Dynamic Economics with Quantile and General Operator Preferences for Additive Aggregators
PhD Thesis
Instituto de Matemática Pura e Aplicada

Daniel Nunes[1]

October 14, 2022

[1]Instituto Nacional de Matematica Pura e Aplicada (IMPA), Brazil. dsnunes.rj@gmail.com

Abstract

This thesis consists of two parts. Firstly, it studies the dynamic quantile model for intertemporal decisions under uncertainty, and it is shown how quantile preferences behave in five standard economic models. It is shown that quantiles are powerful for calculations, giving, for instance, closed-form expressions for the value function in the intertemporal consumption problem.

Then, we generalize the theoretical settings where one can use the dynamic quantile model. First, we allow endogenous and exogenous variables to belong to metric spaces, which permits choice variables and shocks to be either discrete or continuous. Second, the future state is not determined exclusively by agent's choice, but can be determined by a nontrivial law of motion. Third, shocks can follow a more general Markov process. Finally, we increase the reach of the principle of optimality and correct an issue concerning continuity of the value function from previous work.

We also investigate dynamic programming models for general operators with additive temporal aggregators. Under this environment, we provide conditions for the general intertemporal model to be dynamically consistent, the corresponding dynamic problem to yield a value function, and this value function be concave and differentiable, and also subjected to the principle of optimality. Additionally, we derive the corresponding Euler equation. We discuss a few examples such as expected utility, quantile, expectile, cumulative prospect theory, variational preferences and Choquet integral.

Keywords: Quantile preferences, dynamic programming, recursive model, growth model, intertemporal consumption, investment under uncertainty, expectile, cummulative prospect theory, variational preferences, Choquet integral

JEL: C61, D1, D2, E2

Contents

Chapter 1

Introduction

Dynamic programming is a basic tool for intertemporal economic analysis that allows economists to examine a wide variety of problems. This framework has been extensively used because it is sufficiently rich to model problems involving sequential decision making over time and under uncertainty. See, among others, Stokey et al. (1989), Rust (1996), Ljungqvist and Sargent (2012).

Many applications of intertemporal maximization use the standard recursive expected utility (EU). These models have been workhorses in several economic fields. EU is simple and amenable to theoretical modeling. The assumption of maximization of average utility, the average being a simple measure of centrality, has intuitive appeal as a behavioral postulate. Nevertheless, the usual EU framework has been subjected to a number of criticisms, including in the dynamic version.[1] A segment of the literature proposes alternative recursive models. We refer the reader to Epstein and Zin (1989, 1991), Weil (1990), Grant et al. (2000), Epstein and Schneider (2003), Hansen and Sargent (2004), Maccheroni et al. (2006b), Klibanoff et al. (2009), Marinacci and Montrucchio (2010), Bommier et al. (2017), Sarver (2018), and Dejarnette et al. (2020) among others. Although these models capture some important economic and behavioral features in dynamic models, they are not very tractable and flexible. In many instances, they are difficult to solve analytically, and, in general, it is hard to apply them to different contexts.

Recently, de Castro and Galvao (2019) suggested a new recursive model for an economic agent, who, when selecting among uncertain alternatives, chooses the one with the highest τ-quantile of the stream of future utilities for a fixed $\tau \in (0, 1)$ in an feasible set, instead of the standard EU. The dynamic quantile preferences for intertemporal decisions are represented by an additively separable quantile model with standard discounting. The associated recursive equation is characterized by the sum of the current period utility function and the discounted

[1] For example, it has been well documented in the literature that it is not possible to separate the intertemporal substitution from the risk attitude parameters when using standard dynamic models based on the EU (see, e.g., Hall, 1988). In addition, the study of the temporal resolution of uncertainty as in Kreps and Porteus (1978) requires a framework beyond the standard EU.

value of the certainty equivalent, which is obtained from a quantile function. This intertemporal model is tractable and simple to interpret, since the value function and Euler equation are transparent, and easy to calculate (analytically or numerically), as this thesis aims to illustrate. This model substantially broadens the scope of economic applications, because it posses desirable features as allowing for separation of the risk attitude from the intertemporal substitution while maintaining important features of the standard model, such as dynamic consistency and monotonicity.[2] Static quantile preferences were first studied by Manski (1988) and axiomatized by Chambers (2009), Rostek (2010), and de Castro and Galvao (2021). Recently, there are several different applications of quantile preferences models, see, e.g., Bhattacharya (2009), Giovannetti (2013), de Castro et al. (2021), Barunik and Cech (2021), Long et al. (2021), and Chen et al. (2021). From an experimental point of view, de Castro et al. (2022) find that the behavior of between 30% and 50% of the individuals can be better described with quantile preferences rather than the standard EU.

This thesis is divided in two more chapters. In the next one, we revisit some important models from classical economic literature, such as intertemporal consumption (see, e.g. Ljungqvist and Sargent, 2012), one-sector growth (Brock and Mirman (1972)), investment under uncertainty with convex adjustment costs (Adda and Cooper (2003)), industry investment under demand uncertainty (Lucas and Prescott (1971)) and job search with unemployment (McCall (1970)). Instead of the classical expected utility based preference, we investigate how those models behave under dynamic quantile preferences, and also compare our results with the classical ones. Moreover, this chapter also provides further theoretical generalizations for the original model from de Castro and Galvao (2019), allowing shocks, choices and states to be in a metric space. This allows the shocks to be, for instance, discrete, and do not require them to be convex, thus removing some limitations from the previous work from de Castro and Galvao (2019). Moreover, the shocks may follow a more general Markov process than it was assumed before. Another contribution of this thesis is allowing the shocks to affect the decision after a choice is made, so the decision maker no longer directly chooses the next period state, but otherwise, this state can be affected by the chosen action, current state and future shock as well. We extend the theory to this more general setting, providing existence and uniqueness of the value function, as well as increasingness, concavity, differentiability and providing its Euler equations. Finally, we make an extension of the principle of optimality, this time broadening the admissible plans to be measurable (instead of continuous) and also not necessarily time-invariant, in opposition to the previous result presented in de Castro and Galvao (2019).

The final chapter deals with dynamic programming in a more abstract way, dealing with general operators instead of quantiles or expected utility. We provide a general theory (which

[2] In the quantile model, the risk attitude is captured by the τ, as discussed by Rostek (2010). Therefore, the model allows a separation of risk attitude and EIS, because the EIS is determined by the parameters of the utility function, as de Castro and Galvao (2019) discuss.

encompasses those two) made around a Bellman equation for general operators under additive agregation. Our theory allows the derivation of a value function, which can have desirable properties such as increasingness, concavity and differentiability. We also provide Euler equations for this general model. Furthermore, conditions to produce general dynamic consistent preferences are provided, and we show how to properly define a sequential problem related to the Bellman equation by a principle of optimality for general operators. Finally, we illustrate those general methods by discussing how they suit the known models for expected utility and quantiles. We also discuss how our general theory apply to expectiles, providing a complete dynamic economics theory for expectile preferences. Also, we show where mode, prospect theory and confidence preferences fail to suit to our general methods. Finally, we investigate how the general theory suits cumulative prospect theory, variational preferences, smooth ambiguity preferences and Choquet integral.

Chapter 2

Dynamic Quantile Preferences Applications and Extensions

In this chapter, we generalize the quantile dynamic programming model originally presented in de Castro and Galvao (2019). We extend existing theoretical results in important directions that are useful for practical applications. First, we relax the strict assumption that the agent's choice is the future state variable. We allow the choice variable to be not directly related with the state variable. This is an important generalization because includes situations where the next period state is not directly chosen by the economic agent, being possibly affected by an unknown shock. Second, we allow the choice variable to be either discrete or continuous. This is also an important extension because it encompasses models for both continuous and discrete choices, and hence broadens the scope of applications substantially. Third, we extend existing results by allowing the random shock in the dynamic model to belong to a metric space and follow a more general type of Markov process. Fourth, we provide a new proof of continuity of the value function in this more general setting in Lemma 2.3.1, correcting a mistake presented in the former proof of this result (which was also done in a more restrictive setting) in de Castro and Galvao (2019). Finally, a new definition for recursive quantile preferences is given, which enables us to increase the reach of the principle of optimality in comparison to the former version in de Castro and Galvao (2019).

Given these generalizations, we show that the theoretical properties of the dynamic quantile model remain valid in this more general setup, which encompasses the former one. In particular, we first show that the optimization problem leads to a contraction, which therefore has a unique fixed-point. This fixed point is the value function of the problem and satisfies the Bellman equation. Second, we prove that the value function is concave and differentiable, thus establishing the quantile analog of the envelope theorem. Third, using these results, we derive the corresponding Euler equation for the infinite horizon problem. Moreover, we establish dynamic consistency of the quantile preferences, define properly a sequential problem, and show it satisfies the principle of optimality.

The second main contribution of this chapter is to provide canonical examples to illustrate the usefulness of the recursive quantile model with different applications, as well as to show that quantile models are amenable to calculations. In particular, five models that are central to contemporary economics and finance are revisited and adapted to the dynamic quantile model.

First, we illustrate the methods with a simple intertemporal consumption model with a single asset (see, e.g. Ljungqvist and Sargent, 2012). Following a large body of literature, we specify an isoelastic utility function and derive several properties of the model. The quantile model is characterized by three parameters: the discount factor, the risk attitude, and the EIS. Interestingly, we are able to obtain closed form expressions for the fixed point value function, and the optimal consumption and asset allocation. The ability of obtaining these closed form solutions contrasts with the EU case, where given the difficulty, it is usual to solve this problem numerically.

In the second example, we extend the recursive quantile model to the well-known one-sector growth model of Brock and Mirman (1972). The economic agent maximizes the recursive utility function subject to the budget constraint, which incorporates both the production technology and the depreciation of capital. For comparison with the standard EU case, we specialize the model using a logarithimic utility function and depreciation equals to one. As in the first example, when using the quantile model, we are able to derive an explicit closed form solution for the value function.

We provide two examples studying investment under uncertainty (see, e.g., Adda and Cooper (2003)). First, we analyze a general dynamic optimization problem for invest with a convex adjustment cost. We derive the Euler equation, that equates the measure of the marginal cost of capital accumulation – which includes the direct cost of new capital as well as the marginal adjustment cost – with the τ-*quantile* of the marginal gains of more capital. In the literature, this is conventionally termed "marginal q" or Tobin's q, after Tobin (1969). Second, we revisit the industry investment under uncertainty model in Lucas and Prescott (1971) with the dynamic quantile preferences. We investigate the total surplus maximization problem. This example illustrates the case where the marginal cost is equal to the demand, the marginal consumer surplus.

Finally, we discuss a quantile-based version of the job-search model discussed in McCall (1970). The analysis is directed to the employee's job-searching strategy. This model illustrates the use of the quantile framework when the decision variable is discrete and shocks are also discrete, as well as the endogenous state follows a nontrivial law of motion. Therefore, this example fully benefits from our theoretical expansions and could not be carried with the former theory from de Castro and Galvao (2019).

Overall, these examples illustrate the usefulness of the recursive quantile preferences as an attractive alternative to the standard EU to model dynamic economic behavior. Recursive quantile preferences have several desirable properties as dynamic consistency and monotonicity. The quantile model also allows for separation between the risk attitude and intertemporal substitution. In addition, it is possible to compute closed form solutions for the value and policy functions in several cases, and Euler equations characterizing the equilibrium can be derived.

The remaining of the chapter is organized as follows. Section 2.1 describes the dynamic economic model and introduces the dynamic programming approach for determining the optimal solution of the recursive quantile model. Section 2.2 illustrates the empirical usefulness of the the new approach by providing different examples of the dynamic quantile model. Section 2.3 presents the main theoretical results, say existence of the value function associated to the dynamic programming problem. In addition, we establish additional properties of the value function related to monotonicity, concavity, differentiability, and the Euler equation. Section 2.4 deals with the definition of the sequential problem for quantile preferences, where we present an improvement over the former definition given in de Castro and Galvao (2019) which broadens the reach of the results, especially the principle of optimality. Finally, Section 2.5 concludes. We relegate the majority of proofs to the Appendix.

2.1 Dynamic Programming with Quantile Preferences

This section introduces the dynamic programming approach for determining the optimal solution of the recursive quantile model, which was introduced by de Castro and Galvao (2019). The objective is to write a recursive problem and solve the infinite horizon sequence problem, subject to a given constraint.

2.1.1 Quantile

We begin by stating some preliminary definitions. For a given random variable Z and $\tau \in (0,1)$,[1] we define its τ-quantile as

$$Q_\tau[Z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr[Z \leqslant \alpha] \geqslant \tau\} \tag{2.1}$$

and its τ-quantile* (or right quantile) as

$$Q^*_\tau[Z] = \sup\{\alpha \in \mathbb{R} : \Pr[Z \leqslant \alpha] \leqslant \tau\}. \tag{2.2}$$

In the appendix, we state and prove a number of results about quantiles and right quantiles, some of which are well-known. For instance, if $g : \mathbb{R} \to \mathbb{R}$ is weakly increasing and left-continuous, then $g(Q_\tau[Z]) = Q_\tau[g(Z)]$.

[1] In this chapter we will not consider the cases in which $\tau \in \{0,1\}$.

2.1.2 States, Decisions and Shocks

Let $\mathcal{X}$ denote the state space, $\mathcal{Y}$ the set of possible actions the decision-maker (DM) may take, and $\mathcal{Z}$ the range of the shocks (random variables) in the model. We require these sets to be metric spaces. Let $x_t \in \mathcal{X}$ denote the state in period t, and $z_t \in \mathcal{Z}$ the shock at the end of period $t-1$, both of which are known by the DM at the beginning of period t. In each period t, the DM chooses a feasible action y_t from a constraint subset $\Gamma(x_t, z_t) \subset \mathcal{Y}$.

In the model above, the resolution of uncertainty at period t occurs after the DM chooses an action so the next period's state x_{t+1} may be affected by the shocks z_t and z_{t+1}, as discussed in Stokey et al. (1989, p. 240). This influence is described by a law of motion function ϕ from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ to $\mathcal{X}$ that determines the next period state variable x_{t+1} as function of the current state x_t, the choice y_t, the shock z_t that is known at the beginning of period t, and the shock z_{t+1} realized at the beginning of period $t+1$, that is,

$$x_{t+1} = \phi(x_t, y_t, z_{t+1}). \tag{2.3}$$

It is common in the literature to write the law of motion in equation (2.3) as simply a function of x_t, y_t and z_{t+1}; see, e.g., Stokey et al. (1989, p. 256). In most models, this is even simpler and we could write $\phi(x_t, y_t, z_{t+1}) = y_t$.

Let $\mathcal{Z}^t = \mathcal{Z} \times \cdots \times \mathcal{Z}$ (t-times, for $t \in \mathbb{N}$), $\mathcal{Z}^\infty = \mathcal{Z} \times \mathcal{Z} \times \cdots$ and $\mathbb{N}^0 \equiv \mathbb{N} \cup \{0\}$. Given $z \in \mathcal{Z}^\infty$, $z = (z_1, z_2, \ldots)$, we denote $(z_t, z_{t+1}, \ldots)$ by ${}_t z$ and $(z_t, z_{t+1}, \ldots, z_{t'})$ by ${}_t z_{t'}$. A similar notation can be used for $x \in \mathcal{X}^\infty$ and $y \in \mathcal{Y}^\infty$.

The random shocks will follow a time-invariant (stationary) Markov process. In this chapter we allow the random shocks $\mathcal{Z}$ to be a metric space, either connected or finite. For instance, this encompasses the possibility of $\mathcal{Z}$ be continuous or discrete. Further results will require $\mathcal{Z}$ to be Euclidean, that is, $\mathcal{Z} \subseteq \mathbb{R}^k$, and also allow countable $\mathcal{Z}$ endowed with the discrete topology.

Stationary Markov processes will be modeled by a transition function $K : \mathcal{Z} \times \Sigma \to [0,1]$, where Σ is the Borel σ-algebra of the metric space $\mathcal{Z}$. This means that

$$K(z, \cdot) \in \Delta(\Sigma) \text{ for all } z \in \mathcal{Z},$$

where $\Delta(\Sigma)$ denotes the set of probability measures over the measurable space $(\mathcal{Z}, \Sigma)$, and

$$z \in \mathcal{Z} \mapsto K(z, A) \in [0,1] \text{ is } \Sigma\text{-measurable for all } A \in \Sigma. \tag{2.4}$$

In our framework, we will restrict the allowed transition functions in Assumption 1. Two major requirements are the imposition that (2.4) is continuous instead of just measurable, as well as $K(z, A) > 0$ for each nonempty open set A and all $z \in \mathcal{Z}$.

For $\mathcal{Z}$ continuous, the reader can keep in mind a particular example of a Markov transition which uses a conditional probability density function $f : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_+$ and the usual Lebesgue measure in the form $K(z, dz') \equiv f(z'|z)dz'$. This setting was used before in de Castro and Galvao (2019) (requiring also that $f(z'|z) > 0$). For simplicity of notation, we will frequently represent Z_t and Z_{t+1} by Z and Z', respectively.[2]

Therefore, in this setting, the probability that $Z' \in A \subset \mathcal{Z}$ given $Z = z$ is

$$\Pr\left(Z' \in A | Z = z\right) = \int_A f(z'|z)dz'.$$

In the discrete case, $f(z'|z) = P[z'|z]$ stands for the conditional probability mass function. We will also refer to the probability mass function (p.m.f.) of $\mathcal{Z}$ as f. That is, $f(z) = P[Z = z]$. In this setup, all integrals on $\mathcal{Z}$ are to be taken with respect to the discrete measure. For example,

$$\int_A f(z'|z)dz' = \sum_{z' \in A} f(z'|z).$$

Some results in this chapter will require a monotonicity condition on the shocks. Namely, we will sometimes assume that if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant \hat{z}$, then $\mathrm{E}\left[h(w)\middle| z\right] \leqslant \mathrm{E}\left[h(w)\middle| \hat{z}\right]$. When $\mathcal{Z}$ is continuous, this means

$$\int_{\mathcal{Z}} h(\alpha)f(\alpha|z)d\alpha \leqslant \int_{\mathcal{Z}} h(\alpha)f(\alpha|\hat{z})d\alpha. \tag{2.5}$$

The above assumptions are formally stated in Assumption 1 below.

Before we present the recursive model, we introduce the concept of quantile martingale process. This class of processes will be especially useful later to investigate particular examples of the model with closed form solutions. Consider the following definition.

Definition 2.1.1. *We say that Z is a τ-quantile martingale if*

$$Q_\tau[Z_{t+1}|Z_t = z_t] = z_t. \tag{2.6}$$

A simple example of the process in (2.6) is the following.

Example 2.1.2. *Let $Z_{t+1} = Z_t + e_t$, where e_t satisfies $Q_\tau[e_t|Z_t = z_t] = 0$. Then, (2.6) holds:*

$$Q_\tau[Z_{t+1}|Z_t = z_t] = Q_\tau[Z_t + e_t|Z_t = z_t] = z_t + Q_\tau[e_t|Z_t = z_t] = z_t + 0 = z_t.$$

Thus, the best τ-th conditional quantile predictor of the random variable Z_{t+1} is the current value z_t.

Remark 2.1.3. *Some assumptions that we require below are not automatically satisfied for τ-quantile martingales. However, most of the proofs can be adapted to those processes, as we comment in the proofs in the appendix (see Remark A.2.2 in Appendix A).*

[2] This structure implies, in particular, that the information filtration is fixed throughout.

2.1.3 The Dynamic Model

Given the current state x_t and current shock z_t, $\Gamma(x_t, z_t)$ denotes the set of possible choices y_t, that is, the budget set. Given x_t, z_t and $y_t \in \Gamma(x_t, z_t)$, $u(x_t, y_t, z_t)$ denotes the current-period utility function, that is, the instantaneous utility obtained in the current period. If there were no uncertainty, that is, if the sequence $({}_t x, {}_t y, {}_t z)$ were completely known, the utility of the DM beginning on time t would be

$$V_t({}_t x, {}_t y, {}_t z) = \sum_{s=t}^{\infty} \beta^{s-t} u(x_s, y_s, z_s),$$

where $\beta \in (0, 1)$ is the discount factor.

