseteq \Delta_{\mathbb{C}}^\perp$. Notice that for distinct indices we have that

$$\beta(v_i, v_j) = B(v_i)\beta^X(v_j) \in W_j \quad \forall v_i \in V_i, \forall v_j \in V_j.$$

On the other hand, by symmetry,

$$\beta(v_i, v_j) \in W_i \cap W_j = 0, \quad \forall v_i \in V_i, \forall v_j \in V_j, i \neq j. \tag{B.1}$$

So $\beta(V_i, V_j) = 0$ for $i \neq j$ and $\beta(V_i, V_i) = W_i$. This and the flatness of β imply that the W_i's are orthogonal since

$$\langle \beta^X(v_i), \beta^X(v_j) \rangle = \langle \beta(X, X), \beta(v_i, v_j) \rangle = 0, \quad \forall v_i \in V_i, \forall v_j \in V_j, \forall i \neq j.$$

Take $Z \in D := \Delta_\gamma \cap \Delta^\perp$, and decompose it as $Z = \sum_j Z_j$ where $Z_j \in V_j$. Given $v = \sum_j v_j \in \Delta^\perp$ arbitrarily, $v_j \in D_j$, by (B.1) we have that

$$\gamma(Z_j, v) = \gamma(Z_j, v_j) = \gamma(Z, v_j) = 0, \quad \forall v \in \Delta^\perp,$$

which proves that $Z_j \in (\Delta_\gamma)_\mathbb{C}$. Hence, $D = \bigoplus_j D_j$ for $D_j \subseteq (\Delta_\gamma)_\mathbb{C} \cap V_j$.

For $Z \in D_j$ with $j \neq \overline{j}$, the orthogonality of W_j and $W_{\overline{j}}$ gives

$$0 = \langle \beta^X(Z), \beta^X(\overline{Z}) \rangle = \langle \alpha(X, Z), \alpha(X, \overline{Z}) \rangle = ||\alpha(X, Z)||^2.$$

So $\alpha(X, Z) = 0$ and $\beta^X(Z) = 0$. This shows that $D_j = 0$ for all j with $j \neq \overline{j}$.

Suppose that $\dim(V_i) = m + 1 \geq 2$, and set $K := \ker(\lambda_i) \cap V_i \neq V_i$. We claim that

$$\beta(K, K) \perp \beta(K, V_i). \tag{B.2}$$

To prove this, recall the well-known Engel's Theorem for a Lie algebra of nilpotent matrices. It states that there is a basis $\beta^X(x_0), \ldots, \beta^X(x_m)$ of W_i such that all $N_i(y)$ are simultaneously upper triangular. Denote by $n_{ab}(y)$ the coefficients of $N_i(y)$ with respect to this basis, that is

$$\beta(x_r, x_s) = B(x_r)\beta^X(x_s) = \lambda_i(x_r)\beta^X(x_s) + \sum_{j<s} n_{js}(x_r)\beta^X(x_j).$$

The symmetry on r, s of the last expression implies that

$$K = \mathrm{span}\{x_0, \ldots, x_{m-1}\},$$

and $\beta(x_0, x_s) = 0$ for $s < m$.

Due to a process similar to Gram-Schmidt, we can replace x_i by $x_i + \sum_{j<i} a_j x_j$ in such a way that

$$\langle \beta(X, X), \beta(X, x_i) \rangle = \langle \beta(X, x_i), \beta(X, x_j) \rangle = 0, \quad \forall\, 1 \leq i \neq j < m. \tag{B.3}$$

