Shabat Correspondences and One-Dimensional Families of Matings

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Introduction

In [BP94], Bullett and Penrose introduced a family of $2:2$ holomorphic correspondences on $\mathbb{C}$ and showed that this family contains matings of holomorphic quadratic polynomials and the modular group $\mathrm{PSL}(2,\mathbb{Z})$. Recently in [LMM22a], a large class of examples of $d:d$ antiholomorphic correspondences that give rise to matings of anti-rational maps and $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/(d{+}1)\mathbb{Z}$ were produced. We examine some natural one parameter slices of this family, arising from Shabat polynomials. These form an anti-holomorphic analogue to a class of correspondences produced in [BF05].

Belyi parabolic maps and critical trees

We say that an anti-rational map R is a **parabolic Belyi map** if the following hold:

- R has at most three critical values.
- R has a parabolic fixed point at ∞, with a completely invariant, simply connected basin.
- If R has three critical values then one of them is the parabolic fixed point ∞.

We denote the collection of these maps as $\mathcal{L}$. To obtain one parameter families of such maps we impose critical orbit relations. Let $\mathcal{T}$ be a given planar embedded bi-colored rooted tree. We define

$$\mathcal{L}_{\mathcal{T}} = \{R \in \mathcal{L} \mid R^{-1}(\gamma) \cong \mathcal{T}\},$$

where γ is an arc connecting the critical value of R not in the parabolic basin to ∞. $\mathcal{T}$ is known as the **dessin d'enfant** of R.