In our model, the uncertainty with respect to the future realizations of z_t are evaluated by a quantile. In the dynamic quantile model, the intertemporal choices can be represented by the maximization of a value function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ that satisfies the recursive equation:

$$V(x_t, z_t) = \max_{y_t \in \Gamma(x_t, z_t)} \Big\{ u(x_t, y_t, z_t) + \beta Q_\tau [V(x_{t+1}, z_{t+1}) | z_t] \Big\}, \tag{2.7}$$

where x_{t+1} is given by (2.3). Note that this is similar to the usual dynamic programming problem, in which the expectation operator $E[\cdot]$ is in place of $Q_\tau[\cdot]$. In Section 2.4, we develop in details the dynamic quantile model and establish several properties of the model in (2.7).[3] In Section 2.3 below, we provide sufficient conditions for the the uniqueness of the solution to the problem (2.7) above. In this case, this unique optimal solution defines a policy function $y^* : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$, that associates to each (x_t, z_t) the optimal solution $y^* = y^*(x_t, z_t)$.

This basic model encompasses many examples, as Section 2.2 below illustrates. We adopt in those examples the standard notation, and point out only how they relate to the above notation, trying to avoid repetitions as much as possible.

2.1.4 Main Assumptions

Now we state the main assumptions concerning the shocks, used for establishing the results. The following basic assumption is assumed throughout the chapter.

Assumption 1 (Basic). *$\mathcal{Z}$ is a metric space which is either connected or finite, Σ its Borel σ-algebra, and the shocks follow a stationary Markov process with transition function $K : \mathcal{Z} \times \Sigma \to [0, 1]$ satisfying the following:*

(i) for a fixed $z \in \mathcal{Z}$ and $0 < \eta < 1$, there exists a compact $\mathcal{Z}' \subset \mathcal{Z}$ such that

$$K(z, \mathcal{Z}') > 1 - \eta;$$

[3] In particular, Section 2.4 defines plans, the preference, and the sequence of recursive functions. It shows that the recursive quantile preference is well defined, and establishes dynamic consistency of the preferences. It also shows that the principle of optimality holds.

(ii) for each $\mathcal{Z}' \subset \mathcal{Z}$ compact, the function

$$z \in \mathcal{Z} \mapsto K(z, \mathcal{Z}') \in [0,1].$$

is continuous;

(iii) for each $A \subset \mathcal{Z}$ open and nonempty,

$$K(z, A) > 0 \textit{ for all } z \in \mathcal{Z}.$$

Note that Assumption 1 allows an unbounded multidimensional Markov process. Condition (i) is equivalent to require that, for each $z \in \mathcal{Z}$, $K(z,\cdot)$ is a tight measure, that is,

$$K(z, A) = \sup\{K(z, D); \text{ D is compact, } D \subset A\} \text{ for all } A \in \Sigma.$$

When $\mathcal{Z}$ is compact, this condition is trivially satisfied by choosing $\mathcal{Z}' = \mathcal{Z}$. It is worth noting that Assumption 1 extends the setting in de Castro and Galvao (2019) by allowing the random shock $\mathcal{Z}$ to be a metric space, which does not need to be convex anymore but must be either connected or finite. This encompasses the case where $\mathcal{Z}$ is discrete [4]. Later, with further hypotheses, we will also allow $\mathcal{Z}$ to be countable, endowed with the discrete topology. We also impose some weaker hypotheses over the distribution of the shocks, which encompasses the previous assumption from de Castro and Galvao (2019) as a special case. This expansion is one of the major theoretical contributions from the current chapter.

Assumption 1 is adopted in all result of this chapter, even if it is not explicitly mentioned, but occasionally we need to strengthen it with extra requirements. One of them imposes a monotonicity structure in the shocks z, which are then assumed to be Euclidean:

Assumption 2. *$\mathcal{Z} \subset R^k$. Moreover, if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant z'$, then*

$$E\left[h(w)\middle| z\right] = \int_{\mathcal{Z}} h(\alpha) K(z, d\alpha) \leqslant \int_{\mathcal{Z}} h(\alpha) K(z', d\alpha) = E\left[h(w)\middle| z'\right]. \tag{2.8}$$

This Assumption implies that whenever $z \leqslant z'$, the conditional distribution given z' first-order stochastically dominated the conditional distribution given z, that is,

$$\int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} K(z', d\alpha) \leqslant \int_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}} K(z, d\alpha) \tag{2.9}$$

for all w.[5]

Remark 2.1.4. *For τ-quantile martingales processes (see Definition 2.1.1 above), Assumptions*

[4] Discrete topological spaces are metrizable with the trivial metric $d(x, y) = 1$ if $x \neq y$, $d(x, x) = 0$

[5] To obtain (2.9), it is enough to use $h(z) = -1_{\{\alpha \in \mathcal{Z} : \alpha \leqslant w\}}(z)$ in (2.8).

1 and 2 are not easily verifiable. To see how the necessary results hold in this case, see Remark A.2.2 in Appendix A.

2.1.5 Risk Attitude and Axiomatization

We conclude this section with two remarks. The first on the notion of risk attitude, and the second on the axiomatization of the dynamic quantile preferences.

Manski (1988) and Rostek (2010) argue that the risk attitude of a quantile maximizer can be captured by τ in static models. Using the notion of quantile-preserving spread introduced by Mendelson (1987), de Castro and Galvao (2021) adapt the definition of risk for dynamic models in Epstein and Zin (1989) to show that the single dimensional parameter $\tau \in (0,1)$ captures the risk attitude in dynamic quantile models. Hence, this model admits a notion of comparative risk attitude, where an agent with quantile given by τ_1 is more risk preferring than another agent with quantile given by τ_2 if $\tau_1 > \tau_2$, independently of the functional form of the utility function.

Regarding axiomatization, Manski (1988) was the first to study quantile preferences, which was recently axiomatized. Rostek (2010) axiomatized the quantile preferences in the context of Savage (1954)'s subjective framework. Rostek (2010) modifies Savage's axioms to show that they are equivalent to the existence of a $\tau \in (0,1)$, a probability measure and a quantile function. Chambers (2009) works in a risk setting where the probability distribution of the random variables, and shows that the preference satisfies monotonicity, ordinal covariance, and continuity if and only if the preference is a quantile preference. de Castro and Galvao (2021) formally axiomatize both the static and dynamic quantile preferences. For the former case, a finite state space is considered and the main axioms that provide the quantile preferences representation are Monotonicity, Ordinality, and Betting Consistency. These axioms are extended to the dynamic context to axiomatize the dynamic version of the model. The dynamic preferences induce an additively separable quantile model with standard discounting, that is, the recursive equation is characterized by the sum of the current period utility function and the discounted value of the certainty equivalent, which is a quantile function.

2.2 Analysis of Some Canonical Economic Models

In this section we discuss five well-known economic models that can be adapted to quantile preferences. The analysis of these canonical models are useful to illustrate the techniques and advantages of the recursive quantile model, as well as the new theoretical results in this chapter.

2.2.1 Intertemporal Consumption

In a seminal work, Modigliani and Brumberg (1954) investigated intertemporal consumption and life-cycle analysis. This framework has been used as a standard economic approach to the study of consumption behavior and served as basis for a very large literature and subsequent models of intertemporal consumption (see, e.g., Deaton (1992) and Ljungqvist and Sargent (2012)).

This first example uses a consumption-based model to illustrate the dynamic quantile preferences methods. This example also appears in de Castro and Galvao (2019). Nevertheless, here we derive additional results as an explicit formula for the value function, the optimal consumption and asset hold, as well as their corresponding paths. The ability of obtaining these closed form solutions contrasts with the EU case, where given the difficulty, it is usual to solve this problem numerically.

Consider the following economy. At the beginning of period t, the decision-maker (DM) has $x_t \in \mathcal{X} \subset \mathbb{R}_+$ units of the risky asset, with return $z_t \in \mathcal{Z} \subseteq \mathbb{R}_{++}$. With wealth $x_t z_t$ at the beginning of period t, the DM decides the number of units y_t of the risk asset, which is equal to the next period's state $y_t = x_{t+1}$ and $c_t = x_t z_t - y_t$ is the amount consumed in period t. From this, the next period how many units of the risky asset x_{t+1} is given by the law of motion $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ introduced in (2.3), as follows:

$$x_{t+1} = \phi(x_t, y_t, z_{t+1}) = y_t.$$

The dynamic problem of interest is to choose a sequence $y_t = x_{t+1}$ to maximize the following recursive equation:

$$V(x_t, z_t) = \max_{x_{t+1} \in \Gamma(x_t, z_t)} \left\{ U(c_t) + \beta Q_\tau \left[V(x_{t+1}, z_{t+1}) | z_t \right] \right\}, \tag{2.10}$$

where

$$c_t = x_t z_t - x_{t+1}, \tag{2.11}$$

$\Gamma(x_t, z_t) = [0, x_t z_t]$ is the budget set and $U : \mathbb{R}_+ \to \mathbb{R}$ defines the utility function $u(x_t, y_t, z_t) = U(x_t z_t - y_t) = U(c_t)$.

Consider the following assumption.

Assumption 3. *The following hold:*

(i) $\mathcal{X} = [0, \bar{x}] \subseteq \mathbb{R}_+$*;*

(ii) $\mathcal{Z} \subseteq \mathbb{R}_{++}$*;*

(iii) $U : \mathbb{R}_{++} \to \mathbb{R}$ *is given by*

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{2.12}$$

where $\gamma > 0, \gamma \neq 1$;[6]

(iv) $0 < \beta < \min\{1, \sup_{z \in \mathcal{Z}} z^{\gamma-1}\}$.

Assumption 3 is standard in economic applications. It specifies the utility function and will guarantee that the value function converges. This is the only known primitive function. The consumption literature has often worked with an isoelastic utility function (constant elasticity of substitution–CES) also known as Constant Relative Risk Aversion (CRRA). But, naturally, other forms of utility function could be used as we will discuss below.

The model in (2.10), together with Assumption 3, is simple and the DM decides on the stream of consumption, which is equivalent to deciding on the future state $x_{t+1} \in \Gamma(x_t, z_t)$. Later in this chapter we will formally relax this requirement. The optimization problem takes the form

$$V(x_t, z_t) = \max_{x_{t+1} \in [0, x_t z_t]} \left\{ \frac{(x_t z_t - x_{t+1})^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[V(x_{t+1}, z_{t+1}) \,|\, z_t \right] \right\}. \tag{2.13}$$

Model (2.13) is characterized by three parameters: the discount factor (β), the risk attitude (τ), and the parameter in the CES utility function (γ). The discount factor characterizes consumer's patience, is used to discount future payments of intertemporal utility functions, and allows to obtain the present value of future consumption. The risk attitude parameter – given by the quantile τ, as discussed in Section 2.1.5 – describes consumer's reluctance to substitute consumption across states of the world under uncertainty and is meaningful even in an atemporal setting. In the dynamic quantile model, the reciprocal of the parameter γ captures the elasticity of intertemporal substitution (EIS). The EIS is defined as elasticity of consumption growth with respect to marginal utility growth.[7] An important feature of the recursive quantile model is that it allows for the complete separation of the risk and EIS parameters, while maintaining important properties as dynamic consistency and monotonicity. This is in sharp contrast with the standard expected utility (EU) case, where the model is characterized by only two parameters and the risk attitude cannot be separated from the EIS.[8]

Now we use the results from Section 2.3 below to show that the recursive quantile problem in (2.13) possesses a value function that satisfies all standard properties. We are able to establish the following results:

[6] The case $\gamma = 1$, that is, $U(c) = \log(c)$ is treated in a slightly more general setup in Section 2.2.2 below.

[7] Under time separable utility, this definition is equivalent to the percent change in consumption growth per percent increase in the net interest rate.

[8] See Hall (1978) for a discussion on the separation of these two parameters in the EU case.

Theorem 2.2.1. *Denote y the choice variable (future state), and w the future shock. Let Assumptions 1 and 3 hold. Then, there exists a function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfying (2.13), that is,*

$$V(x,z) = \max_{y \in [0,xz]} \left\{ \frac{(xz-y)^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[V(y,w) | z \right] \right\}.$$

With an additional assumption concerning the distribution of the shocks, we can establish further properties of the value function:

Theorem 2.2.2. *Let Assumptions 2 and 3 hold. Then, the unique function $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ satisfying (2.13) is differentiable in x, strictly increasing in z and satisfies*

$$\frac{\partial V}{\partial x}(x,z) = U'(xz - y^*)z = (xz - y^*)^{-\gamma} z \tag{2.14}$$

if y^* is interior to $\mathcal{X}$, where $\{y^*\} = \arg\max_{y \in \mathcal{Y}} \{U(xz - y) + \beta Q_\tau \left[V(y,w) | z \right]\}$.

A similar result appears in de Castro and Galvao (2019). From Theorem 2.3.12 below, the Euler equation in the intertemporal consumption model is simply:

$$U'(c_t) = Q_\tau[\beta U'(c_{t+1}) z_{t+1} | z_t].$$

This equation has a very simple intertemporal substitution interpretation. Suppose the DM decreases the consumption by dc_t at time t, invests dc_t in the asset and consumes the proceeds at time $t+1$. The decrease in utility at time t is $U'(c_t)$. The increase in utility at time $t+1$ is uncertain because of the shock, but viewed at t, it is evaluated as the τ-quantile $Q_\tau[\beta U'(c_{t+1}) z_{t+1} | z_t]$. Together with part (ii) of Assumption 3 the expression simplifies to

$$Q_\tau \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} z_{t+1} \Big| z_t \right] = 1.$$

In fact, we are able to say more than that, namely, in this chapter we obtain an interesting explicit expression for the value function. In addition, we derive closed form solutions for the optimal asset allocation and consumption, as well as to the consumption path.

The following functions will be useful in the statement below. Let $r_{\tau,s}(z)$ be defined recursively by

$$\begin{aligned} r_{\tau,0}(z) &= 1, \\ r_{\tau,s}(z) &= r_{\tau,s-1}\left(Q_\tau[w|z]\right) \cdot Q_\tau[w|z] \quad (s \geqslant 1). \end{aligned} \tag{2.15}$$

Given this, define the function:

$$S(z) \equiv \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} \left[r_{\tau,s}(z) \right]^{\frac{1-\gamma}{\gamma}}. \tag{2.16}$$

Observe that $S(z)$ depends on β, τ and γ.

Theorem 2.2.3. *Let Assumptions 2 and 3 hold. Then, the unique value function* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *satisfying* (2.10) *is given by*

$$V(x,z) = \frac{1}{1-\gamma} \cdot x^{1-\gamma} \cdot \left[(1+S(z))^{\gamma} z^{1-\gamma} \right]. \tag{2.17}$$

Moreover, the optimal y^* *is interior and given by the policy function* $y^* : \mathcal{X} \times \mathcal{Z} \to \mathcal{Y}$*:*

$$y^* = y^*(x,z) = \frac{zS(z)}{1+S(z)} \cdot x, \tag{2.18}$$

so that the consumption is given by

$$c^* = c^*(x,z) = \frac{z}{1+S(z)} \cdot x. \tag{2.19}$$

Therefore, the optimal consumption path $\{c_t\}_{t=1}^{\infty}$ *is given by*

$$c_{t+1} = m_{\tau}(z_t, z_{t+1}) \cdot c_t, \tag{2.20}$$

where

$$m_{\tau}(z_t, z_{t+1}) \equiv \frac{S(z_t)}{1+S(z_{t+1})} \cdot z_{t+1}. \tag{2.21}$$

Theorem 2.2.3 provides explicit solutions for the value function, and optimal asset allocation and consumption. We first observe that both the optimal policy functions, in equations (2.18) and (2.19), are liner functions of x. In particular, the optimal policy rules $y^*(x,z)$ and $c^*(x,z)$ are functions of current state x multiplied by a factor that captures the uncertainty, given by the shock z, through the quantile. The uncertainty is resolved through the recursive quantile function $r_{\tau,s}(z)$ in (2.15). The expressions in (2.18) and (2.19) also show that the optimal asset allocation and consumption are functions of the three parameters characterizing the model, the discount factor β, the EIS $1/\gamma$, and the risk attitude (quantile) τ. Finally, we observe that, in contrast with the example above, it is difficult to obtain closed form expressions in the standard recursive EU case.[9]

If we assume that the shocks are independent and identically distributed (iid), we can specialize the above results as follows:

Example 2.2.4 (The iid case)**.** *If the shocks are independent, then* $Q_{\tau}[w|z]$ *becomes a constant,* $Q_{\tau}[w]$*, such that* (2.15) *reduces to* $r_{\tau,s} = r_{\tau,s}(z) = Q_{\tau}[w]^s$*. Similarly, let* $a_{\tau,\gamma} =$

[9]In the EU case, the solution is also separable in the form $v(x,z) = \frac{x^{1-\gamma}}{1-\gamma} L(z)$, where $L(z)$ is the fixed point of the operator $T(L(z)) = z^{1-\gamma} \left\{ 1 + \beta^{\frac{1}{\gamma}} \left(E[L(w)|z] \right)^{\frac{1}{\gamma}} \right\}^{\gamma}$. However, the fact that $E[\cdot]$ does not commute with increasing functions makes it hard to find a simple closed form for L as we did in the quantile case, so a numerical approach seems unavoidable.

$\beta^{\frac{1}{\gamma}}(Q_\tau[w])^{\frac{1-\gamma}{\gamma}}$. *Then,*

$$1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left[r_{\tau,s}(z)\right]^{\frac{1-\gamma}{\gamma}} = 1+\sum_{s=1}^{\infty}a_{\tau,\gamma}^{s} = 1+\frac{a_{\tau,\gamma}}{1-a_{\tau,\gamma}} = \frac{1}{1-a_{\tau,\gamma}},$$

where $|a_{\tau,\gamma}|<1$ *by Assumption 3. With this, the above results simplify to the following:*

$$\begin{aligned} V(x_t,z_t) &= \frac{(1-a_{\tau,\gamma})^{-\gamma}}{1-\gamma}(x_t z_t)^{1-\gamma}, \\ x_{t+1} &= a_{\tau,\gamma}x_t z_t, \\ c_t &= (1-a_{\tau,\gamma})x_t z_t. \end{aligned}$$

Another case of interest is when the shocks are τ-quantile martingales (see Definition 2.1.1):

Example 2.2.5 (τ-quantile martingales)**.** *Assume that* z *follows a* τ*-quantile martingale process (see Definition 2.1.1 and equation (2.6)). Then* $Q_\tau[w|z]=z$ *for all* z*, so*

$$r_{\tau,s}(z)=z^s \text{ for all } s\geqslant 1.$$

Therefore, Theorem 2.2.3 implies that the value function is explicitly given by

$$\begin{aligned} V(x,z) &= \frac{1}{1-\gamma}(xz)^{1-\gamma}\left\{\sum_{s=0}^{\infty}\left(\beta^{\frac{1}{\gamma}}z^{\frac{1-\gamma}{\gamma}}\right)^{s}\right\}^{\gamma} \\ &= \frac{1}{1-\gamma}(xz)^{1-\gamma}\left\{\frac{1}{1-\beta^{\frac{1}{\gamma}}z^{\frac{1-\gamma}{\gamma}}}\right\}^{\gamma}, \end{aligned} \tag{2.22}$$

with optimal choice

$$y^*(x,z)=(\beta z)^{\frac{1}{\gamma}}x, \tag{2.23}$$

and optimal consumption

$$c^*(x,z)=\left(1-\beta^{\frac{1}{\gamma}}z^{\frac{1-\gamma}{\gamma}}\right)xz. \tag{2.24}$$

Notice that the general formulas for the value function and the optimal assets and consumption, equations (2.17), (2.18), and (2.19) respectively, explicitly depend on all three parameters. The corresponding equations in (2.22), (2.23), and (2.24) are explicit functions of β *and* γ*. But they are functions of* τ *implicitly, because of the* τ*-quantile martingale process condition, which means that for a given risk attitude* τ*, the uncertainty is solved as* $Q_\tau[w|z]=z$*.*

To emphasize a point made earlier, we highlight that the results in Theorem 2.2.3 provide closed form solutions to the dynamic optimization quantile problem and may be very useful in empirical analysis. In contrast, it has been standard in the literature (see, e.g., Adda and Cooper (2003)) to use numerical methods to solve dynamic programming problems under the EU model. The need for numerical tools arises from the fact that, under the EU case, even very simple dynamic programming problems do not possess tractable closed form solutions.