Indeed, as $\beta(x_0, x_s) = 0$ for $s < m$ we have $\langle \beta^X(x_m), \beta^X(x_0) \rangle \neq 0$, and hence $\langle \beta(X, X), \beta(X, x_0) \rangle \neq 0$. So, we can replace x_1 by $x_1 + a_0 x_0$ for some a_0 to obtain

$$\langle \beta(X, X), \beta(X, x_1) \rangle = 0.$$

If $m = 1$ then have (B.3), so assume $m > 1$. Notice that $\langle \beta(X, x_j), \beta(X, x_j) \rangle \neq 0$ for all $0 < j < m$. Indeed, if that is not the case then $\mathrm{span}\{\beta(X, x_0), \beta(X, x_j)\} \subseteq W_i$ would be a two-dimensional null subspace, with W_i Lorentz or Euclidean. In particular, $\langle \beta(X, x_1), \beta(X, x_1) \rangle \neq 0$, so we replace x_2 by $x_2 + a_0 x_0 + a_1 x_1$ for some a_0 and a_1 to have $\langle \beta(X, X), \beta(X, x_2) \rangle = \langle \beta(X, x_1), \beta(X, x_2) \rangle = 0$. Analogously, we replace x_3 by some $x_3 + a_0 x_0 + a_1 x_1 + a_2 x_2$, and so on to prove (B.3).

We proceed by proving (B.2) by induction on r, that is, $\beta(x_r, K) \perp \beta(K, V_i)$. This holds for $r = 0$ since $\beta(x_0, x_s) = 0$ for $s < m$, so assume that this is true for $r < m - 1$. For $v \in K$ and $s, t < m$, then by induction

$$\langle \beta(x_{r+1}, v), \beta(x_s, x_t) \rangle = \Big\langle \sum_{j \leq r} n_{j(r+1)}(v)\beta^X(x_j), \beta(x_s, x_t) \Big\rangle = 0.$$

Thus, to finish the induction we need to show that

$$\langle \beta(x_{r+1}, v), \beta(x_s, X) \rangle = 0, \quad \forall s < k.$$

This is true for $s \leq r$ by flatness and induction, so assume $r < s < m$. Then, by (B.3), we have

$$\langle \beta(x_{r+1}, v), \beta(x_s, X) \rangle = \Big\langle \sum_{j \leq r} n_{j(r+1)}(v)\beta^X(x_j), \beta(x_s, X) \Big\rangle = 0,$$

which finishes the induction, and proves (B.2).

We show now that if $\dim(V_i) \geq 2$ then $D_i = 0$. Assume the contrary and take $Z \in D_i \setminus \{0\}$. Notice that

$$(\alpha(Z, x_0), 0) = \beta(Z, x_0) = B(Z)\beta^X(x_0) = \lambda_i(Z)\beta^X(x_0) = \lambda_i(Z)(\alpha(X, x_0), \gamma(X, x_0)).$$

Therefore, if $\lambda_i(Z) \neq 0$ then $\gamma(X, x_0) = 0$ and

$$\langle \alpha(X, x_0), \alpha(X, x_0) \rangle = \langle \beta(X, x_0), \beta(X, x_0) \rangle = \langle \beta(x_0, x_0), \beta(X, X) \rangle = 0,$$

by flatness of β and the fact that $\beta(x_0, x_0) = 0$. This shows implies $\beta^X(x_0) = 0$ which is a contradiction. Hence, $\lambda_i(Z) = 0$ and $Z \in K$. By (B.2) we have

$$\langle \alpha(Z,Z), \alpha(Z,Z) \rangle = \langle \beta(Z,Z), \beta(Z,Z) \rangle = 0,$$

so $\alpha(Z,Z) = 0$. Hence,

$$\langle \alpha(Z,X), \alpha(Z,X) \rangle = \langle \alpha(Z,Z), \alpha(X,X) \rangle = 0,$$

which shows that $\beta^X(Z) = 0$. This is again a contradiction since β^X is a bijection. We conclude that $D_i = 0$ for $\dim(V_i) \geq 2$.

This discussion shows that if $J = \{j : D_j \neq 0\}$ $D_j \neq 0$ then $D_j = V_j$ and $\dim_{\mathbb{R}}(D_j) = 1$ for all $j \in J$. Then the subbundle

$$\mathcal{S}(\alpha|_{TM \times \Delta_\gamma}) = \mathcal{S}(\beta|_{TM \times \Delta_\gamma}) = \bigoplus_{j \in J} W_j \subseteq T_g^\perp M,$$

has rank ℓ. Hence, g is a composition by Lemma 5.1.4. $\square$

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