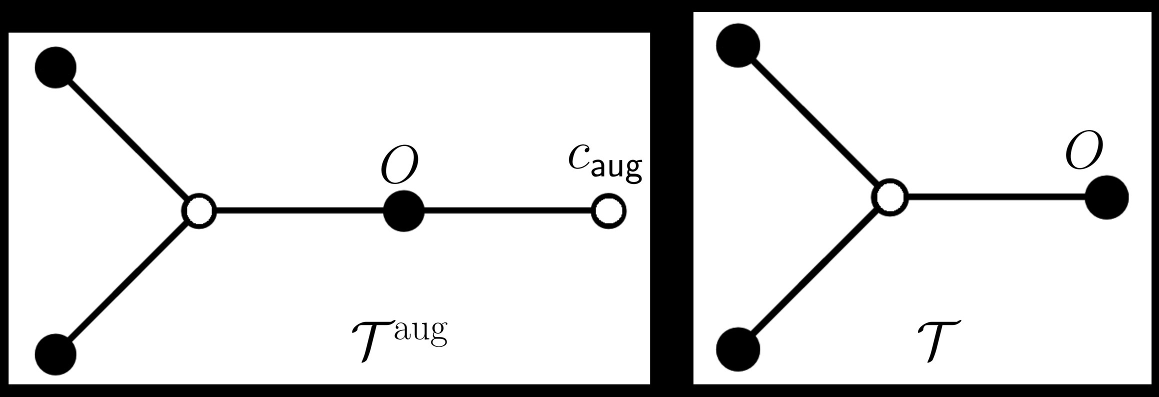


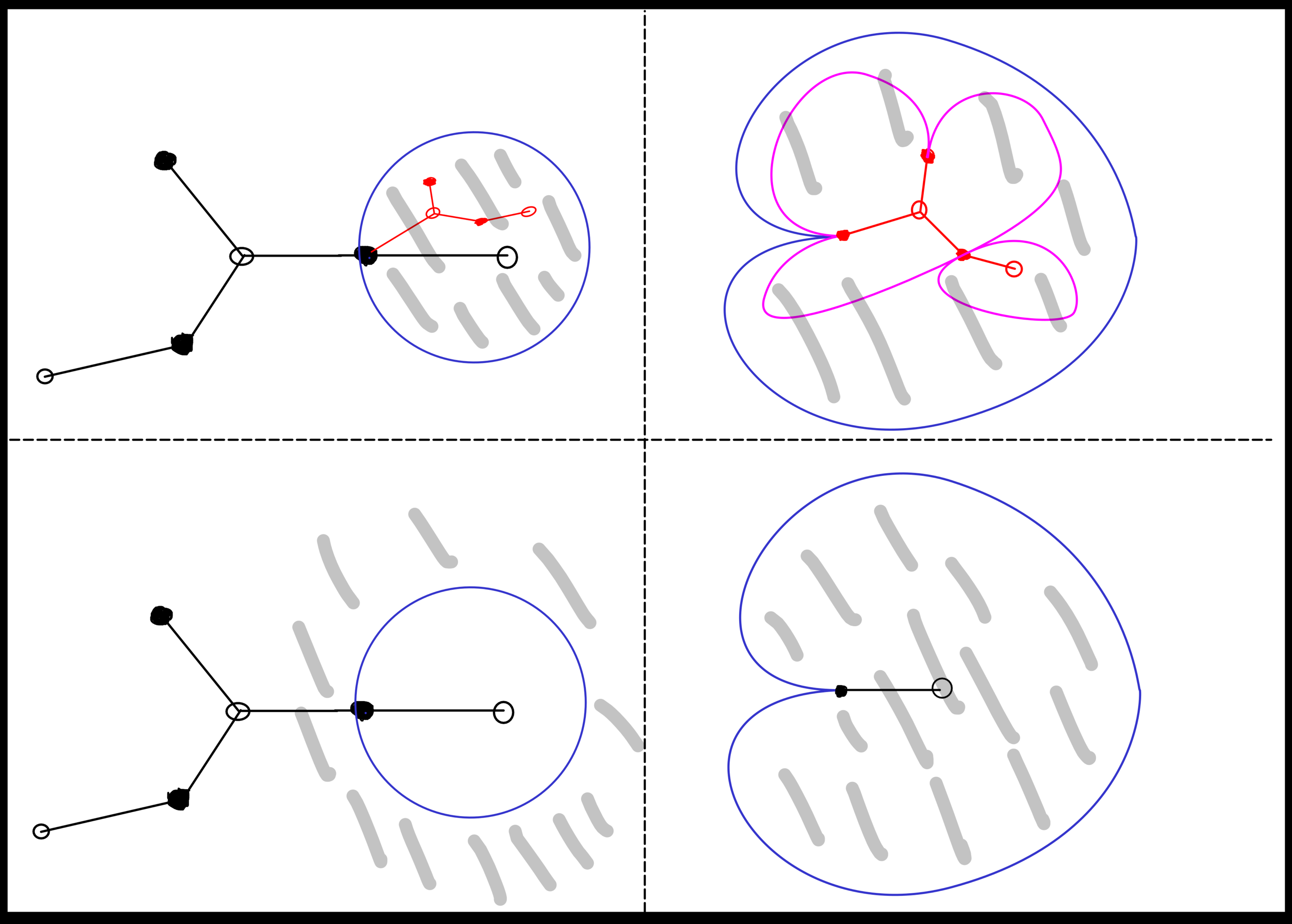
Figure 1: A 3-star tree $\mathcal{T}$ and the corresponding augmented dessin $\mathcal{T}^{\text{aug}}$ are displayed. $\mathcal{T}^{\text{aug}}$ is realized by the Shabat polynomial $\boldsymbol{f}(z) = z^3(4-z)$.

Add an edge to the root of $\mathcal{T}$ to produce the augmented tree $\mathcal{T}^{\text{aug}}$. There is a polynomial $\boldsymbol{f}_{\mathcal{T}}$, called a Shabat polynomial for which $\boldsymbol{f}_{\mathcal{T}}^{-1}([0,1]) \cong \mathcal{T}^{\text{aug}}$.

Schwarz Reflections

If D is a disk such that $(f)_{\mathcal{T}}$ is univalent on D, then $\Omega = (f)_{\mathcal{T}}$ is a quadrature domain, and admits a Schwarz refleciton map $\sigma\colon \Omega \to \mathbb{C}$ which satisfies:

- σ is anti-holomorphic map on Ω
- $\sigma|_{\partial\Omega} \equiv Id$.