Naturally, a different utility function could be used in place of the CRRA in Assumption 3-(iii). For instance, we could have considered the following conditions as a replacement for Assumption 3:

Assumption 4. *The following hold:*

(i) $\mathcal{X} = [0, \bar{x}] \subseteq \mathbb{R}_+$*;*

(ii) $\mathcal{Z} = [0, \bar{z}] \subseteq \mathbb{R}_{++}$*;*

(iii) $U : \mathbb{R}_{++} \to \mathbb{R}$ *is given by*

$$U(c) = -\frac{1}{\gamma} e^{-\gamma c} \tag{2.25}$$

where $\gamma > 0$*.*

Assumption 4-(iii) considers a Constant Absolute Risk Aversion (CARA) utility function. In this case, we have the following result:

Theorem 2.2.6. *Under Assumptions 2 and 4, there exists a unique continuous and bounded function* $V(x, z)$ *such that*

$$V(x, z) = \max_{y \in [0, zx]} \left\{ -\frac{1}{\gamma} e^{-\gamma(zx-y)} + \beta Q_\tau \left[V(y, z') | z \right] \right\}. \tag{2.26}$$

Moreover, V *is strictly increasing in both variables, strictly concave and differentiable in the first variable, and satisfies*

$$\frac{\partial V}{\partial x}(x, z) = z e^{-\gamma(zx - y^*(x,z))},$$

where $y^*(x, z)$ *denotes the optimal choice, which is single valued and continuous.*

Remark 2.2.7. *If one were concerned only with existence and uniqueness of a continuous and bounded* V *in Theorem 2.2.6, Assumption 2 would not be required.*

We now specialize the model for iid distributions and obtain a sharper characterization of the DM's behavior:

Theorem 2.2.8. *Let Assumptions 2 and 4 hold, and assume that the shocks are iid. Then the optimal policy* $y^*(x, z)$ *from (2.26) is strictly increasing in both variables, and the optimal consumption* $c^*(x, z)$ *is increasing in both variables, where*

$$c^*(x, z) = zx - y^*(x, z).$$

We conclude with an interesting expression relating the optimal consumption at times $t+1$ and t, c_{t+1} and c_t, and the risky asset return z_t, when the optimal consumption function $c^*(x, z)$ is increasing in the shock z. As seen in Theorem 2.2.8, this is the case, for instance, if we assume iid shocks:

Theorem 2.2.9. *Let Assumptions 2 and 4 hold. Moreover, assume that the optimal consumption $c^*(x, z)$ is increasing with respect to z. Then*

$$Q_\tau[c_{t+1}|z_t] = \frac{1}{\gamma}\log\left(Q_\tau[z_{t+1}|z_t]\right) + c_t + \frac{1}{\gamma}\log\beta. \tag{2.27}$$

The model in equation (2.27) is very similar to the well-known permanent income hypothesis (PIH) model in Hall (1978, 1988) and Flavin (1981) for the conditional expectations. Indeed, Hall (1988, equation (1), p. 342) writes the following equation resulting from an EU model and lognormal returns:

$$E[c_{t+1}|z_t] = \frac{1}{\gamma}\log\left(E[z_{t+1}|z_t]\right) + c_t + k, \tag{2.28}$$

adapting his notation to ours. Notice the similarities and differences between equations (2.27) and (2.28). First, expectations in (2.28) are substituted by conditional quantiles in (2.27). These quantiles obviously vary with τ, so that the actual number will vary with the quantile level τ chosen. This is true both for the quantile of the future consumption (c_{t+1}) and for the returns (z_{t+1}). Also, to obtain (2.28), Hall needs to assume not only the form of the utility, but also that the returns have log-normal distribution. In contrast, we do not need this extra assumption.

Generally the PIH predicts that consumption depends on permanent income, which is the annuity value of lifetime resources. If rational expectations are also assumed, the PIH implies that consumption follows a random walk, so that only consumption in the previous period contains information which can predict current consumption. Therefore, the DM adjusts current consumption immediately to the point where consumption is not expected to change, smoothing the consumption path. If one assumes that $z_t = z$ for all t, as it is common the literature, then equation (2.27) becomes a quantile (regression) version of the unit root model for the conditional average widely analyzed in the literature.

2.2.2 One-Sector Growth Model

Now we discuss another model to illustrate the dynamic quantile maximization framework: the one sector-growth model (see, e.g., Brock and Mirman (1972)). This model is fundamental in economic growth and development, see, e.g., Acemoglu (2009). We first present a model with capital and labor with general production and utility functions. Second, we specialize the model for a particular Cobb-Douglas production and a logarithmic utility function.

One-Sector Growth Model with Capital and Labor

Let $Y_t = F(K_t, L_t, z_t)$ denote the aggregate production function at time t, where $K_t \in \mathcal{X} \subseteq \mathbb{R}_+$ represents the capital, $L_t \in \mathbb{R}_{++}$ stands for labor, and z_t represents the shock. Assuming that F is homogeneous of degree one in both K and L, so that

$$\frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1, z_t\right) \equiv g(k_t, z_t), \tag{2.29}$$

where $k_t \equiv K_t/L_t \in \mathbb{R}_+$.[10]

At the beginning of period t, the decision-maker (DM) has $k_t \in \mathbb{R}_+$ units of the stock of capital normalized by labor, which depreciates at ratio δ. Given the technology $g(\cdot)$ and productivity shock $z_t \in \mathcal{Z} \subset \mathbb{R}_{++}$, the DM decides on the amount of consumption good, c_t, and the amount of capital for next period k_{t+1}. In this context, the DM problem with quantile preferences can be written as the following maximization:

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \left\{ U(c_t) + \beta Q_\tau \left[V(k_{t+1}, z_{t+1}) | z_t \right] \right\},$$

subject to the following constraints:

$$\begin{aligned} c_t + k_{t+1} &= g(k_t, z_t) + (1-\delta)k_t \\ c_t, k_{t+1} &\geqslant 0, \end{aligned} \tag{2.30}$$

where $U(\cdot)$ is the utility function, δ is the fraction from the existing capital stock that depreciates at each date, $g(\cdot)$ is the technology from equation (2.29), $\beta \in (0,1)$ is the discount factor, and the parameter $\tau \in (0,1)$ captures the risk attitude.

The corresponding functional equation to this problem can be written as

$$V(k_t, z_t) = \max_{k_{t+1} \in \Gamma(k_t, z_t)} \left\{ U\Big(g(k_t, z_t) + (1-\delta)k_t - k_{t+1}\Big) + \beta Q_\tau \left[V(k_{t+1}, z_{t+1}) | z_t \right] \right\}, \tag{2.31}$$

where $\Gamma(k_t, z_t) \equiv [0, g(k_t, z_t) + (1-\delta)k_t]$.

We have the following basic result:

Theorem 2.2.10. *Let Assumption 1 hold. Also, assume that U and g are both continuous and bounded. Then, there exists a unique value function V satisfying (2.31).*

To establish further properties of the model, we consider the following conditions.

Assumption 5. *The following hold:*

[10] We follow the usual notation in the literature to denote the production function as $F(K_t, L_t, A_t)$, although substituting the shock A_t by our z_t notation. However, when we divide F by L in (2.29), we define g instead of using f, to prevent confusion with the p.d.f. f that governs the Markov process.

(i) U *is continuously differentiable, bounded, strictly increasing and strictly concave;*

(ii) g *is continuously differentiable, nonnegative, strictly increasing in both variables and strictly concave in the first variable.*

Now we can establish the following:

Theorem 2.2.11. *Let Assumptions 1, 2 and 5 hold. Then,*

1. *The unique solution* V *to* (2.31) *is strictly increasing in both variables, strictly concave and differentiable in* k*, with optimal policy* $y^* = y^*(k,z)$ *strictly increasing in* k*;*

2. *if* $z_t \mapsto U'(g(k_t,z_t) + (1-\delta)k_t - k_{t+1})(\partial_k g(k_t,z_t) + 1 - \delta)$ *is an increasing function of* z_t*, then the solution satisfies the following Euler equation*

$$Q_\tau\left[\beta\frac{U'(c_{t+1})}{U'(c_t)}\Big(\partial_k g(k_{t+1},z_{t+1}) + 1 - \delta\Big)\Big|z_t\right] = 1, \tag{2.32}$$

where $\partial_k g$ *denotes the derivative of* g *with respect to* k *and* $k_{t+1} = y^*(k_t,z_t)$ *for* $t \geqslant 0$*, for a given* $(k_0,z_0) \in \mathcal{X}\times\mathcal{Z}$*.*

For relating the results in Theorem 2.2.11 with those of the standard recursive expected utility model, we write the latter problem as

$$V(x_t,z_t) = \max_{c_t,k_{t+1}}\Big\{U(c_t) + \beta E\left[V(x_{t+1},z_{t+1})|\,z_t\right]\Big\},$$

subject to the same constraints in (2.30). This problem can be rewritten and the associated value function is:

$$V(k_t,z_t) \quad = \quad \max_{k_{t+1}\in\Gamma(k_t,z_t)}\Big\{U\Big(g(k_t,z_t) + (1-\delta)k_t - k_{t+1}\Big) + \beta E\left[V(k_{t+1},z_{t+1})|\,z_t\right]\Big\}.$$

Finally, the Euler equation can be derived as

$$\begin{aligned}&-U'\Big(g(k_t,z_t) - k_{t+1} + (1-\delta)k_t\Big)\\&+\beta E\Big[U'\Big(g(k_{t+1},z_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}\Big)\Big(\partial_k g(k_{t+1},z_{t+1}) + 1 - \delta\Big)|\,z_t\Big] = 0.\end{aligned}$$

From this, we obtain

$$E\left[\beta\frac{U'(c_{t+1})}{U'(c_t)}\Big(\partial_k g(k_{t+1},z_{t+1}) + 1 - \delta\Big)\Big|z_t\right] = 1. \tag{2.33}$$

When comparing the Euler equations in (2.32) and (2.33) one can notice similarities and differences. The expressions inside the conditional quantile in (2.32) and the conditional expectation in (2.33) are the same. Therefore, inside the brackets, the number of parameters are equal. Nevertheless, notice that, for the quantile model, τ is a parameter that captures the risk attitude. This was discussed in some detail in the previous section.

Specializing the One-Sector Growth Model

Now we specialize the one sector-growth model with the quantile preferences above, and use a Cobb-Douglas production function and logarithmic utility function. This exercise is useful to illustrate the model further and also compare the proposed methods with the well-known expected utility (EU) case.

Consider a Cobb-Douglas production function for equation (2.29) as

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha},$$

where the parameter α captures the returns to scale. Recalling that $k_t \equiv K_t/L_t$ and dividing both sides by L_t, we obtain

$$\frac{Y_t}{L_t} = z_t k_t^{\alpha}. \tag{2.34}$$

Notice that this function satisfies Assumption 5 above for $\alpha \in (0, 1)$.

Using the notation introduced in the previous section, we can write the DM problem as

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \Big\{ U(c_t) + \beta Q_\tau \left[V(k_{t+1}, z_{t+1}) | z_t \right] \Big\},$$

subject to the following constraints:

$$\begin{aligned} c_t + k_{t+1} &= z_t k_t^{\alpha} + (1-\delta) k_t \\ 0 &\leqslant k_{t+1} \leqslant z_t k_t^{\alpha} + (1-\delta) k_t, \end{aligned}$$

where the variables and parameters are as in the previously.

The corresponding functional equation to this problem can be written as

$$V(k_t, z_t) = \max_{k_{t+1} \in \Gamma(k_t, z_t)} \Big\{ U\Big(z_t k_t^{\alpha} + (1-\delta) k_t - k_{t+1} \Big) + \beta Q_\tau \left[V(k_{t+1}, z_{t+1}) | z_t \right] \Big\}, \tag{2.35}$$

where $\Gamma(k_t, z_t) \equiv [0, z_t k_t^{\alpha} + (1-\delta) k_t]$.

Theorem 2.2.11 can be applied, and there is a unique solution to (2.35), which under the hypotheses of Theorem 2.2.11, satisfies the following Euler equation

$$Q_\tau \left[\beta \frac{U'(c_{t+1})}{U'(c_t)} \Big(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \Big) \Big| z_t \right] = 1. \tag{2.36}$$

Now we specialize the utility function with a logarithmic utility as

$$U(c) = \log c.$$

We also consider $\delta = 1$, and $\alpha, \beta \in (0, 1)$. The functional equation (2.35) becomes

$$V(k_t, z_t) = \sup_{k_{t+1} \in [0, z_t k_t^\alpha)} \left\{ \log \left(z_t k_t^\alpha - k_{t+1} \right) + \beta Q_\tau \left[V(k_{t+1}, z_{t+1}) | z_t \right] \right\}. \tag{2.37}$$

For this example, the next result derives an explicit formula for the value function as well as the optimal policy function. Notice that, even having $U(c) = \log(c)$ unbounded, we can still provide a solution.

Theorem 2.2.12. *Let Assumptions 1 and 2 hold. Then there is a unique solution to (2.37) given by*

$$V(k, z) = \sum_{s=0}^{\infty} \frac{\beta^s}{1 - \alpha\beta} \log q_{\tau,s}(z) + \frac{\alpha \log k}{1 - \alpha\beta} + \frac{\log \left[(\alpha\beta)^{\alpha\beta} (1 - \alpha\beta)^{1 - \alpha\beta} \right]}{(1 - \beta)(1 - \alpha\beta)}, \tag{2.38}$$

where $q_{\tau,s}(z)$ is given recursively by

$$\begin{aligned} q_{\tau,0}(z) &= z \\ q_{\tau,s}(z) &= q_{\tau,s-1}\left(Q_\tau[w|z]\right) \quad (s \geqslant 1). \end{aligned} \tag{2.39}$$

Moreover, the optimal policy is given by

$$y^*(k, z) = \alpha\beta z k^\alpha. \tag{2.40}$$

First, we note that the optimal policy result in equation (2.40) of Theorem 2.2.12 is the same as that for the EU case; see Acemoglu (2009, Example 17.1, p. 571). The optimal stochastic behavior is simply a fraction of the stock of capital k. Second, it is interesting to notice that although the value function in (2.38) depends on the risk attitude parameter, the optimal policy in (2.40) does not depend on τ. This is a consequence of the special logarithmic utility function. The log utility function fixes the elasticity of intertemporal substitution at one, and in this very special case, a recursive quantile term capturing the uncertainty separates from the decision variable, and hence, the optimal capital of equilibrium does not depend on the risk attitude τ. We highlight that this independence is a particular feature of the restrictions made above. Indeed, recall that in the example of intertemporal consumption in Section 2.2.1 above, one can see from equation (2.18) that when using a more general CRRA function, the optimal policy depends of the risk (τ), the elasticity of intertemporal substitution (γ), and the discount factor (β) parameters.

Remark 2.2.13. *Although the optimal policies from Theorems 2.2.3 and 2.2.12 are of the form* $y^*(x,z) = f(z)x^{\alpha}$, *where* α *is determined by the consumption* $c = (zx^{\alpha} - y)$, *this is a special case that only occurs when either* $\alpha = 1$ *and the utility is given by* $U(c) = c^{1-\gamma}/(1-\gamma)$ *or when* $U(c) = \log(c)$. *We refer the reader to a more detailed discussion in Discussion on Remark 2.2.13 in Appendix 2.2 – after the proof of Theorem 2.2.12.*

Now we note that equation (2.38) simplifies further when the shocks z are iid:

Example 2.2.14. *(iid) Assume that the shocks are iid. In this case,* $q_{\tau,s}(z) = Q_{\tau}[Z]$ *for all* $s \geqslant 1$. *Hence,* (2.38) *can be written as*

$$\begin{aligned} V(k,z) &= \sum_{s=1}^{\infty} \frac{\beta^s}{1-\alpha\beta} \log Q_{\tau}[Z] + \frac{\log z}{1-\alpha\beta} + \frac{\log k^{\alpha}}{1-\alpha\beta} + \frac{\log\left[(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\beta)(1-\alpha\beta)} \\ &= \frac{\log zk^{\alpha}}{1-\alpha\beta} + \bar{\kappa}, \end{aligned}$$

where

$$\bar{\kappa} = \frac{\log\left[(Q_{\tau}[Z])^{\beta}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\beta)(1-\alpha\beta)}$$

is a constant.

Analogously, we can treat the case of τ-quantile martingale process.

Example 2.2.15 (τ-quantile martingale process)**.** *When* Z *follows a* τ*-quantile martingale process (see equation* (2.6)*), the recursive functions from* (2.39) *are* $q_{\tau,s}(z) = z$ *for all* s, *so* (2.38) *takes the form*

$$\begin{aligned} V(k,z) &= \sum_{s=0}^{\infty} \frac{\beta^s}{1-\alpha\beta} \log z + \frac{\log k^{\alpha}}{1-\alpha\beta} + \frac{\log\left[(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\beta)(1-\alpha\beta)} \\ &= \frac{\log z}{(1-\beta)(1-\alpha\beta)} + \frac{(1-\beta)\log k^{\alpha}}{(1-\beta)(1-\alpha\beta)} + \frac{\log\left[(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\beta)(1-\alpha\beta)} \\ &= \frac{\log\left\{zk^{\alpha(1-\beta)}(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right\}}{(1-\beta)(1-\alpha\beta)}. \end{aligned}$$

2.2.3 Investment under Uncertainty with Convex Adjustment Costs

In this section, we first analyze a general dynamic optimization problem for investment with a convex adjustment costs. We start by presenting a standard investment model (see, e.g., Adda and Cooper (2003, Chapter 8) and Miao (2013, Chapter 8)) in a quantile setting. In this section, we follow the notation by Adda and Cooper (2003, Chapter 8) as close as possible. In particular, we denote next period variables by a prime symbol, instead of carrying the dependence with t.

The firm aims to maximize the value of the plant by choosing the variable factors of production that are rented for the production period, such as labor. Let K represent the stock of capital used by the firm; L, the variable factors, such as labor; and A, the shock to revenue and/or productivity. Let p denote the purchasing price of new capital, while w denotes the price of the variable factors L.

Demand for the variable inputs L is optimally determined given its factor prices w and the state variable and shock, represented by (A, K). The result of the firm's optimization leaves a profit function, denoted by $\Pi(A, K)$. This can be modeled using a profit function $\Pi(A, K)$ as

$$\Pi(A, K) = \max_{L} R(A, K, L) - wL,$$

where $R(A, K, L)$ denotes revenues given the capital input K and variable factors' price w. However, for simplicity, we are not interested in L and w and the analysis will not depend on them. Here, A and p are the shocks considered in this model.

To complete the model, we consider costs of adjustment given by $C(K', A, K)$, where K' stands for the next period choice of capital, while K denotes the current level of capital. The decision maker (DM) problem has the following functional equation

$$\begin{aligned} V(K, (A, p)) \quad = \quad & \max_{K' \in \Gamma(K)} \left\{ \Pi(A, K) - C(K', A, K) - p\left(K' - (1-\delta)K\right) + \right. \\ & \left. \beta Q_{\tau}\left[V(K', (A', p')) \middle| (A, p) \right] \right\}, \end{aligned} \tag{2.41}$$

where $\Gamma(K) = [(1-\delta)K, M]$ is the budget set such that M is just an upper bound for the capital, $\delta \in [0, 1]$ represents the rate of depreciation of capital stock, $\beta \in (0, 1)$ is the discounting factor, and τ-quantile parameter captures the risk attitude of the firm. Moreover, the capital accumulation equation is

$$K' = (1-\delta)K + I, \tag{2.42}$$

where I denotes the investment.

This model translates to the general dynamic quantile model in equation (2.7) with the following utility function

$$u(K, K', (A, p)) \equiv \Pi(A, K) - C(K', A, K) - p\left(K' - (1-\delta)K\right).$$

We have the following result concerning the existence of a solution to the dynamic programming problem:

Theorem 2.2.16. *Let Assumption 1 hold for the shock* (A, p) *and assume that* $\Pi(A, K) - C(K', A, K)$ *is continuous and bounded. Then, there exists a unique solution* V *to* (2.41), *which is continuous.*

Additional assumptions will enable us to tell more about the solution V. The major change is the condition that A and p are both functions of a common shock z, which will satisfy the following hypotheses, together with some properties of the profit and cost functions, Π and C:

Assumption 6. *The following hold:*

(i) $A : \mathcal{Z} \to \mathbb{R}$ *is strictly increasing and* $p : \mathcal{Z} \to \mathbb{R}$ *is strictly decreasing and both are continuous;*

(ii) z *is unidimensional and has a distribution satisfying Assumptions 1 and 2;*

(iii) $\Pi(A, K) - C(K', A, K)$ *is continuous in all variables, strictly increasing in* A*, continuously differentiable and strictly concave in both* K *and* K'*, and strictly increasing in* K*.*

With this assumption, (2.41) can be written as

$$V(K, z) = \max_{K' \in \Gamma(K)} \left\{ \Pi(A, K) - C(K', A, K) - p\left(K' - (1-\delta)K\right) + \beta Q_\tau \left[V(K', z') \middle| z \right] \right\}. \tag{2.43}$$

Moreover, we are now able to provide further characterizations of the value function and derive the Euler Equation for the investment model:

Theorem 2.2.17. *Under Assumption 6, the unique solution* $V(K, z)$ *to (2.43) is strictly increasing in both variables, strictly concave in* K*, and differentiable with respect to* K*. Furthermore, if* $K \mapsto \partial_2 \Pi(A, K) - \partial_3 C(K', A, K)$ *is increasing, it satisfies the Euler Equation*

$$\partial_1 C(K', A, K) + p = \beta Q_\tau \left[\partial_2 \Pi(A', K') - \partial_3 C(K'', A', K') + (1-\delta) p' \middle| z \right], \tag{2.44}$$

where $\partial_1 C$ *and* $\partial_3 C$ *denote the derivatives of* C *with respect to the first and last variables, respectively,* $\partial_2 \Pi$ *denotes the derivative of* Π *with respect to* K *and* $A' = A(z')$*,* $p' = p(z')$*, where* z' *denotes the next period shock.*

The Euler equation in (2.44) is similar to the usual expectation case, and also has a natural interpretation. First, notice that the main difference is that uncertainty here is resolved through the τ-quantile, which is the measure of risk attitude for the firm. Now, the left hand side of (2.44) is a measure of the marginal cost of capital accumulation, which includes the direct cost of new capital (p) as well as the marginal adjustment cost ($\partial_1 C(\cdot)$). The right hand side of the expression measures the τ-*quantile* of the marginal gains of more capital through the derivative of the value function. In the literature, this is conventionally termed "marginal q" or Tobin's q, after Tobin (1969).

As mentioned above, and discussed in the previous examples, equation (2.44) is closely related to the corresponding one from the expected utility (EU) model, where uncertainty is resolved by taking expectation in (2.43) instead of quantile. For completeness, we now present the Euler equation for the EU, which is given by

$$\partial_1 C(K', A, K) + p = \beta E\left[\partial_2 \Pi(A', K') - \partial_3 C(K'', A', K') + (1-\delta)p' \middle| z\right]. \tag{2.45}$$

Detailed discussion of model (2.45) can be found, for example, in Adda and Cooper (2003, p. 204). The main difference, however, is that in the quantile Euler equation (2.44) the τ parameter captures the risk attitude of the firm, and hence the model is able to account for this. On the other hand, the EU case requires that the firm is risk neutral.

To make further comparisons between the models, consider a particular case where Π is proportional to K, C is quadratic, and p is constant. That is, assume the following:

Assumption 7. *The following hold:*

(i) $\Pi(A, K) = AK$*;*

(ii) $C(K', A, K) = \frac{\gamma}{2}\left(\frac{K'-(1-\delta)K}{K}\right)^2 K$*;*

(iii) $A(z) = z$ *and* p *is constant;*

(iv) z *has a distribution satisfying Assumptions 1 and 2.*

In face of Assumption 7-(iii), we will simply write A for the shock instead of z. With this convention, we have the following results:

Theorem 2.2.18. *Under Assumption 7, the functional equation (2.43) has a solution of the form*

$$V(K, A) = \varphi(A)K,$$

where φ *is an increasing function satisfying the implicit relation*

$$\begin{aligned}\varphi(A) \quad = \quad & A - \frac{1}{2\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right]^2 - \frac{p}{\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right] \\ & +\beta\varphi\left(Q_\tau\left[A'|A\right]\right)\left[\frac{1}{\gamma}\left(\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right) + 1 - \delta\right].\end{aligned} \tag{2.46}$$

Moreover, the rate of investment,

$$i \equiv I/K$$

does not depend on the current capital level K*.*

The results in Theorem 2.2.18 show that the value function is proportional to the stock of capital and the investment rate is independent of the current level of the capital stock, under the linear-quadratic formulation of the capital accumulation assumption. These results are the same as in the EU case. In the empirical literature on investments, these separability results form the basis for a wide range of empirical exercises since they allow researchers to substitute the marginal q (unobservable) for the average value of Tobin's q (observable from the stock market). An important implication of Theorem 2.2.18 is that the the marginal q is equal to the average q. That is, the marginal q is $\partial_1 V(K, A)\varphi(A)$ and the average q is $V(K, A)/K = \varphi(A)$. This justifies, for this model, the usual practice of identifying the marginal q and the average q.

As in the previous sections, we consider the cases of iid and τ-quantile martingale process shocks.

Example 2.2.19 (iid)**.** *When A is iid, then $Q_\tau[A'|A] \equiv Q_\tau[A]$ is constant, so (2.46) can be solved and φ will have the form*

$$\varphi(A) = A + b$$

for some constant b.

Example 2.2.20 (τ-quantile martingale)**.** *If A follows a τ-quantile martingale process (see (2.6)), we have $Q_\tau[A'|A] = A$, so (2.46) implies a qudratic expression for φ of the form*

$$\varphi(A) = a - \sqrt{b - cA}$$

for some constants a, b, c.

These two examples illustrate that the function φ simplifies and can be linear (in the iid case).

2.2.4 Industry Investment under Demand Uncertainty

Now we modify the previous model to consider shocks in the demand rather than in the productivity, following the analysis of investment under uncertainty by Stokey et al. (1989, Example 10.4), which is a simplification of Lucas and Prescott (1971).

Consider an industry in which costs of production and of investment are certain and time-invariant, but with shocks to demand forming a stationary, first-order Markov process. The exogenous shocks $z \in \mathcal{Z}$ represent an index of the strength of demand. Demand itself will be specified by the inverse demand function $D : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+$, that is, $p = D(q, z)$ is the market clearing price when q is the aggregate quantity supplied and z determines the state of demand. We will include assumptions about D below. Let us define the function $U : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+$ by

$$U(q, z) = \int_0^q D(s, z)ds \quad \text{for all } q \in \mathbb{R}_+, z \in \mathcal{Z}. \tag{2.47}$$

Therefore, $U(q, z)$ is the gross consumer surplus (the total area below the demand curve) when q is the quantity consumed and z is the state of demand.

The endogenous state variable is the total industry capital stock, denoted by x. We assume that the output is produced using only capital as input. Therefore, without loss of generality, we can choose units so that aggregate industry output is equal to the aggregate industry capital stock, that is, $q = x$.

Costs are assumed to depend only on the rate of increase in the capital stock. If the current capital stock is $x > 0$ and next period's stock is $y > 0$, then the cost of investment is given by $xc(y/x)$, which is zero if next period's stock of capital is below the current level minus depreciation. This is formalized in the assumption below. To relate this model with the model in Section 2.2.3, we could say that

$$Kc\left(\frac{K'}{K}\right) = C(K', A, K) + p(K' - (1-\delta)K), \tag{2.48}$$

where we are using $x = K$, $y = K'$ and A is a constant in the current model and therefore omitted.

We require the following:

Assumption 8. *The following hold:*

(i) $D : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+$ *is continuous, strictly decreasing in* q *and strictly increasing in* z*, with*

$$D(0, z) > 0 \quad \textit{and} \quad \lim_{q\to\infty} D(q, z) = 0, \ \textit{for all } z \in \mathcal{Z};$$

(ii) there exists some $B < \infty$ *such that* $U : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}$ *defined in* (2.47) *satisfies*

$$U(q, z) \leqslant B, \ \textit{for all } (q, z) \in \mathbb{R}_+ \times \mathcal{Z};$$

(iii) $c : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuously differentiable and, for some* $\delta \in (0, 1)$*, satisfies* $c(a) = 0$ *on* $[0, 1-\delta]$ *and* c *is strictly increasing and strictly convex on* $(1-\delta, +\infty)$*;*

Lucas and Prescott (1971, Section 5) shows that the equilibrium can be obtained by maximizing the total surplus. We will not reproduce their derivation here, departing directly from the corresponding dynamic problem to our case, that is, we consider the value function that solves the following functional equation:

$$V(x, z) = \sup_{y\in\mathbb{R}_{++}} \left\{ U(x, z) - xc\left(\frac{y}{x}\right) + \beta Q_\tau \left[V(y, z') \middle| z \right] \right\}. \tag{2.49}$$

Our first task is to establish the existence of the value function satisfying (2.49). This is the content of the following:

Theorem 2.2.21. *Under Assumptions 1 and 8, there exists a unique bounded and continuous function* $V : X \times \mathcal{Z} \to \mathbb{R}_+$ *satisfying* (2.49).

Notice that the set where y is maximized in (2.49) is not compact. However, the proof of Theorem 2.2.21 shows that the optimal y indeed belongs to a compact set $\Gamma(x)$, and the standard techniques can be applied. Details are given in the appendix.

After obtaining the value function, we need an extra assumption to obtain the characterization

Theorem 2.2.22. *Let Assumptions 1, 2 and 8 hold. The optimal policy $y^* : \mathcal{X} \times \mathcal{Z} \mapsto \mathcal{X}$ to (2.49) is continuous, single-valued and strictly increasing in x.*

Moreover, the growth rate in aggregate capacity, $y^*(x,z)/x$, is strictly decreasing in current capacity x.

Furthermore, if $z \mapsto D(x,z) - c(y^*(x,z)/x) + (y^*(x,z)/x)\, c'(y^*(x,z)/x)$ is increasing, the solution satisfies the following Euler Equation:

$$c'\left(\frac{y^*(x,z)}{x}\right) = \beta Q_\tau\left[\partial_x U(x,z)|z\right] = \beta Q_\tau[D(x,z')|z]. \tag{2.50}$$

The result given in (2.50) shows that the marginal cost, c', is equal to the demand, the marginal consumer surplus. This is similar to the result in the previous section in equation (2.44), with the difference that as equation (2.48) shows, there is no explicit time to build in the investment in the present model, such that $p = 0$. Moreover, on the right hand side of equation (2.50) we have the marginal consumer surplus, instead of marginal profits.

2.2.5 Search with Unemployment

We now present a quantile-based version of the job-search model discussed in McCall (1970), see also Lippman and McCall (1976) and Albrecht and Axell (1984). This model strongly benefits from the new theoretical contributions from this chapter, and could not be dealt under the former theory from de Castro and Galvao (2019), due to the presence of a nontrivial law of motion for the endogenous state, as well as the discreteness of choices and shocks. In a labor market characterized by uncertainty and costly information, both employers and employees will be searching. The analysis presented here is directed to the employee's job-searching strategy.

The worker begins each period t with a wage offer w_t and has to decide if accepts the offer and works at that wage ($y_t = 1$) or refuses the offer ($y_t = 0$) and searches for a new one. Hence, the decision variable y_t takes discrete values in $\{0, 1\}$. If she decides to search, she earns nothing during the current period t, and a new wage offer $w_{t+1} \in [0, \bar{w}]$ will be her best option for the next period, when she will be making another choice between searching or working. This new wage offer w_{t+1} is modeled as a shock. If the worker chooses to work at period t, there is a chance that she looses her job ($e_{t+1} = 0$) in the next period $t + 1$, or keeps it ($e_{t+1} = 1$) and thus maintains the same wage x_t as in the previous period, where x_t denotes the effectively earned money at period t. The probability of losing the job (e_t), which can be interpreted as employer's decision, is also modeled as a shock.

Therefore, as discussed in equation (2.3) in Section 2.1.2, x_t satisfies the following law of motion:

$$x_{t+1} = \phi(x_t, y_t, e_{t+1}, w_{t+1}) = e_{t+1} x_t y_t + (1 - y_t) w_{t+1}. \tag{2.51}$$

The decision maker's problem can be represented by the the following functional equation:

$$V(x_t, z_t) = \sup_{y_t \in \{0,1\}} \{ y_t U(x_t) + \beta Q_\tau [V(\phi(x_t, y_t, z_{t+1}), z_{t+1}) | z_t] \}, \tag{2.52}$$

where $U(\cdot)$ denotes the utility over consumption.

We assume that the worker cannot lend nor borrow, so consumption will equal earnings x_t at each period t. The variable z is a vector $z_t = (e_t, w_t)$ representing the shocks concerning the employer's decision e_t of keeping the worker and the wage offer w_t resulting from the search.

Notice that this model presents a discrete decision variable y_t and a composite shock, which has a discrete part e_t and a continuous one w_t. Also, the state variable x_t is not directly given by the choice y_t. It follows, indeed, the law of motion (2.51), so the next period state x_{t+1} is affect by the shock $z_{t+1} = (e_{t+1}, w_{t+1})$, which occurs after the decision y_t is taken.

To establish the properties of the model we consider the following conditions:

Assumption 9. *$U : \mathcal{X} \to \mathbb{R}$ is continuous and bounded.*

Then, we can prove a result regarding the value function:

Theorem 2.2.23. *Under Assumptions 1 and 9, there exists an unique solution V to (2.52).*

Now we impose additional assumption on the model to provide further analysis:

Assumption 10. *The following hold:*

(i) *e is iid, with $P[e_t = 0] = \theta$, $P[e_t = 1] = (1 - \theta)$;*

(ii) *w is iid;*

(iii) *e and w are independent;*

(iv) *$Q_\tau[w'] > 0$;*

(v) *$U(x)$ is strictly increasing, $U(0) = 0$ and $\lim_{x\to\infty} U(x) = +\infty$;*

(vi) *$\mathcal{X} \subseteq [0, \bar{w}]$.*

Under Assumption 10, after the substitution $(e, w) = z$, the functional equation (2.52) can be written as

$$V(x_t, e_t, w_t) = \max \{ \beta Q_\tau [V(w_{t+1}, e_{t+1}, w_{t+1})], U(x_t) + \beta Q_\tau [V(e_{t+1} x_t, e_{t+1}, w_{t+1})] \}. \tag{2.53}$$

A closer inspection on (2.53) shows that its right-hand side does not depend on (e_t, w_t). Therefore, V is a function of x_t alone, and we can rewrite the functional equation as

$$V(x_t) = \max\left\{\beta Q_\tau\left[v(w_{t+1})\right], U(x_t) + \beta Q_\tau\left[v(e_{t+1}x_t)\right]\right\}. \tag{2.54}$$

We have the following result characterizing the value function as a function of the wage:

Theorem 2.2.24. *Under Assumptions 1, 9 and 10, there is a unique solution to (2.54) which is increasing and is given by*

$$V(x) = \begin{cases} A, & \text{if } x \leqslant x^* \\ \left(1 + \frac{\beta}{1-\beta}Q_\tau[e]\right)U(x) + (1 - Q_\tau[e])\beta A, & \text{if } x > x^* \end{cases} \tag{2.55}$$

where A is a constant given by

$$A = \frac{\beta\left(1 + \beta Q_\tau[e]\right)}{1 - \beta^2} U\left(Q_\tau[w]\right), \tag{2.56}$$

and x^* is the unique value satisfying

$$U(x^*) = A(1 - \beta). \tag{2.57}$$

This solution agrees with intuition. The DM has in mind a benchmark salary x^* given by (2.57). Whenever the wage offer is below this level, the worker decides to reject the offer and search for a new one. If, on the contrary, the DM receives an offer greater than x^*, the offer is accepted. It is worth notice that this critical wage x^* is directly proportional to τ, as deduced from expression (2.56) and the increasing Assumption 10-(v) for the utility U. Since the parameter τ captures the risk attitude of the DM, with greater values of τ meaning that the agent is more risk lover, this also agrees with intuition, since a risk loving DM will have a higher benchmark wage level x^*, and thus she will be more likely to engage in searching for a better salary, whereas a more risk averse DM is more likely to accept wage offers, since her benchmark x^* is lower.

2.3 General Theoretical Results

This section generalizes results in de Castro and Galvao (2019) and provides theoretical foundations for the results derived in the previous sections. Such generalizations are important for potential applications of dynamic economic models, thus substantially enlarging the scope of applicability of the recursive quantile model.

We begin by establishing the existence of the value function associated to the dynamic programming problem for the quantile preferences. We then present results on monotonicity, concavity, and differentiability of the value function. Finally, we derive the Euler equation.

Discussions and derivations for the dynamic consistency and the principle of optimality are provided in Sections 2.4.4 and 2.4.5, respectively.

2.3.1 Existence of the Value Function

We prove the existence of the value function through a contraction fixed point theorem. The first step is to define the contraction operator. For $\tau \in (0,1)$, define the transformation $\mathbb{M}^\tau : \mathcal{C} \to \mathcal{C}$ as

$$\mathbb{M}^\tau(v)(x,z) = \sup_{y \in \Gamma(x,z)} u(x,y,z) + \beta Q_\tau[v(\phi(x,y,w),w)|z]. \tag{2.58}$$

The functional in (2.58) is similar to the usual dynamic programming problem with the expectation operator $E[\cdot]$ instead of $Q_\tau[\cdot]$. We show that the above transformation has a fixed point, which is the value function of the dynamic programming problem. We need the following Assumption:

Assumption 11 (Contraction). *The following hold:*

(i) $\mathcal{X}$ *is a metric space;*

(ii) $\mathcal{Y}$ *is a metric space;*

(iii) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded;*

(iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is continuous;*

(v) The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ *is continuous, with nonempty, compact values.*[11]

Note that in the Assumption 11, the state space $\mathcal{X}$ is not required to be compact, neither convex nor Euclidean at this point by property (i). This allows the case where $\mathcal{X}$ is finite, for instance. The same is true for the action space $\mathcal{Y}$ due to condition (ii). Property (iii) is the standard continuity assumption. Condition (v) and the continuity of u and ϕ required in properties (iii) and (iv) guarantee that an optimal choice always exist. Moreover, Assumption 11 requires a condition on the law of motion in part (iv). This is required because now the choice variable is completely separated from the state variable, so now the agent chooses a contingent action plan, which could be influenced by the shock.

The main theoretical contribution of this chapter is the following result, whose proof, due to its extensiveness, can be found in Appendix A:

Lemma 2.3.1. *Let Assumptions 1 and 11 hold, and let* $\tau \in (0,1)$*. If* $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is bounded and continuous, then the map* $(x,y,z) \mapsto Q_\tau[v(\phi(x,y,w),w)|z]$ *is continuous.*

[11]Since at this point convexity is not required, we may have Γ finite-valued, representing the case where only finitely many options are available to the decision-maker at each period

The equivalent version of Lemma 2.3.1 in de Castro and Galvao (2019) was their Lemma A.5. However, their result was based in more restrictive hypotheses. Firstly, they assumed that $\mathcal{Z}$ was a convex subset of $\mathbb{R}^k$. Secondly, these shocks had to follow a Markov process based on a continuous, symmetric and strictly positive pdf $f(z'|z)$, that is, they considered transitions of the form $K(z, dz') = f(z'|z)dz'$. Finally, the argument carried there had an issue. The reader can check Remark A.2.1 in the Appendix A for details on this issue.

Nonetheless, the more general Assumption 1 is satisfied by the previous corresponding assumption considered in de Castro and Galvao (2019), hence their results are in fact true, as they benefit from an application of Lemma 2.3.1. This Lemma is central in the proof of the following result, which establishes the existence and uniqueness of the contraction $\mathbb{M}^\tau$ and whose proof, as the remaining ones from this chapter, can be found in Appendix A.