Analogous to the theory of polynomial like maps, we define the **non-escaping set** $K(\sigma)$ to be those points for which orbits are defined for all times. We may then define the **connectednes locus** $\mathcal{C}_{\mathcal{T}}$ as he space of parameters σ with connected nonescaping set $K(\sigma)$. The main result of [LLM22a] in the context is the following:

Theorem A [LMM22a]

There is a bijection $\chi\colon \mathcal{C}_{\mathcal{T}} \to \mathcal{L}_{\mathcal{T}}$ with every $\sigma \in \mathcal{C}_{\mathcal{T}}$ hybrid conjugate to $\chi(\sigma)$.

Parameter Equivalence

Each element of $\mathcal{C}_{\mathcal{T}}$ can be realized, via the Bullett-Penrose correspondence $\frac{(f)_{\mathcal{T}}(w)-(f)_{\mathcal{T}}(\iota_D(z))}{w-\iota_D} = 0$ as a mating of a Belyi parabolic map with a reflection group. We call this a **Shabat correspondence**. In [BL20] and [LLMM22] parameter spaces for similar correspondeces were shown to be combinatorially equivalent to spaces of rational maps.

Question

Do parameter spaces of Shabat correspondences and parabolic Belyi maps have homeomorphic combinatorial models?

Parameter Space

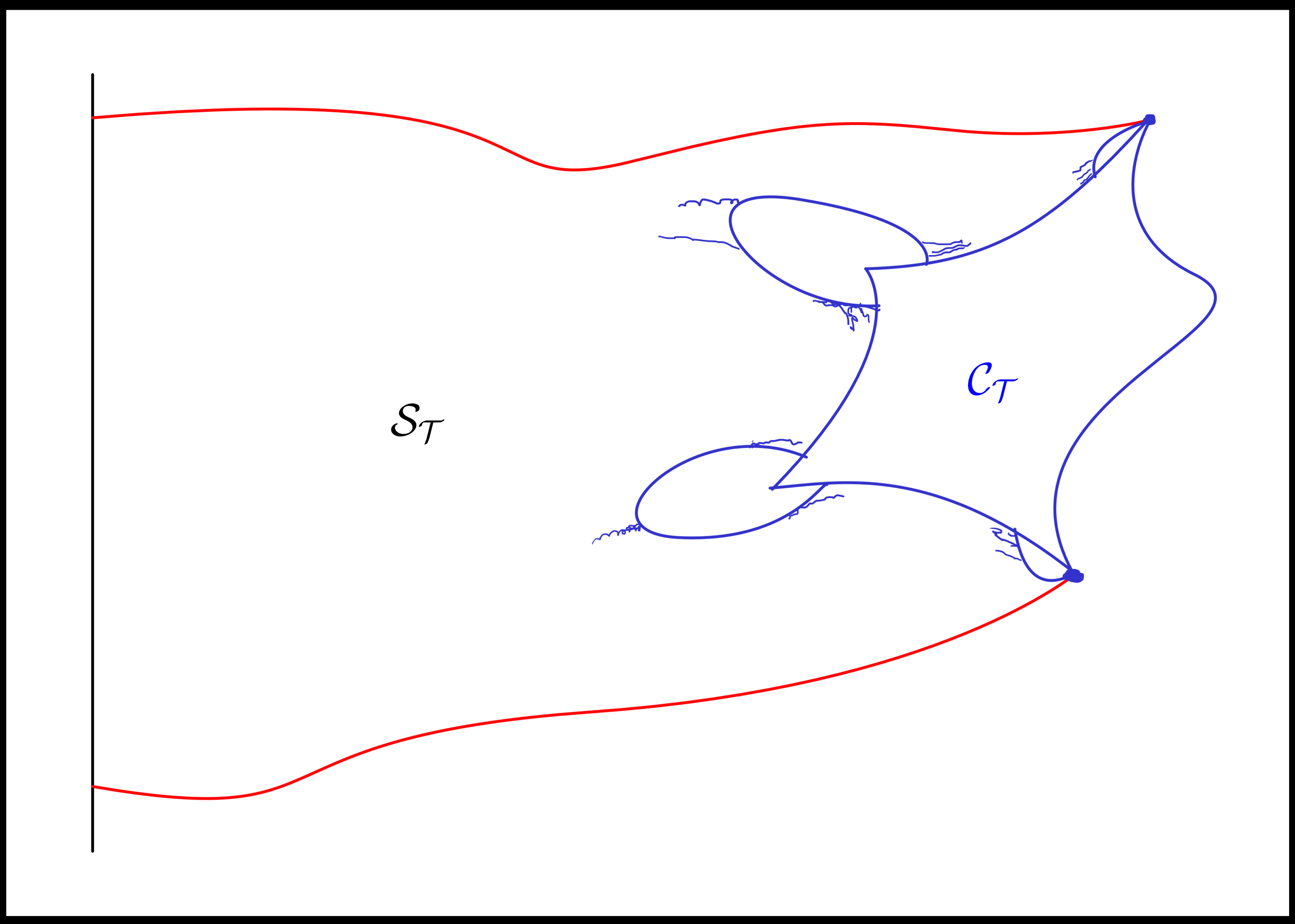
Our parameter space of Schwarz reflections of relevant parameters is described by the set $\mathcal{S}_{\mathcal{T}}$ of $a \in \mathbb{C}$ such that