Theorem 2.3.2. *Under Assumptions 1 and 11, $\mathbb{M}^\tau$ is a contraction and has a unique fixed point $V \in \mathcal{C}$.*

An important restriction made in Assumption 1, which is used in Lemma 2.3.1 to ensure continuity of the solution V from Theorem 2.3.2, is that the metric space of shocks $\mathcal{Z}$ must be either connected or finite. Thus, whenever $\mathcal{Z}$ is continuous, it must be connected. This restriction is necessary, as the following counterexample shows:

Example 2.3.3. *Let $\mathcal{Z} = [0, a] \cup [b, \tau/2]$, where $0 < a < b < \tau/2$, $\tau \in (0, 1)$. Consider a transition function $K : \mathcal{Z} \times \Sigma \to [0, 1]$ given by*

$$K(z, A) = \int_A f(w|z)dw,$$

where

$$f(w|z) = \begin{cases} \frac{\tau - z}{a}, & \text{if } w \in [0, a] \\ \frac{1-\tau+z}{\tau/2-b}, & \text{if } w \in [b, \tau/2] \end{cases}$$

Notice that Assumption 1 is satisfied, since $\mathcal{Z}$ is compact, $f(w|z) > 0$ for all (w, z) and

$$|K(z, A) - K(z', A)| \leqslant \int_{\mathcal{Z}} |f(w|z) - f(w, |z')|dw \leqslant 2|z - z'|$$

for all $z, z' \in \mathcal{Z}$ and all $A \in \Sigma$.

Let $v(x, z) \equiv z$, and consider a sequence $z_n \equiv 1/n \to 0 = z^*$. We have

$$\Pr[w \leqslant \alpha|z] = \begin{cases} \frac{\tau - z}{a}\alpha, & \text{if } \alpha \in [0, a] \\ \tau - z, & \text{if } \alpha \in [a, b] \\ \tau - z + \frac{1-\tau+z}{\tau/2-b}(\alpha - b), & \text{if } \alpha \in [b, \tau/2] \end{cases}$$

Therefore,

$$Q_\tau[w|z^*] = a,$$

while

$$Q_\tau[w|z_n] = b + \frac{z_n(\tau/2 - b)}{1 - \tau + z_n} \to b.$$

Thus, $Q_\tau[v(x, w)|z]$ *is not continuous in* z.

Remark 2.3.4. *Later in Lemma 2.3.8 we will establish the same existence result from Theorem 2.3.2 for countable* $\mathcal{Z}$ *endowed with the discrete topology under Assumption 16.*

The fixed point from Theorem 2.3.2 is known as the value function. Below, we derive some sharper properties of this function, namely, monotonicity, concavity and differentiability, as well as single-valuedness of the policy correspondence. As will be seen, this will require progressive imposition of more structure over $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$.

2.3.2 Monotonicity

In this section we establish monotonicity of the value function with respect to the x and z variables. This section imposes only that the metric spaces $\mathcal{X}$ and $\mathcal{Y}$ are Euclidean, so monotonicity has a natural meaning. Thus, our results apply, for example, in the case where either $\mathcal{X}$ or $\mathcal{Y}$ (or both) are discrete, that is, the sates are discrete or the choices available to the decision maker are discrete. We start with some assumptions which will be used to prove strict increasingness of the value function concerning the state variable x:

Assumption 12 (Monotonicity in x)**.** *The following holds:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in the first variable;*

(ii) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is non-decreasing in the first variable;*

(iii) For every $z \in \mathcal{Z}$ *and* $x \leqslant x'$, $\Gamma(x, z) \subseteq \Gamma(x', z)$*;*

(iv) $\mathcal{X} \subset \mathbb{R}^p$;

(v) $\mathcal{Y} \subset \mathbb{R}^m$.

We have the following result:

Theorem 2.3.5. *Under Assumptions 1, 11 and 12, the value function* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in the first variable.*

It is also possible to establish strict increasingness of the value function with respect to the shocks z. In order to do this, since we are working with a general law of motion ϕ which may depend on z, it is necessary to have also increasingness of the value function in x, already treated in Theorem 2.3.5. That is the reason why the next assumption encloses the former one:

Assumption 13 (Monotonicity in both x and z)**.** *The following hold:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in the first and last variables;*

(ii) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is non-decreasing in the first and last variables;*

(iii) For every $x \in \mathcal{X}$ *and* $z \leqslant z'$, $\Gamma(x, z) \subseteq \Gamma(x, z')$*;*

(iv) For every $z \in \mathcal{Z}$ *and* $x \leqslant x'$, $\Gamma(x, z) \subseteq \Gamma(x', z)$*;*

(v) $\mathcal{X} \subset \mathbb{R}^p$;

(vi) $\mathcal{Y} \subset \mathbb{R}^m$.

Assumption 13 is mild. Assumption 13-(iv), as well as increasingness in x of the function u from part (i), are not necessary in the case where $\phi(x, y, z) = y$, that is, when the decision is taken after the shock and, therefore, corresponds to directly choosing next period's state. That is, in this special case one does not need to prove that V is increasing in x in order to have increasingness in z.

Theorem 2.3.6. *Under Assumptions 1, 2, 11 and 13, the value function* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in both variables.*

It should be noted that Theorem 2.3.6 holds not only for very general Euclidean $\mathcal{X}$ and $\mathcal{Y}$, which may be even discrete, but also for any $\mathcal{Z} \subset \mathbb{R}^k$ satisfying Assumptions 1 and 2, which allow, for instance, a multidimensional shock z. In the next section, where we establish concavity, more restrictions will be imposed over the sets $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$.

2.3.3 Concavity

In this section we establish concavity of the value function for both the continuous and discrete shock cases. Moreover, we show that the policy correspondence is single-valued and continuous for both cases.

Although different treatments are used depending on the nature of $\mathcal{Z}$, the following assumptions are common to both continuous and discrete shock scenarios. Since concavity depends on monotonicity, we will repeat the content of Assumptions 12 and 13.

Assumption 14 (Monotonicity and Concavity)**.** *The following holds:*

(i) $\mathcal{X} \subset \mathbb{R}^p$ *is convex;*

(ii) $\mathcal{Y} \subset \mathbb{R}^m$ *is convex;*

(iii) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in the first and last variables. Also, it is strictly concave in the first two variables.*

(iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is non-decreasing and concave in the first two variables, and non-decreasing in the last variable;*

(v) *For every* $x \in \mathcal{X}$ *and* $z \leqslant z'$, $\Gamma(x, z) \subseteq \Gamma(x, z')$*;*

(vi) *For every* $z \in \mathcal{Z}$ *and* $x \leqslant x'$, $\Gamma(x, z) \subseteq \Gamma(x', z)$*;*

(vii) *For all* $z \in \mathcal{Z}$ *and all* $x, x' \in \mathcal{X}$, $y \in \Gamma(x, z)$ *and* $y' \in \Gamma(x', z)$ *imply*

$$\theta y + (1 - \theta) y' \in \Gamma[\theta x + (1 - \theta) x', z], \text{ for all } \theta \in [0, 1].$$

Assumption 14-(vii) implies that $\Gamma(x, z)$ is a convex set for each $(x, z) \in \mathcal{X} \times \mathcal{Z}$, and that there are no increasing returns. Convexity in item (i) discards the possibility of finite-valued options of actions available to the decision-maker. Moreover, Assumption 14-(ii) excludes an only finite (neither countable) set of possible states for the endogenous variable. These limitations are not a concern since Assumption 14 is used to prove concavity of the value function on the first variable and also differentiability (with extra restrictions on the law of motion ϕ), and these properties are meaningless on a finite state setup. Once more, Assumption 14-(vi) is not necessary in the case where $\phi(x, y, z) = y$, that is, when the decision is taken after the shock and, therefore, corresponds to directly choosing next period's state.

The Continuous Shock Case

In addition to the common Assumption 14, to deal with the continuous shock scenario, we also assume the following:

Assumption 15. $\mathcal{Z} \subseteq \mathbb{R}$ *is an interval.*

To work with concavity, we restrict the dimension of the Markov process to $k = 1$. The next result establishes concavity of the value function in the continuous case.

Theorem 2.3.7. *If Assumptions 1, 2, 11, 14 and 15 hold, then* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *strictly increasing in both* x *and* z*, and strictly concave in* x*. Moreover, the policy correspondence* $\Upsilon(x, z) \subset \Gamma(x, z)$*, which maximizes* (2.58)*, is single-valued and continuous.*

The Discrete Shock Case

This time we will consider not only finite $\mathcal{Z}$, but also $\mathcal{Z}$ with at most countable many elements and endowed with the discrete topology. This extension in comparison with previous results for finite $\mathcal{Z}$ is due to the interchangeability property between quantiles and continuous, increasing functions, which is the content of Lemma A.1.1. Some modifications related to topological structure of functions acting on $\mathcal{Z}$ are needed in the case of discrete, at most countable shocks. In this case we will use the following condition.

Assumption 16. *The following holds:*

(i) $\mathcal{Z} \subset \mathbb{R}$ *is at most countable, endowed with the discrete topology, and it is also closed as a subset of* $\mathbb{R}$ *with respect to its usual topology (we say that* $\mathcal{Z}$ *is* $\mathbb{R}$*-closed);*

(ii) $z \mapsto Q_\tau\left[Z \middle| z\right]$ *is* $\mathbb{R}$*-continuous, that is, it is continuous in the usual topology on* $\mathbb{R}$*;*

(iii) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is* $\mathbb{R}$*-continuous (that is, continuous with respect to the usual topology on* $\mathbb{R}$*) in the last variable;*

(iv) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *is* $\mathbb{R}$*-continuous in the last variable;*

The $\mathbb{R}$-continuity in Assumption 16(i) is automatically satisfied if $\mathcal{Z}$ is finite, since in this case $\mathcal{Z}$ is closed in $\mathbb{R}$. The assumption is restrictive only for countable $\mathcal{Z}$, in which case, its accumulation points are required to also belong to $\mathcal{Z}$. This condition is used to prove Lemma A.1.1, which is used to prove concavity of the value function, a key property that allows further characterizations. If Assumption 16(i) is not required for countable $\mathcal{Z}$, one could find a counterexample where Lemma A.1.1 fails, since every function is continuous in the trivial discrete topology. Assumption 16(ii) is analogous to the Feller property for expectations, and it is used in the discrete case to prove concavity of the value function and single valuedness of the policy correspondence.

Since now we are dealing also with countable and discrete $\mathcal{Z}$, an existence result for a solution to the value function (as in Theorem 2.3.2) is need for this case. This is the content of the following Lemma, which requires monotonicity to hold:

Lemma 2.3.8. *Under Assumptions 2, 11, 13 and 16,* $\mathbb{M}^\tau$ *is a contraction and has a unique fixed point* $V \in \mathcal{C}$*.*

Now we state the analog of Theorem 2.3.7 for the discrete shock case and establish concavity of the value function.

Theorem 2.3.9. *If Assumptions 1, 2, 11, 14 and 16 hold, then* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in both* x *and* z*, and strictly concave in* x*. Moreover, the policy correspondence* $\Upsilon(x, z) \subset \Gamma(x, z)$*, which maximizes (2.58), is single-valued and continuous.*

2.3.4 Differentiability

Now we present results for differentiability of the value function with respect to the state variable x. In this case, two different approaches are needed depending on whether the choice space, $\mathcal{Y}$, is continuous or discrete. Nevertheless, both cases rely on the following common basic assumption:

Assumption 17. *The following hold:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is* C^1 *in the first variable;*

(ii) $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *does not depend on* x*, that is,* $\phi = \phi(y, z)$.

The second part of Assumption 17 imposes that the next period state can depend on the taken action y and the observed shock z, but not on the currently state x. The set of actions $\Gamma(x, z)$ available to the decision-maker may depend on x. The requirement is that in no other way the current state x can affect the next period state after an action y is picked and a shock z is realized. It is important to note that Assumption 17 is also required in the expected utility context; see Stokey et al. (1989, p. 270, item f).[12]

We note that, although we present separate results for the cases when $\mathcal{Y}$ is continuous or discrete below, no separation is needed for a continuous or discrete shock z, in opposition to the results in Section 2.3.3 presented above.

The Continuous Choice Case

For continuous $\mathcal{Y}$, the goal is to pursue the classical Benveniste and Scheinkman (1979)'s argument for differentiability. In order to achieve this, we must take a step back in generality concerning the law of motion ϕ, as seen in Assumption 17-(ii). This restriction let us prove the following:

Theorem 2.3.10. *Let Assumptions 1, 2, 11, 14 and 17 hold, together with Assumption 15 or 16, depending on whether* $\mathcal{Z}$ *is continuous or discrete, respectively. Then,* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *is differentiable in* x*, and*

$$\frac{\partial V}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z),$$

where $y^* \in \Gamma(x, z)$ *is the unique maximizer of* (2.58)*, assumed to be interior.*

This theorem, together with Theorems 2.3.7 and 2.3.9, delivers interesting and important properties of the value function. It shows that the value function that one obtains from quantile functions possesses, essentially, the same basic properties of the value function of the corresponding expected utility problem. Theorem 2.3.10 is very important for the characterization of the problem. It is the extension of the standard envelope theorem for the quantile preferences case. We adapt Benveniste and Scheinkman (1979)'s argument for showing differentiability of the value function from the expectation to the quantile case.

We note that Assumption 17 part (ii) is restrictive. Nevertheless, in the same way as in the expected utility case, in many practical applications this requirement is not necessary to establish differentiability of the value function.

[12] Blume et al. (1982) assume that the shock z_t is an argument of the law of motion ϕ, but z_t is not in Γ or the instantaneous utility function. Nevertheless, they apply different techniques to show that optimal plans can be obtained by an application of the Implicit Function Theorem to first order conditions.

The Discrete Choice Case

When the choice space $\mathcal{Y}$ is discrete, less assumptions are needed to establish differentiability. With discrete choices, we can no longer use the concavity of the value function v, so a different approach is needed to prove the result.

Theorem 2.3.11. *Let Assumptions 1, 11 and 17 hold. Fix $x \in \mathcal{X}$, $z \in \mathcal{Z}$. Assume that $x \in \mathcal{X}$ is an interior point where the optimal correspondence $\Upsilon(x,z) \subset \Gamma(x,z)$ is lower hemi-continuous.Then, $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ is differentiable in x and*

$$\frac{\partial V}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*,z),$$

where $y^* \in \Upsilon(x,z)$ is a maximizer of (2.58).

The lower hemi-continuity at (x,z) in Theorem 2.3.11 means that for every sequence $x_n \to x$, and every $y^* \in \Upsilon(x,z)$, there exists some sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $y_n \in \Upsilon(x_n,z)$ for every $n \in \mathbb{N}$ and $y_n \to y^*$. This condition is satisfied, for example, if $\Upsilon(x,z)$ is single valued at (x,z), that is, there is a unique maximizer y^* to (2.58) at (x,z).

2.3.5 Euler Equation

The final step is to characterize the solutions of the quantile recursive problem through the Euler equation. Let $v = V$ be the unique solution of the functional equation:

$$V(x,z) = \sup_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta Q_\tau[V(\phi(x,y,w),w)|z]\}. \tag{2.59}$$

By Theorem 2.3.10, if ϕ does not depend on x, V is differentiable in its first coordinate, satisfying $\frac{\partial V}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*,z)$.

Given that we have shown the differentiability of value function, we are able to apply the standard technique to obtain the Euler equation, as formalized in the following theorem:

Theorem 2.3.12. *Let Assumptions 1, 2, 11, 14 and 17 hold, together with Assumption 15 or 16, depending on whether $\mathcal{Z}$ is continuous or discrete, respectively. In addition, assume that both $u(x,y,z)$ and $\phi(y,z)$ are continously differentiable in each coordinate of the y variable. Let $(x_t,y_t,z_t)_{t\in\mathbb{N}}$ be a sequence of states, optimal decisions and shocks, such that $y_t \in \operatorname{int}\Gamma(x_t,z_t)$, and $z_t \mapsto \frac{\partial u}{\partial x}(x_t,y_t,z_t)\cdot\frac{\partial\phi}{\partial y_i}(y_{t-1},z_t)$ is strictly increasing. Then, the following first order condition (called Euler equation in this setting) necessarily holds for every $t \in \mathbb{N}$ and $i = 1,\ldots,m$:*

$$\frac{\partial u}{\partial y_i}(x_t,y_t,z_t) + \beta Q_\tau\left[\frac{\partial u}{\partial x}(x_{t+1},y_{t+1},z_{t+1})\cdot\frac{\partial\phi}{\partial y_i}(y_t,z_{t+1})\,\middle|\,z_t\right] = 0. \tag{2.60}$$

In the expression above, $\frac{\partial u}{\partial y_i}$ represents the derivative of u with respect to the i-th coordinate of its second variable (y) (that is, an unidimensional value) and $\frac{\partial u}{\partial x}$ represents the derivative of u with respect to its first variable (x) (that is, a p-dimensional vector). Since ϕ takes value on $\mathcal{X} \subset \mathbb{R}^p$, $\frac{\partial \phi}{\partial y_i}$ stands for the p-dimensional derivative vector of ϕ with respect to the i-th coordinate of y.

We could also rewrite (2.60) as follows:

$$\frac{\partial u}{\partial y_i}(x_t, y_t, z_t) + \beta Q_\tau \left[\sum_{j=1}^{p} \frac{\partial u}{\partial x_j}(x_{t+1}, y_{t+1}, z_{t+1}) \frac{\partial \phi_j}{\partial y_i}(y_t, z_{t+1}) \middle| z_t \right] = 0, \tag{2.61}$$

where ϕ_j stands for the j-th component of ϕ.

Theorem 2.3.12 provides the Euler equation, that is the optimality conditions for the quantile dynamic programming problem. This result is the generalization of the traditional expected utility to the quantile preferences. The Euler equation in (2.60) is displayed as an implicit function, nevertheless for any particular application, and given utility function, one is able to solve it explicitly as a conditional quantile function.

When $\phi(y, z) = y$ and we identify $\mathcal{X} \equiv \mathcal{Y}$, as in the model where the shock occurs before the decision-maker chooses his action, so in practice it is the same as considering his choice being directly the next period state, (2.60) simplifies to

$$\frac{\partial u}{\partial y_i}(x_t, y_t, z_t) + \beta Q_\tau \left[\frac{\partial u}{\partial x_i}(x_{t+1}, y_{t+1}, z_{t+1}) \middle| z_t \right] = 0.$$

The proof of Theorem 2.3.12 relies on a result about the differentiability inside the quantile function. Indeed, if h is differentiable and the derivative $\frac{\partial h}{\partial y_i}(y, Z)$ is integrable, then

$$\frac{\partial}{\partial y_i} \mathrm{E}[h(y, Z)] = \mathrm{E}\left[\frac{\partial h}{\partial y_i}(y, Z)\right], \text{ but } \frac{\partial}{\partial y_i} Q_\tau[h(y, Z)] \neq Q_\tau\left[\frac{\partial h}{\partial y_i}(y, Z)\right],$$

in general. However, de Castro and Galvao (2019) establish conditions under which the commutability of the two operations holds. See their paper for details.

2.4 Quantile Sequential Problem

This section provides additional characterization of the dynamic quantile model, especially in terms of the sequential problem. First, we define plans and the preference. Second, we define the sequence of recursive functions to show that the recursive quantile preference is well defined. Third, we establish dynamic consistency of the preferences. Finally, we show that the principle of optimality holds. These results are parallel extensions of de Castro and Galvao (2019) to the cases studied in this chapter, except for the definition of the recursive preferences, which are now carried in a different way. This new definition allows the DM to consider broader types of plans in comparison to de Castro and Galvao (2019), and thus the principle of optimality becomes stronger in this new setting.

2.4.1 Plans

At the beginning of period t, the decision-maker knows the current state x_t, and decides (according to preferences defined below) an action $y_t \in \Gamma(x_t, z_t) \subset \mathcal{Y}$, where $\Gamma(x, z)$ is the constraint set. We notice that, in de Castro and Galvao (2019) the agent's choice is restricted to be the future state variable. Here, the choice variable is completely separate from the state variable, and the agent chooses a contingent action plan, which could be influenced by the shock. From this, we can define plans as follows:

Definition 2.4.1. *A plan h is a profile $h = (h_t)_{t\in\mathbb{N}}$ where, for each $t \in \mathbb{N}$, h_t is a measurable function from $\mathcal{X} \times \mathcal{Z}^t$ to $\mathcal{Y}$.*[13] *The set of plans is denoted by H.*

In the Definition 2.4.1, a plan $h_t(x_t, z^t)$ represents the choice that the individual makes at time t upon observing the current state x_t and the sequence of previous shocks z^t. The following notation will simplify statements below.