- $(f)_{\mathcal{T}}$ is univalent on $D(a, |O-a|)$
- $c_aug \in D(a, |O-a|)$.

The parameter space then has a relatively simple form:

Lemma [LMM22b]

$S_{\mathcal{T}}$ is the union of a Jordan domain whose boundary consists of four analytic arcs, together with one of those arcs.

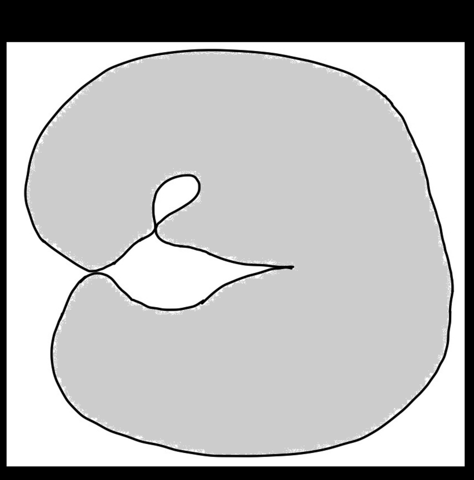


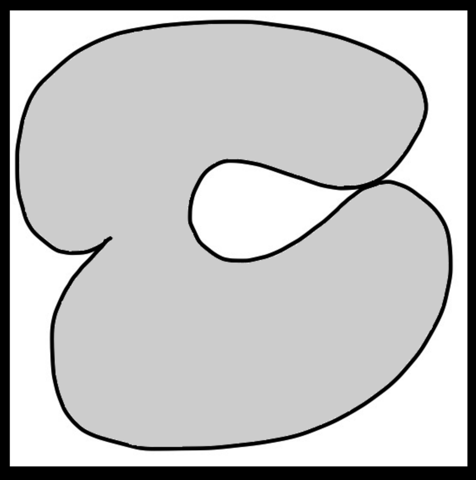
Theorem B, together with a uniformization for the escape locus of $\mathcal{S}_{\mathcal{T}}$ gives the following:

Theorem B [LMM22b]

The connectedness locus $\mathcal{C}(\mathcal{S}_{\mathcal{T}})$ is a hull; that is, it is a compact, connected set with connected exterior.

The proof of the lemma above follow from an analysis of how quadrature domains can degenerate. In particular, we show that the degenerations are fairly well controlled in this setting. For example, the images below **cannot be** degenerations of our quadrature domain.





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