Definition 2.4.2. *Given a plan $h = (h_t)_{t\in\mathbb{N}} \in H$, $x \in \mathcal{X}$ and realization $z^\infty = (z_1, \ldots) \in Z^\infty$, the sequence associated to (x, z^∞) is the sequence $(x_t^h)_{t\in\mathbb{N}^0} \in \mathcal{X}^\infty$ defined recursively by $x_1^h = x$ and $x_t^h = \phi(x_{t-1}^h, h_{t-1}(x_{t-1}^h, z^{t-1}), z_t)$, for $t \geqslant 2$. Similarly, given $h \in H$, $(x, z^t) \in \mathcal{X} \times Z^t$, the t-sequence associated to (x, z^t) is $(x_l^h)_{l=1}^t \in \mathcal{X}^t$ defined recursively as above. We write $y_t^h = h_t(x_t^h, z^t)$ for the choice taken at period t.*

We may write $x_t^h(\cdot)$, $x_t\left(x, z^t\right)$ or $x_t^h\left(x, z^\infty\right)$ to emphasize that x_t^h depends on the initial state x and on the sequence of shocks z^∞, up to time t.

Definition 2.4.3. *A plan h is feasible from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ if $h_t(x_t^h, z^t) \in \Gamma\left(x_t^h, z_t\right)$ for every $t \in \mathbb{N}$ and $z^\infty \in \mathcal{Z}^\infty$ such that $x_1^h = x$ and $z_1 = z$.*

We denote by $H(x, z)$ the set of feasible plans from $(x, z) \in \mathcal{X} \times \mathcal{Z}$. Let H denote the set of all feasible plans from some point, that is, $H \equiv \cup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} H(x, z)$.

2.4.2 Preferences

Now we briefly review the dynamic quantile preferences as discussed in de Castro and Galvao (2019).

Let Ω_t represent all the information revealed up to time t.[14] We assume that in time t with revealed information Ω_t, the consumer/decision-maker has a preference $\succcurlyeq_{t,\Omega_t}$ over plans

[13] In the expressions below, $h_0(z^0)$ should be understood as just $h_0 \in \mathcal{Y}$.

[14] With the knowledge of a fixed h, Ω_t reduces to the initial state x_1 and the sequence of shocks z^t. More generally, we could take the sequence of states and shocks (x^t, z^t).

$h, h' \in H(x, z)$, which is represented by a function $V_t : H \times \mathcal{X} \times Z^t \to \mathbb{R}$, that is,

$$h' \succcurlyeq_{t,x,\Omega_t} h \iff V_t(h', x, z^t) \geqslant V_t(h, x, z^t). \tag{2.62}$$

Notice that the preferences in (2.62) are time, information, and state contingent.[15]

de Castro and Galvao (2019) adapt the recursive equation for the expected utility case by replacing the expectation operator E with the quantile operator Q_τ, that is, the recursive quantile model is defined as:

$$V_t(h, x, z^t) = u(x_t^h, y_t^h, z_t) + \beta Q_\tau \left[V_{t+1}(h, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right], \tag{2.63}$$

where $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the current-period utility function.

The recursive equation (2.63) is the foundation of the dynamic quantile preferences and it leads to dynamically consistent preferences. In Section 2.4.3 below, we explicitly define a sequence of functions V_t for the more general cases this chapter examine that satisfy (2.63) and will specify the preferences (2.62). Nevertheless, we review the intuition on how the recursive equation (2.63) leads to an expression in quantiles that would be different from the standard expected utility case.

To see this, let $t = 1$ and substitute the expression of $V_{t+1} = V_2$ into the expression in (2.63) for V_1, and by continuing this process recursively we obtain:

$$\begin{aligned}
V_1(h, x, z^t) &= u(x_1^h, y_1^h, z_1) + \beta Q_\tau \left[V_2(h, x, z^t) \middle| Z_1 = z \right] \\
&= u(x_1^h, y_1^h, z_1) + \beta Q_\tau \left[u(x_2^h, y_2^h, z_2) + \beta Q_\tau \left[V_3(h, x, z^t) \middle| Z_2 = z_2 \right] \middle| Z_1 = z \right] \\
&= Q_\tau \left[Q_\tau \left[u(x_1^h, y_1^h, z_1) + \beta u(x_2^h, y_2^h, z_2) + \beta^2 V_3(h, x, z^t) \middle| Z_2 = z_2 \right] \middle| Z_1 = z \right] \\
&= Q_\tau \left[Q_\tau \left[Q_\tau \left[\sum_{t=1}^{3} \beta^{t-1} u(x_t^h, y_t^h, z_t) + \beta^3 V_4(h, x, z^t) \middle| Z_3 = z_3 \right] \middle| Z_2 = z_2 \right] \middle| Z_1 = z \right] \\
&= Q_\tau \left[\cdots Q_\tau \left[\sum_{t=1}^{n} \beta^{t-1} u(x_t^h, y_t^h, z_t) + \beta^n V_{n+1}(h, x, z^t) \middle| Z_n = z_n \right] \middle| \cdots \middle| Z_1 = z \right]
\end{aligned} \tag{2.64}$$

where the operator $Q_\tau[\cdot]$ and corresponding conditionals $Z_t = z_t$ appear n times in the last line above. In order to simplify the above equation, we use the following notation:

$$Q_\tau^n[\cdot] \equiv Q_\tau \left[\cdots \left[Q_\tau \left[\, \cdot \, \middle| Z_n = z_n \right] \middle| \cdots \right] \middle| Z_1 = z \right], \tag{2.65}$$

where the operator Q_τ and corresponding conditionals appear n times. Therefore, by using

[15] A special case of this model corresponds to the standard case of expected utility, that is, $V_t(h, x, z^t) = u(x_t^h, y_t^h, z_t) + \beta E\left[V_{t+1}(h, x, (Z^t, z_{t+1})) \middle| Z^t = z^t \right]$.

the notation defined by (2.65), we are able to rewrite (2.64) as

$$V_1(h,x,z^t) \;=\; Q^n_\tau\left[\sum_{t=1}^{n}\beta^{t-1}u(x^h_t,y^h_t,z_t)+\beta^n V_n(h,x,z^t)\right]. \tag{2.66}$$

The next step is to take the limit as n goes to ∞. The formalization of such limit will be made in Section 2.4.3 below, but one can now intuitively understand the following:

$$V_1(h,x,z^t) \;=\; Q^\infty_\tau\left[\sum_{t=1}^{\infty}\beta^{t-1}u(x^h_t,y^h_t,z_t)\right], \tag{2.67}$$

as a notation for an (infinite) sequence of applications of $Q^n_\tau[\cdot|\,Z^t=z^t]$.

It is worth mentioning that in the particular case in which the z_t are independent, (2.66) and (2.67) can be simplified. Notice that independence implies

$$Q_\tau[u(x_t,y_t,z_t)|\,z_{t-1}]=Q_\tau[u(x_t,y_t,z_t)],$$

which is a number, not a random variable. Being a number, it can be taken out of the quantile. Thus, (2.66) simplifies to:

$$V_1(h,x,z^t) \;=\; \sum_{t=1}^{n}\beta^{t-1}Q_\tau\left[u(x^h_t,y^h_t,z_t)\right]+\beta^n Q_\tau\left[V_n(h,x,z^t)\right],$$

and (2.67) simplifies to

$$V_1(h,x,z^t) \;=\; \sum_{t=1}^{\infty}\beta^{t-1}Q_\tau\left[u(x^h_t,y^h_t,z_t)\right].$$

2.4.3 The Sequence of Recursive Functions

In this section, we follow a different approach than de Castro and Galvao (2019). Our goal is the same: define the sequence of functions V_t that satisfy (2.63) and specify the preferences (2.62). In order to achieve this, de Castro and Galvao (2019) fixed a plan $h\in H$ and defined a particular transformation $\mathbb{T}^h$ acting over continuous an bounded functions $\mathcal{C}$ from $\mathcal{X}\times\mathcal{Z}$ to $\mathbb{R}$, endowed with the sup norm. Their definition of $\mathbb{T}^h$, which were carried out in a context where the law of motion was simply $\phi(x,y,z)=y$, can be rephrased in our more general context in the following way:

$$\mathbb{T}^h(V)(x,z)=u\left(x^h_1,y^h_1,z_1\right)+\beta Q_\tau[V(x^h_2,Z_2)|\,Z_1=z],$$

where $\left(x^h_1,z_1\right)=(x,z)$, $y^h_1=h_1(x,z)$ and $x^h_2=\phi\left(x^h_1,y^h_1,z\right)$.

Then de Castro and Galvao (2019) showed that $\mathbb{T}^h(\mathcal{C})\subset\mathcal{C}$ and that $\mathbb{T}^h$ is a contraction, so it has a unique fixed point V^h. Thereafter, they defined V_t as follows:

$$V_t(h, x, z^t) = V^h(x_t^h, z_t), \tag{2.68}$$

where $(x_l^h)_{l=1}^t$ is the associated t-sequence to (x, z^t) (see Definition 2.4.2). This completes the definition of the preferences (2.62) from de Castro and Galvao (2019).

The first problem with the usage of the transformation $\mathbb{T}^h$ acting over $\mathcal{C}$ is that, in order to ensure $\mathbb{T}^h(\mathcal{C}) \subset \mathcal{C}$, we must restrict ourselves to plans $h = \{h_t\}_{t\in\mathbb{N}} \in H$ where h_t is continuous for each $t \in \mathbb{N}$. This represents a step back in generality, since we defined plans imposing only that h_t is measurable.

Nevertheless, this is a minor restriction which could be bypassed by taking $\mathcal{B}$, the set of bounded functions with the sup norm, to be the domain of $\mathbb{T}^h$. In this scenario, V^h would be a fixed point of $\mathbb{T}^h$ on $\mathcal{B}$. Unfortunately, though, we would still have another issue with that.

Since our final goal is to define the functions V_t, where t may be, of course, greater than 1, we are in trouble whenever we deal with plans h which are not stationary, that is, whenever $h_t \neq h_1$. This is because the recursive equation satisfied by the fixed point V^h is

$$\begin{aligned} V^h(x,z) &= u\left(x_1^h, y_1^h, z_1\right) + \beta Q_\tau[V^h(x_2^h, Z_2) | Z_1 = z] \\ &= u(x, h_1(x,z), z) + \beta Q_\tau[V^h(\phi(x, h_1(x,z), Z_2) | Z_1 = z]. \end{aligned} \tag{2.69}$$

Thus, taking $x = x_t^h$, $z = z_t$ for $t > 1$ in (2.69) may lead to trouble, since we could have $y_t^h = h_t(x_t^h, z_t) \neq h_1(x_t^h, z_t)$ if $h_t \neq h_1$, that is, when $h \in \mathcal{H}$ is not stationary. Therefore, the definition (2.68) would result in preferences satisfying the recursive relation (2.63) only for plans $h \in H$ such that $h_t \equiv h_1$ for all $t \in \mathbb{N}$.

This is a strong restriction which weakens the Principle of Optimality. This principle ideally states that, essentially, the best plans that the decision maker can take are stationary plans. So, departing already from stationary plans represents a serious issue in establishing this principle.

Fortunately, these restrictions can be overcome. Namely, it is possible to define preferences $V_t(h, x, z)$ which work with measurable and non-stationary h_t, and attain a Principle of Optimality where the optimal decision is to take stationary plans, that is, plans such that $h_t \equiv h_1$.

The way to do this is just defining directly V_t without using the operator $\mathbb{T}^h$, which is responsible for the restrictions in the model from de Castro and Galvao (2019).

Based on previous ideas already contained in de Castro and Galvao (2019), we start defining V_t:

Proposition 2.4.4. *Let $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ be a bounded function, and let $\beta \in (0,1)$. Fix some* *plan* $h \in \mathcal{H}$*. For* $n \in \mathbb{N}$*, let*

$$\begin{aligned} V_t(h,x,z^t) &\equiv \lim_{n\to\infty} Q_\tau^n\left[\sum_{s=t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right] \\ &\equiv \lim_{n\to\infty} Q_\tau\left[\ldots\left[Q_\tau\left[\sum_{s=t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\,\middle|\,Z_{n+t-1}=z_{n+t-1}\right]\ldots\right]\middle|\,Z_t=z_t\right] \end{aligned} \tag{2.70}$$

Then the limit in (2.70) *exists, so it is well defined, and satisfies the recursive relation* (2.63)*.*

Proof of Proposition 2.4.4: Since u is bounded, its sup norm $\|u\|_\infty$ is finite. Let

$$V_t^n(h,x,z^t) \equiv Q_\tau^n\left[\sum_{s=t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right].$$

We will show that $\left\{V_t^n(h,x,z^t)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume that $n > m$ are natural numbers. Then

$$\begin{aligned} V_t^n(h,x,z^t) &= Q_\tau^n\left[\sum_{s=t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right] \\ &= Q_\tau^n\left[\sum_{s=t}^{m+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)+\sum_{s=m+t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right] \\ &\leqslant Q_\tau^n\left[\sum_{s=t}^{m+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)+\frac{\beta^m}{1-\beta}\|u\|_\infty\right] \\ &= Q_\tau^n\left[\sum_{s=t}^{m+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right]+\frac{\beta^m}{1-\beta}\|u\|_\infty \\ &= Q_\tau^m\left[\sum_{s=t}^{m+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right]+\frac{\beta^m}{1-\beta}\|u\|_\infty \\ &= V_t^m(h,x,z^t)+\frac{\beta^m}{1-\beta}\|u\|_\infty, \end{aligned} \tag{2.71}$$

where the inequality comes from the monotone property of quantiles, namely, $Q_\tau[f(w)|z] \leqslant Q_\tau[g(w)|z]$ if $f \leqslant g$, and the change from Q_τ^n to Q_τ^m is due to the nonoccurence of $z_{m+t},\ldots,z_{n+t-1}$ inside the $Q_\tau^n[\cdot]$ operator after our manipulations.

Analogously, one proves that

$$V_t^n(h,x,z^t) \geqslant V_t^m(h,x,z^t) - \frac{\beta^m}{1-\beta}\|u\|_\infty. \tag{2.72}$$

Therefore, (2.71) and (2.72) imply

$$\left|V_t^n(h,x,z^t)-V_t^m(h,x,z^t)\right| \leqslant \frac{\beta^m}{1-\beta}\|u\|_\infty,$$

which clearly establishes that $\left\{V_t^n(h,x,z^t)\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, expression (2.70) is well defined. Indeed, we proved that $V_t^n(h,\cdot,\cdot)$ converges uniformly to $V_t(h,\cdot,\cdot)$.

To prove that V_t satisfies (2.63), we compute

$$\begin{aligned}
V_t(h,x,z^t) &= \lim_{n\to\infty} Q_\tau^n\left[\sum_{s=t}^{n+t-1}\beta^{s-t}u(x_s^h,y_s^h,z_s)\right]\\
&= \lim_{n\to\infty} Q_\tau^n\left[u(x_t^h,y_t^h,z_t)+\beta\sum_{s=t+1}^{(n-1)+(t+1)-1}\beta^{s-(t+1)}u(x_s^h,y_s^h,z_s)\right]\\
&= u(x_t^h,y_t^h,z_t)+\beta\lim_{n\to\infty} Q_\tau^n\left[\sum_{s=t+1}^{(n-1)+(t+1)-1}\beta^{s-(t+1)}u(x_s^h,y_s^h,z_s)\right]\\
&= u(x_t^h,y_t^h,z_t)+\beta\lim_{n\to\infty} Q_\tau\left[Q_\tau^{n-1}\left[\sum_{s=t+1}^{(n-1)+(t+1)-1}\beta^{s-(t+1)}u(x_s^h,y_s^h,z_s)\right]\middle|z_t\right]\\
&= u(x_t^h,y_t^h,z_t)+\beta\lim_{n\to\infty} Q_\tau\left[V_{t+1}^{n-1}(h,x,z^{t+1})\middle|z_t\right]\\
&= u(x_t^h,y_t^h,z_t)+\beta Q_\tau\left[\lim_{n\to\infty} V_{t+1}^{n-1}(h,x,z^{t+1})\middle|z_t\right]\\
&= u(x_t^h,y_t^h,z_t)+\beta Q_\tau\left[V_{t+1}(h,h,z^{t+1})\middle|z_t\right],
\end{aligned}$$

where we took out $u(x_t^h,y_t^h,z_t)$ from inside the $Q_\tau^n[\cdot]$ operator because the first conditional is over z_t. Moreover, we could pass the limit into the quantile because of Lemma A.1.5, since $V_{t+1}^{n-1}(h,\cdot,\cdot)$ converges uniformly to $V_{t+1}(h,\cdot,\cdot)$ by the first part. This concludes the proof.

□

We turn now to verify that this preference is dynamically consistent.

2.4.4 Dynamic Consistency

In this Section we formally define dynamic consistency and show that it is satisfied by the above defined dynamic quantile preferences. The following definition is from Maccheroni et al. (2006b); see also Epstein and Schneider (2003).

Definition 2.4.5 (Dynamic Consistency). *The system of preferences* $\succcurlyeq_{t,\Omega_t}$ *is* dynamically consistent *if for every* t *and* Ω_t *and for all plans* h *and* h', $h_{t'}(\cdot)=h'_{t'}(\cdot)$ *for all* $t'\leqslant t$ *and* $h'\succcurlyeq_{t+1,\Omega'_{t+1},x} h$ *for all* Ω'_{t+1},x, *implies* $h'\succcurlyeq_{t,\Omega_t,x} h$.

To show dynamic consistency for the expected utility preferences it is standard to appeal to the law of iterated expectations. Unfortunately, an analogue of such law does not hold for quantiles, see e.g. Examples 3.7 and 3.8 in de Castro and Galvao (2019).

Failure of law of iterated quantiles could suggest that quantile preferences would be dynamically inconsistent. To avoid this dynamic inconsistency, we adopted the iterated quantile preference (2.70), that is $Q_\tau^\infty\left[\sum_{t=0}^{\infty}\beta^t u(x_t^h, y_t^h, z_t)\right]$, which involves an infinite sequence of nested conditional quantiles – see Section 2.4.2 for the notation $Q_\tau^\infty[\cdot]$. This sequence is exactly what allows to obtain dynamic consistency. Indeed, in our framework, quantile preferences are dynamically consistent and amenable to the use of the standard techniques of dynamic programming, as the following result establishes.

Theorem 2.4.6. *The quantile preferences defined by* (2.70) *are dynamically consistent.*

The proof of this theorem is completely analogous to the former version stated at de Castro and Galvao (2019), so we omit it here. Moreover, de Castro and Galvao (2019) provide an example illustrating Theorem 2.4.6 and show the recursive structure guarantees dynamic consistency. This result is important, because many preferences that departure from the expected utility framework do not satisfy dynamic consistency.

Our approach to establish dynamic consistency is similar to that taken by Epstein and Schneider (2003) for the maximin expected utility dynamic preferences, in the sense that the filtration of events where decisions are made is fixed. As discussed by Strzalecki (2013, p. 1048), this is one of the main approaches that have been used to obtain dynamic consistency for different preferences.

We also note that Epstein and Le Breton (1993) essentially prove that dynamic consistent preferences are "probabilistic sophisticated" in the sense of Machina and Schmeidler (1992). Probabilistic sophistication *roughly* means that the preference is "based" in a probability.[16] Extending Machina-Schmeidler's definition, Rostek (2010) shows that the static quantiles preferences are probabilistic sophisticated for $\tau \in (0,1)$. Her observation is also valid for our dynamic quantile preference. However, we do not use these developments, since Theorem 2.4.6 offers a direct proof of dynamic consistency.

2.4.5 The Principle of Optimality

This section establishes that the principle of optimality holds in our model. That is, optimizing period after period, as in the recursive problem in equation (2.58), yields the same result as choosing the best plan for the whole horizon of the problem. This principle and its proof follows the same basis as stated in de Castro and Galvao (2019), to where we send the reader interested in its proof.

As pointed out in Section 2.4.3, the method used to define the recursive functions in de Castro and Galvao (2019) allowed only stationary and continuous plans. Since the Principle of Optimality ideally departs from non-stationary and measurable plans and aims to prove that the best policy is to take stationary plans (which under further hypotheses will also be

[16] See also Karni and Schmeidler (1991).

continuous), here we provide a more general derivation of the recursive functions in Section 2.4.3. The methods employed by de Castro and Galvao (2019) to prove their Principle of Optimality remain valid in the present case, since all the arguments they used rely solely in the recursive equation (2.63), not in the definition of the recursive functions. Therefore, there is no need to repeat the proof of the Principle of Optimality here, since the only difference is due to the introduction of the law of motion ϕ, which inserts y_t^h in place of x_{t+1}^h in their equations. But it is important to keep in mind that, in this new context with a new derivation of the recursive functions in Section 2.4.3, the reach of the Principle of Optimality is broader.

Although we are not writing the proofs of this Section, we find it useful to reproduce some equations that are being used in the current chapter and illustrate the derivations made in de Castro and Galvao (2019). Let us begin by defining the set of feasible plans departing from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ at time t:

$$H_t(x,z) \equiv \{h \in H(x,z) : \exists (x, z^t) \in \mathcal{X} \times \mathcal{Z}^t, \text{ with } z_t = z, \text{ such that } x_t^h(x, z^t) = x\}.$$

Thus we can define a supremum function as:

$$v_t^*(x,z) \equiv \sup_{h \in H_t(x,z)} V_t(h, x, z). \tag{2.73}$$

We first observe that t plays no role in the above equation (2.73), that is, we are able to drop the subscript t from (2.73) and write $v^*(x,z)$ instead of $v_t^*(x,z)$.

The next step is to relate v^* to V, the solution of the functional equation studied in Section 2.3, which was proved to exist in Theorem 2.3.2 and satisfies the Bellman equation (2.59). In this direction, we have the following result:

Proposition 2.4.7. *Let V be a bounded and continuous solution to* (2.59). *Let $y^* \in \Upsilon(x,z)$ be a maximizer of V at (x,z), that is,*

$$V(x,z) = u(x, y^*(x,z), z) + \beta Q_\tau \left[V(\phi(x, y^*(x,z), z'), z') | z \right].$$

Consider the plan $h \in H$ given by

$$h_t(x_t, z^t) = y^*(x, z_t).$$

Then $v^* = V$, and h defined above attains the supremum in (2.73).

Proof of Proposition 2.4.7: See the proof of Proposition 3.17 in de Castro and Galvao (2019, p. 1935). □

Thus, the principle of optimality provides sufficient conditions for a solution v to the functional equation be the supremum function. Again, we refer the reader to de Castro and Galvao (2019) for details.

2.5 Summary and Open Questions

This chapter develops a dynamic model of rational behavior under uncertainty for an agent maximizing the quantile function indexed by $\tau \in (0,1)$. More specifically, an agent maximizes the stream of future τ-quantile utilities. We show dynamic consistency of the recursive quantile preferences and that this dynamic problem yields a value function, using a fixed-point argument. We also obtain desirable properties of the value function. In addition, we derive the corresponding Euler equation.

We use five canonical economic models, that are central to contemporary economics and finance, to illustrate the importance of economic dynamics with a recursive quantile model.

Many issues remain to be investigated. Extensions of the methods to general equilibrium models and to the cases $\tau = 0$ and $\tau = 1$ are left for future research. In addition, other types of aggregation of the quantile preferences is also an interesting direction for future research. Another interesting avenue would be the relationship between the quantile preferences model and more general rank-dependent models of choice under uncertainty.

Chapter 3

Dynamic Economics with General Operator

3.1 Introduction

The previous chapter developed extensively dynamic programming for quantile preferences. There, we investigated five classical examples from the literature under the quantile framework. It is of interest to know how other types of operators instead of quantile and expectation would behave with respect to dynamic programming.

Since dynamic economic models are now routinely used in many fields, such as macroeconomics, finance, international economics, public economics, industrial organization, labor economics, scenario-based analysis, among others, this chapter provides general conditions to study general recursive models for economic analysis.

Previous work from Marinacci and Montrucchio (2010) showed conditions under which a very general recursive setting given by

$$V(c) = I(c, M(V)),$$

where I is an aggregator function and M is a general operator related to the uncertainty resolution, would have a unique solution V via a fixed point argument. We want to specialize this Bellman equation to the case where I is additive, that is, it assumes the form

$$I(c, M(V)) = u(c) + \beta M(V),$$

and establish further properties of the solution V, as well as provide conditions to properly define a sequential problem which is connected to the functional Bellman equation via a principle of optimality.

Thus, the first main contribution of this chapter to the literature is to provide general conditions to dynamic programming models with additive aggregator and a general operator that solves the uncertainty in the model.

Given this general environment, we provide conditions that are verifiable to derive important theoretical properties of the general dynamic model. In particular, we provide conditions to show that the optimization problem leads to a contraction, which therefore has a unique fixed-point. This fixed point is the value function of the problem and satisfies the Bellman equation. Although the fixed point existence is already established in Marinacci and Montrucchio (2010), in our specialized setting we are able to prove that, under some conditions over the utility u, the value function is monotone, concave and differentiable, thus establishing the analog of the envelope theorem. Then, using these results, we derive the corresponding Euler equation for the infinite horizon problem. Finally, we provide conditions to establish a well defined sequential problem whose economic meaning is to give rise to dynamically consistent general operator preferences. Moreover, we show how this general sequential problem connects to the general additive Bellman equation by means of a principle of optimality.

The second main contribution of this chapter is to provide examples to illustrate the usefulness and generality of the recursive model. In particular, we use different statistics measures to solve and model the uncertainty, such as the expectation, the quantile, expectile and cumulative prospect theory. The expectation is a very well-known and simple measure of centrality. It is simple and intuitive, moreover it surmises to the well-known expected utility model. We also investigate how the methods from the previous chapter concerning the quantile as the main statistic solving the uncertainty suits our general methods. We also verify how expectiles, introduced by Newey and Powell (1987), adapts to our techniques, thus providing a complete dynamic programming theory for expectiles. We also investigate where some risk measures fail to fit our model. First we use the mode as a statistic. The mode is the value or number that has the highest frequency. It is also a measure of centrality. We show how usual definitions of mode contradicts our hypotheses for general dynamic programming. We also investigate how prospect theory, introduced by Kahneman and Tversky (1979), also fails to fit our general methods. Nevertheless, the improvement of prospect theory introduced in Tversky and Kahneman (1992), known as cumulative prospect theory, fits our general methods, although some minor adaptions are needed when dealing with the sequential problem.

3.2 General Results

In this section, we treat the functional equation problem in greater generality. Our object of study will be a family of operators $A_x : \mathcal{C}(x,z) \to \mathcal{C}(x,y,z)$, and we will state sufficient conditions on these operators such that the problem

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta(A_x V)(y,z)\} \tag{3.1}$$

has a unique solution. Here, $\mathcal{C}(x,z)$ denotes the set of bounded and continuous real-valued functions $f : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, whereas $\mathcal{C}(x,y,z)$ denotes the corresponding set in three variables, that is, functions $g : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$. When made clear by the context, we will simply write $\mathcal{C}$ for each of these sets.

The family of operators $A = (A_x)_{x\in\mathcal{X}}$ works as follows: for a fixed $x \in \mathcal{X}$, $A_x : \mathcal{C}(x,z) \to \mathcal{C}(y,z)$ will take $f \in \mathcal{C}(x,z)$ and generate a function $A_x f : \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ which is bounded and continuous. When one also varies x, $A_x f(y,z)$ becomes a function in three variables, and it is required that this function is also bounded and continuous in the three variables (x,y,z) together.

Moreover, we will impose further conditions so the solution $V(x,z)$ to (3.1) has some desirable properties, such as increasingness, concavity and differentiability. Finally, we will furnish the Euler equations for this general model.

Remark 3.2.1. *By considering a family of operators A_x, we embrace, for instance, the case where the* *decision maker is affected by a law of motion given by* $x_{t+1} = \phi(x_t, y_t, z_{t+1})$. *In applications following this direction, for example when the decision maker is a* τ*-quantile maximizer, (3.1) is written as*

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta Q_\tau [V(\phi(x,y,z'),z')|z]\}.$$

In our general context, this translates as

$$A_x V(y,z) = Q_\tau [V(\phi(x,y,z'),z')|z].$$

3.2.1 Existence of the Value Function

Consider the operator T_A acting on $\mathcal{C}$ given by

$$T_A f(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta(A_x f)(y,z)\} \tag{3.2}$$

together with the following assumptions:

Assumption 18. *The following hold:*

(i) $\mathcal{Z} \subset \mathbb{R}^k$;

(ii) $\mathcal{X} \subset \mathbb{R}^p$;

(iii) $\mathcal{Y} \subset \mathbb{R}^m$;

(iv) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is continuous and bounded*

(v) The correspondence $\Gamma : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ *is continuous, with nonempty, compact values;*

(vi) $\{A_x\}_{x\in\mathcal{X}}$ *is a family of operators acting on* $\mathcal{C}(x,z)$ *such that* $(A_x f)(y,z) \in \mathcal{C}(x,y,z)$ *if* $f \in \mathcal{C}(x,z)$;

(vii) $f \leqslant g$ *implies* $A_x f \leqslant A_x g$;

(viii) $A_x(f + a) \leqslant A_x f + a$ *for constant* $a \geqslant 0$;

(ix) $0 < \beta < 1$.

With these assumptions, we are able to establish the following:

Theorem 3.2.2. *Under Assumption 18,* T_A *is a contraction on* $\mathcal{C}$ *and has a unique fixed point* $V \in \mathcal{C}$.

Proof of Theorem 3.2.2: Firstly, Assumptions 18 (iv) and (vi) imply that, for all $f \in \mathcal{C}$,

$$u(x,y,z) + \beta(A_x f)(y,z) \in \mathcal{C}(x,y,z).$$

This, together with Assumption 18-(v), allows us to use Berge's Maximum Theorem to prove that $T_A f \in C(x,z)$.

To see that T_A is indeed a contraction, notice that, if $f \leqslant g$, then

$$\begin{aligned} T_A f(x,z) &= \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta(A_x f)(y,z)\} \\ &= u(x,y_f,z) + \beta(A_x f)(y_f,z) \\ &\leqslant u(x,y_f,z) + \beta(A_x g)(y_f,z) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \max_{y \in \Gamma(x,z)} \{u(x,y,z) + \beta(A_x g)(y,z)\} \\
&= T_A g(x,z),
\end{aligned}
$$

where $y_f \in \Gamma(x,z)$ denotes some realization of the maximum for $T_A f(x,z)$ and we used Assumption 18-(vii) in the first inequality.

Moreover, if $f \in \mathcal{C}$ and $a \geqslant 0$ is a constant, then Assumption 18-($viii$) implies that

$$
\begin{aligned}
T_A(f+a)(x,z) &= \max_{y \in \Gamma(x,z)} \{u(x,y,z) + \beta(A_x(f+a))(y,z)\} \\
&\leqslant \max_{y \in \Gamma(x,z)} \{u(x,y,z) + \beta(A_x f)(y,z)\} + \beta a \\
&= T_A f(x,z) + \beta a.
\end{aligned}
$$

Therefore, Blackwell conditions are satisfied, and T_A is a contraction on $\mathcal{C}$, so it has a unique fixed point $V \in \mathcal{C}$.

□

3.2.2 Monotonicity and Concavity

With additional hypothesis, we can make further characterizations of the solution V from (3.1).

Assumption 19 (Monotonicity)**.** *The following hold:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is strictly increasing in the first variable;*

(ii) For every $x \leqslant x'$ *and* $z \in \mathcal{Z}$, $\Gamma(x,z) \subseteq \Gamma(x',z)$*;*

(iii) For every $x \leqslant x'$, $(y,z) \in \Gamma(x,z) \times \mathcal{Z}$ *and* $f \in \mathcal{C}$ *increasing in the first variable,* $(A_x f)(y,z) \leqslant (A_{x'} f)(y,z)$.

With this, we are able to make the following characterization of the value function concerning increasingness:

Theorem 3.2.3. *Under Assumptions 18 and 19, the unique solution* V *to* (3.1) *is strictly increasing in the first variable.*

Proof of Theorem 3.2.3: Let V be the unique solution to (3.1), whose existence is ensured by Theorem 3.2.2. We will show that V is increasing in the first variable. For this, fix $z \in \mathcal{Z}$ and assume that $x_1 \leqslant x_2$. Let y_i be a maximizer for $V(x_i,z)$ in (3.1) for $i = 1,2$, that is,

$$
V(x_i,z) = u(x_i,y_i,z) + \beta(A_{x_i} V)(y_i,z).
$$

Thus,

$$
\begin{aligned}
V(x_1,z) &< u(x_2,y_1,z) + \beta(A_{x_2} V)(y_1,z) \\
&\leqslant \max_{y \in \Gamma(x_2,z)} \{u(x_2,y,z) + \beta(A_{x_2} V)(y,z)\} \\
&= V(x_2,z),
\end{aligned}
$$

where we used Assumption 19. This proves that V is strictly increasing in the first variable.

□

Now we impose some conditions to assure concavity:

Assumption 20 (Concavity)**.** *The following hold:*

(i) $\mathcal{X} \subset \mathbb{R}^p$ *is convex;*

(ii) $\mathcal{Y} \subset \mathbb{R}^m$ *is convex;*

(iii) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ *is strictly concave in the first two variables;*

(iv) For all $z \in \mathcal{Z}$ *and all* $x, x' \in \mathcal{X}$, $y \in \Gamma(x, z)$ *and* $y' \in \Gamma(x', z)$ *imply*

$$\theta y + (1-\theta)y' \in \Gamma[\theta x + (1-\theta)x', z], \text{ for all } \theta \in [0,1];$$

(v) For every $x, x' \in \mathcal{X}$, $z \in \mathcal{Z}$, $y \in \Gamma(x, z)$, $y' \in \Gamma(x', z)$, $\theta \in (0,1)$ *and for all* $f \in \mathcal{C}$ *concave in the first variable, we have*

$$\theta(A_x f)(y, z) + (1-\theta)(A_{x'} f)(y', z) \leqslant (A_{x_\theta} f)(y_\theta, z),$$

where

$$\begin{aligned} x_\theta &= \theta x + (1-\theta)x' \\ y_\theta &= \theta y + (1-\theta)y'. \end{aligned}$$

Now we can prove the following:

Theorem 3.2.4. *Under Assumptions 18 and 20, the unique solution* V *to* (3.1) *is strictly concave in the first variable. Moreover, the policy correspondence* $y^*(x, z) \in \Gamma(x, z)$ *which maximizes* (3.1) *is single-valued and continuous.*

Proof of Theorem 3.2.4: Fix some $f \in \mathcal{C}$ which is concave in the first variable and take $x_1, x_2 \in \mathcal{X}$, $\theta \in (0,1)$. Let y_i be a maximizer for $T_A f(x_i, z)$. Then,

$$\begin{aligned} (1-\theta)(T_A f)(x_1, z) + \theta(T_A f)(x_2, z) &= (1-\theta)u(x_1, y_1, z) + \theta u(x_2, y_2, z) \\ &\quad + \beta(1-\theta)(A_{x_1} f)(y_1, z) + \beta\theta(A_{x_2} f)(y_2, z) \\ &< u(x_\theta, y_\theta, z) + \beta(A_{x_\theta} f)(y_\theta, z) \\ &\leqslant \max_{y \in \Gamma(x_\theta, z)} \{u(x_\theta, y, z) + \beta(A_{x_\theta} f)(y, z)\} \\ &= (T_A f)(x_\theta, z), \end{aligned}$$

where we used Assumptions 20-$(\mathrm{iii}) - (\mathrm{v})$. This proves that $T_A f$ is strictly concave in x whenever f is concave. Hence, if $\mathcal{C}'$ denotes the closed subset of $\mathcal{C}$ consisting of concave functions with respect to the first variable, and $\mathcal{C}'' \subset \mathcal{C}'$ denotes the subset of strictly concave functions in the first variable, then $T_A(\mathcal{C}') \subset \mathcal{C}''$. Since T_A is a contraction, this implies that its fixed point lies in $\mathcal{C}''$, that is, V is strictly concave with respect to x.

Finally, by Berge's Maximum Theorem, the optimal policy $y^*(x, z)$ is upper hemi-continuous and nonempty. Since V is strictly concave, y^* must be single-valued. Therefore, it is continuous, and the proof is complete.

□

3.2.3 Differentiability

Now we state a result concerning differentiability of the value function V from (3.1).

Assumption 21. *The following hold:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ *is* C^1 *in the first variable;*

(ii) Assume that the family of operators $\{A_x\}_{x \in \mathcal{X}}$ *is independent of* x*, that is,* $A_x \equiv A$ *for all* $x \in \mathcal{X}$*.*

We have the following result:

Theorem 3.2.5. *Let Assumptions 18, 20 and 21 hold. Then, the unique solution* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *to (3.1) is differentiable in* x*, and*

$$\frac{\partial V}{\partial x_i}(x, z) = \frac{\partial u}{\partial x_i}(x, y^*, z),$$

where y^* *is the unique maximizer of (3.1), assumed to be interior to* $\Gamma(x, z)$*.*

Proof of Theorem 3.2.5: The proof follows from an easy adaptation of Benveniste and Scheinkman (1979)'s argument. For completeness and reader's convenience, we reproduce it here.

Since the needed assumptions are valid, Theorem 3.2.4 applies. Then, the value function $V(x, z)$ is strictly concave in the first variable and the the correspondence policy $y^*(x, z) \in \Gamma(x, z)$ is single valued.

Thus, for all (x, z), recording that A does not depend on x, we have

$$V(x, z) = u(x, y^*(x, z), z) + \beta(AV)(y^*(x, z), z).$$

Fix $z \in \mathcal{Z}$ and x_0 in the interior of X and define:

$$\bar{w}(x) = u(x, y^*(x_0, z), z) + \beta(AV)(y^*(x_0, z), z).$$

Since Γ is continuous and $y^*(x_0, z) \in \mathrm{int}\Gamma(x_0, z)$, there exists a neighborhood D of x_0 such that $y^*(x_0, z) \in \Gamma(x, z)$ for all $x \in D$. Thus, we have $\bar{w}(x) \leqslant V(x, z)$ whenever $x \in D$, with equality at $x = x_0$, which implies $\bar{w}(x) - \bar{w}(x_0) \leqslant V(x, z) - V(x_0, z)$. Note that $\bar{w}$ is concave and differentiable in x because u is. Thus, any subgradient p of $V(\cdot, z)$ at x_0 must satisfy

$$p \cdot (x - x_0) \geqslant V(x, z) - V(x_0, z) \geqslant \bar{w}(x) - \bar{w}(x_0).$$

Thus, p is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, p is unique. Therefore, $V(\cdot, z)$ is a concave function with a unique subgradient. Therefore, it is differentiable in x (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to x is the same as that of $\bar{w}$, that is,

$$\frac{\partial V}{\partial x_i}(x, z) = \frac{\partial \bar{w}}{\partial x_i}(x) = \frac{\partial u}{\partial x_i}(x, y^*(x, z), z),$$

as we wanted to show. □

3.2.4 Euler Equation

With further hypothesis, are able to provide the Euler equation for the value function V from (3.1).

Assumption 22. *The following hold:*

(i) $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ *is* C^1 *in the first two variables;*

(ii) There exists a continuous function $\phi : \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *which is concave and differentiable in the first variable and an operator* $\bar{A}$ *independent of* x *satisfying Assumptions 18-*$(vi) - (viii)$ *and 20-*(v)*, such that, for all* $f \in \mathcal{C}$*,*

$$Af(y,z) = \bar{A}\left(f(\phi(y',z'),z')\right)(y,z);$$

(iii) For $f \in \mathcal{C}(x,z)$ *differentiable and strictly concave in the first variable we have*

$$\frac{\partial}{\partial y_i}(Af)(y,z) = \bar{A}\left[\frac{\partial}{\partial {y'}_i} f\left(\phi(y',z'),z'\right)\right](y,z)$$

for all $(y,z) \in \Gamma(x,z) \times \mathcal{Z}$*,* $x \in \mathcal{X}$*.*

Remark 3.2.6. *If* $f \in \mathcal{C}$ *is differentiable, strictly concave and strictly increasing in the first variable, an easy adaption in the proof of Theorem 3.2.5 shows that* Af *is differentiable in the first variable. Hence, Assumption 22-*(iii) *makes sense since* Af *is differentiable.*

We have the following result:

Theorem 3.2.7. *Let Assumptions 18, 20, 21 and 22 hold. Then, the unique solution* $V : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *to* (3.1) *satisfies the Euler equation*

$$\frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta\bar{A}\left[\frac{\partial u}{\partial x}(\phi(y',z'), y^*(\phi(y',z'),z'),z') \cdot \frac{\partial \phi}{\partial y_i}(y',z')\right](y^*(x,z),z) = 0. \quad (3.3)$$

Here, $y^*(x,z)$ *denotes the optimal policy, assumed to be interior.*

Proof of Theorem 3.2.7: Let $g(x,y,z) \equiv u(x,y,z) + \beta(AV)(y,z)$ and $y^*(x,z)$ be an interior solution of the problem (3.1).

The differentiability of u and of AV (see Remark 3.2.6) with respect to y imply this same property on g. Since $y^*(x,z)$ is interior, the following first order condition holds:

$$\begin{aligned}
\frac{\partial g}{\partial y_i}(x, y^*(x,z), z) &= \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta\frac{\partial}{\partial y_i}(AV)(y^*(x,z),z) \\
&= \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) + \beta\bar{A}\left[\frac{\partial}{\partial {y'}_i} V\left(\phi(y',z'),z'\right)\right](y^*(x,z),z) \\
&= \frac{\partial u}{\partial y_i}(x, y^*(x,z), z) \\
&\quad + \beta\bar{A}\left[\frac{\partial u}{\partial x}(\phi(y',z'), y^*(\phi(y',z'),z'),z') \cdot \frac{\partial \phi}{\partial y_i}(y',z')\right](y^*(x,z),z) \\
&= 0.
\end{aligned}$$

where we used Assumption 22-(iii) in the second equality and Theorem 3.2.5 in the third.

□

3.3 Sequential Problem and Principle of Optimality

After treating the general recursive problem, we bring some hypotheses in order to define a sequential problem connected to the functional equation problem (3.1) by means of a Principle of Optimality.

Assumption 23. *The following conditions concerning the family of operators* $A = (A_x)_{x \in \mathcal{X}}$ *hold:*

(i) There exists a continuous function $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$ *and an operator* $\bar{A}$ *independent of* x *satisfying Assumptions 18-*$(vi) - (vii)$ *such that, for all* $f \in \mathcal{C}$*,*

$$\begin{aligned} A_x f(y,z) &= \bar{A}\left(f(\phi(x,y,z'),z')\right)(z) \\ &\equiv \bar{A}\left[f(\phi(x,y,z'),z')|z\right] \end{aligned}$$

(ii) $\alpha \bar{A}\left[f(\theta,z')|z\right] = \bar{A}\left[\alpha f(\theta,z')|z\right]$ *for constant* $\alpha > 0$*;*

(iii) $g(z) + \bar{A}\left[f(\theta,z')|z\right] = \bar{A}\left[g(z) + f(\theta,z')|z\right]$ *for any function* $g(z)$*.*

Notice that Assumption 23 introduces a bracket notation $A[f(\phi(x,y,z'),z')|z]$ which will be quite useful below. This notation reminds the conditional notation for expectation or quantiles. In each of these contexts, we have

$$A_x f(y,z) = \bar{A}\left[f(\phi(x,y,z'),z')|z\right] = \mathrm{E}\left[f(\phi(x,y,z'),z')|z\right]$$

or

$$A_x f(y,z) = \bar{A}\left[f(\phi(x,y,z'),z')|z\right] = Q_\tau\left[f(\phi(x,y,z'),z')|z\right],$$

respectively.

We conclude with a useful property of convergence:

Lemma 3.3.1. *Let* $A = (A_x)_{x \in \mathcal{X}}$ *be a family of operators satisfying Assumption 23, and let* $f_n : \mathcal{Z} \to \mathbb{R}$ *be a sequence of functions converging uniformly to a function* $f : \mathcal{Z} \to \mathbb{R}$*. Then*

$$\lim_{n \to \infty} \bar{A}[f_n(z')|z] = \bar{A}[f(z')|z].$$

Proof of Lemma 3.3.1: Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists some $N \in \mathbb{N}$ such that

$$-\frac{\epsilon}{2} + f(z') < f_n(z') < f(z') + \frac{\epsilon}{2}$$

for all $z' \in \mathcal{Z}$ whenever $n \geqslant N$. Taking $\bar{A}$ imply

$$\begin{aligned} -\epsilon + \bar{A}[f(z')|z] &< -\frac{\epsilon}{2} + \bar{A}[f(z')|z] = \bar{A}\left[-\frac{\epsilon}{2} + f(z')\Big|z\right] \\ &\leqslant \bar{A}[f_n(z')|z] \\ &\leqslant \bar{A}\left[f(z') + \frac{\epsilon}{2}\Big|z\right] = \bar{A}[f(z')|z] + \frac{\epsilon}{2} \\ &< \bar{A}[f(z')|z] + \epsilon, \end{aligned}$$

where in the equalities we used Assumption 23-(iii), and the inner inequalities are due to the monotonicity property from Assumption 18-(vii). Therefore,

$$\left|\bar{A}[f_n(z')|z] - \bar{A}[f(z')|z]\right| < \epsilon$$

if $n \geqslant N$. Thus, the result follows. □

Now we establish a notion of plans in our general setting.

3.3.1 Plans

In order to properly define a sequential problem, we imagine a decision-maker at the beginning of period t, knowing the current state x_t and history of shocks z^t, and deciding (according to preferences defined below) an action $y_t \in \Gamma(x_t, z_t) \subset \mathcal{Y}$, where $\Gamma(x, z)$ is the constraint set. From this, we can define plans as follows:

Definition 3.3.2. *A plan h is a profile $h = (h_t)_{t\in\mathbb{N}}$ where, for each $t \in \mathbb{N}$, h_t is a measurable function from $\mathcal{X} \times \mathcal{Z}^t$ to $\mathcal{Y}$. The set of plans is denoted by H.*

In the Definition 3.3.2, a plan $h_t(x_t, z^t)$ represents the choice that the individual makes at time t upon observing the current state x_t and the sequence of previous shocks z^t. The following notation will simplify statements below.

Definition 3.3.3. *Given a plan $h = (h_t)_{t\in\mathbb{N}}$, $x \in \mathcal{X}$ and realization $z^\infty = (z_1, \ldots) \in Z^\infty$, the sequence associated to (x, z^∞) is the sequence $(x_t^h)_{t\in\mathbb{N}} \in \mathcal{X}^\infty$ defined recursively by $x_1^h = x$ and $x_t^h = \phi(x_{t-1}^h, h_{t-1}(x_{t-1}^h, z^{t-1}), z_t)$, for $t \geqslant 2$. We write $y_t^h = h_t(x_t^h, z^t)$ for the choice taken at period t.*

Another important notion is that of a feasible plan:

Definition 3.3.4. *A plan h is feasible from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ if $h_t(x_t^h, z^t) \in \Gamma\left(x_t^h, z_t\right)$ for every $t \in \mathbb{N}$ and $z^\infty \in \mathcal{Z}^\infty$ such that $x_1^h = x$ and $z_1 = z$.*

We denote by $H(x, z)$ the set of feasible plans from $(x, z) \in \mathcal{X} \times \mathcal{Z}$. When x and z are clear from the context, or are not important, we will write only H.

3.3.2 Preferences

Now we define preferences based on family of operators $A = (A_x)_{x\in\mathcal{X}}$ which satisfy Assumption 23.

We assume that in time t with revealed shocks z^t, the decision-maker has a preference $\succcurlyeq_{t,z^t}$ over plans $h, h' \in H(x, z)$, which is represented by a function $V_t : H \times \mathcal{X} \times \mathcal{Z}^t \to \mathbb{R}$, that is,

$$h' \succcurlyeq_{t,x,z^t} h \iff V_t(h', x, z^t) \geqslant V_t(h, x, z^t). \tag{3.4}$$

We aim to define preferences for our general family of operators $A = (A_x)_{x\in\mathcal{X}}$ satisfying Assumption 23 based on a recursive relation given by

$$V_t(h, x, z^t) = u(x_t^h, y_t^h, z_t) + \beta\bar{A}\left[V_{t+1}(h, x, (Z^t, z_{t+1}))\middle| z_t\right], \tag{3.5}$$

where $u : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathbb{R}$ is the current-period utility function and we used the notation introduced in Assumption 23.

The recursive equation (3.5) is the foundation of the dynamic A-preferences.In Section 3.3.3 below, we explicitly define the sequence of functions V_t that satisfy (3.5) and will specify the preferences (2.62). Meanwhile, we show how the recursive equation (3.5) leads to an useful expression that will motivate the definition of V_t.

To see this, let $t = 1$ and substitute the expression of $V_{t+1} = V_2$ into the expression in (3.5) for V_1, and by continuing this process recursively we obtain:

$$\begin{aligned}
V_1(h,x,z^t) &= u(x_1^h, y_1^h, z_1) + \beta\bar{A}\Big[V_2(h,x,z^t)\Big| z\Big] \\
&= u(x_1^h, y_1^h, z_1) + \beta\bar{A}\Big[u(x_2^h, y_2^h, z_2) + \beta\bar{A}\Big[V_3(h,x,z^t)\Big| z_2\Big]\Big| z\Big] \\
&= \bar{A}\Big[\bar{A}\Big[u(x_1^h, y_1^h, z_1) + \beta u(x_2^h, y_2^h, z_2) + \beta^2 V_3(h,x,z^t)\Big| z_2\Big]\Big| z\Big] \\
&= \bar{A}\left[\bar{A}\left[\bar{A}\left[\sum_{t=1}^{3}\beta^{t-1}u(x_t^h, y_t^h, z_t) + \beta^3 V_4(h,x,z^t)\Big| z_3\right]\Big| z_2\right]\Big| z\right] \\
&= \bar{A}\left[\cdots\bar{A}\left[\sum_{t=1}^{n}\beta^{t-1}u(x_t^h, y_t^h, z_t) + \beta^n V_{n+1}(h,x,z^t)\Big| z_n\right]\Big|\cdots\Big| z\right],
\end{aligned} \tag{3.6}$$

where the operator $\bar{A}[\cdot]$ and corresponding $z, z_2, ..., z_n$ appear n times in the last line above. The algebraic manipulations made in (3.6) are justified by Assumption 23. In order to simplify the above equation, we use the following notation:

$$\bar{A}^n[\cdot] \equiv \bar{A}\left[\cdots\Big[\bar{A}\big[\,\cdot\,\big| z_n\big]\Big|\cdots\Big]\Big| z\right], \tag{3.7}$$

where the operator $\bar{A}$ and corresponding $z, z_2, ..., z_n$ appear n times. Therefore, by using the notation defined by (3.7), we are able to rewrite (3.6) as

$$V_1(h,x,z^t) = \bar{A}^n\left[\sum_{t=1}^{n}\beta^{t-1}u(x_t^h, y_t^h, z_t) + \beta^n V_n(h,x,z^t)\right]. \tag{3.8}$$

The next step is to take the limit as n goes to ∞. The formalization of such limit will be made in Section 3.3.3 below, but one can now intuitively understand the following:

$$V_1(h,x,z^t) = \bar{A}^\infty\left[\sum_{t=1}^{\infty}\beta^{t-1}u(x_t^h, y_t^h, z_t)\right], \tag{3.9}$$

as a notation for an (infinite) sequence of applications of $\bar{A}^n[\cdot]$.

3.3.3 The Sequence of Recursive Functions

In this section, we define the sequence of functions V_t that satisfy (3.5) and specify the preferences (2.62). We will make use of the notation introduced in (3.7).

Proposition 3.3.5. *Assume that $A = (A_x)_{x\in\mathcal{X}}$ is a family of operators satisfying Assumption 23. Let $u : \mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\to\mathbb{R}$ be a bounded and continuous function, and let $\beta\in(0,1)$. Fix some plan $h\in\mathcal{H}$. For*

$n \in \mathbb{N}$, *let*

$$\begin{aligned} V_t(h, x, z^t) &\equiv \lim_{n\to\infty} \bar{A}^n \left[\sum_{s=t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] \\ &\equiv \lim_{n\to\infty} \bar{A}\left[\ldots\left[\bar{A}\left[\sum_{s=t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\,\Big|\, z_{n+t-1}\right]\ldots\right]\Big| z_t\right]. \end{aligned} \tag{3.10}$$

Then the limit in (2.70) exists, so it is well defined, and satisfies the recursive relation (3.5).

Proof of Proposition 3.3.5: Since u is bounded, its sup norm $\|u\|_\infty$ is finite. Let

$$V_t^n(h, x, z^t) \equiv \bar{A}^n \left[\sum_{s=t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right].$$

We will show that $\{V_t^n(h, x, z^t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume that $n > m$ are natural numbers. Then

$$\begin{aligned} V_t^n(h, x, z^t) &= \bar{A}^n \left[\sum_{s=t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] \\ &= \bar{A}^n \left[\sum_{s=t}^{m+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s) + \sum_{s=m+t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] \\ &\leqslant \bar{A}^n \left[\sum_{s=t}^{m+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s) + \frac{\beta^m}{1-\beta}\|u\|_\infty\right] \\ &= \bar{A}^n \left[\sum_{s=t}^{m+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] + \frac{\beta^m}{1-\beta}\|u\|_\infty \\ &= \bar{A}^m \left[\sum_{s=t}^{m+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] + \frac{\beta^m}{1-\beta}\|u\|_\infty \\ &= V_t^m(h, x, z^t) + \frac{\beta^m}{1-\beta}\|u\|_\infty, \end{aligned} \tag{3.11}$$

where the inequality comes from the monotone of Assumption 18-(vii), namely, $\bar{A}[f(\theta, z')|z] \leqslant \bar{A}[g(\theta, z')|z]$ if $f \leqslant g$, and the change from $\bar{A}^n$ to $\bar{A}^m$ is due to the nonoccurence of $z_{m+t}, \ldots, z_{n+t-1}$ inside the $\bar{A}^n[\cdot]$ operator after our manipulations, together with Assumption 23-(iii).

Analogously, one proves that

$$V_t^n(h, x, z^t) \geqslant V_t^m(h, x, z^t) - \frac{\beta^m}{1-\beta}\|u\|_\infty. \tag{3.12}$$

Therefore, (2.71) and (2.72) imply

$$\left|V_t^n(h, x, z^t) - V_t^m(h, x, z^t)\right| \leqslant \frac{\beta^m}{1-\beta}\|u\|_\infty,$$

which clearly establishes that $\{V_t^n(h, x, z^t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, expression (2.70) is well defined. Indeed, we proved that $V_t^n(h, \cdot, \cdot)$ converges uniformly to $V_t(h, \cdot, \cdot)$.

To prove that V_t satisfies (2.63), we compute

$$
\begin{aligned}
V_t(h,x,z^t) &= \lim_{n\to\infty} \bar{A}^n\left[\sum_{s=t}^{n+t-1} \beta^{s-t} u(x_s^h, y_s^h, z_s)\right] \\
&= \lim_{n\to\infty} \bar{A}^n\left[u(x_t^h, y_t^h, z_t) + \beta \sum_{s=t+1}^{(n-1)+(t+1)-1} \beta^{s-(t+1)} u(x_s^h, y_s^h, z_s)\right] \\
&= u(x_t^h, y_t^h, z_t) + \beta \lim_{n\to\infty} \bar{A}^n\left[\sum_{s=t+1}^{(n-1)+(t+1)-1} \beta^{s-(t+1)} u(x_s^h, y_s^h, z_s)\right] \\
&= u(x_t^h, y_t^h, z_t) + \beta \lim_{n\to\infty} \bar{A}\left[\bar{A}^{n-1}\left[\sum_{s=t+1}^{(n-1)+(t+1)-1} \beta^{s-(t+1)} u(x_s^h, y_s^h, z_s)\right]\Big| z_t\right] \\
&= u(x_t^h, y_t^h, z_t) + \beta \lim_{n\to\infty} \bar{A}\left[V_{t+1}^{n-1}(h,x,z^{t+1})\Big| z_t\right] \\
&= u(x_t^h, y_t^h, z_t) + \beta \bar{A}\left[\lim_{n\to\infty} V_{t+1}^{n-1}(h,x,z^{t+1})\Big| z_t\right] \\
&= u(x_t^h, y_t^h, z_t) + \beta \bar{A}\left[V_{t+1}(h,h,z^{t+1})\Big| z_t\right],
\end{aligned}
$$

where we took out $u(x_t^h, y_t^h, z_t)$ from inside the $\bar{A}^n[\cdot]$ operator because of Assumption 23-(iii). Moreover, we could pass the limit into the $\bar{A}[\cdot]$ because of Lemma 3.3.1, since $V_{t+1}^{n-1}(h,x,\cdot)$ converges uniformly to $V_{t+1}(h,x,\cdot)$ by the first part. This concludes the proof.

□

We turn now to verify that this preference is dynamically consistent.

3.3.4 Dynamic Consistency

In this Section we formally define dynamic consistency and show that it is satisfied by the above defined dynamic A-preferences. The following definition is from Maccheroni et al. (2006b); see also Epstein and Schneider (2003).

Definition 3.3.6 (Dynamic Consistency). *The system of preferences* $\succcurlyeq_{t,\Omega^t}$ *is* dynamically consistent *if for every* t *and* $\Omega_t \equiv z^t$ *and for all plans* h *and* h', $h_{t'}(\cdot) = h'_{t'}(\cdot)$ *for all* $t' \leqslant t$ *and* $h' \succcurlyeq_{t+1,\Omega'_{t+1},x} h$ *for all* Ω'_{t+1}, x, *implies* $h' \succcurlyeq_{t,\Omega_t,x} h$.

We then have the following result:

Theorem 3.3.7. *The quantile preferences defined by* (2.70) *are dynamically consistent.*

The proof of this Theorem is completely analogous to the former quantile version stated at de Castro and Galvao (2019), so we omit it here. This occurs because the proof relies entirely in the recursive relation of preferences (3.5), which is entirely analogous to the quantile case.

3.3.5 The Principle of Optimality

This section establishes that the principle of optimality holds in our model. That is, optimizing period after period, as in the functional equation problem in equation (3.1), yields the same result as choosing the best plan for the whole horizon of the problem. This principle and its proof follows the same basis as stated in de Castro and Galvao (2019), to where we send the reader interested in its proof, which was carried out in the quantile context. Like the proof of dynamic consistency, it relies solely in the recursive relation (3.5), which have an entirely analogous version for quantiles.

Although we are not writing the proofs of this Section, we find it useful to transcript some equations that are being used in the current chapter and illustrate the derivations made in de Castro and Galvao (2019).

Let us begin by defining the set of feasible plans departing from $(x, z) \in \mathcal{X} \times \mathcal{Z}$ at time t:

$$H_t(x, z) \equiv \{h \in H(x, z) : \exists (x, z^t) \in \mathcal{X} \times \mathcal{Z}^t, \text{ with } z_t = z, \text{ such that } x_t^h(x, z^t) = x\}.$$

Thus we can define a supremum function as:

$$v_t^*(x, z) \equiv \sup_{h \in H_t(x,z)} V_t(h, x, z). \tag{3.13}$$

We first observe that t plays no role in the above equation (2.73), that is, we are able to drop the subscript t from (2.73) and write $v^*(x, z)$ instead of $v_t^*(x, z)$.

The next step is to relate v^* to V, the solution of the functional equation studied in Section 3.2, which was proved to exist in Theorem 3.2.2 and satisfies the Bellman equation (3.1). In this direction, we have the following result:

Proposition 3.3.8. *Let $A = (A_x)_{x \in \mathcal{X}}$ be a family of operators satisfying Assumption 23. Let V be a bounded and continuous solution to (3.1). Let $y^* \in \Upsilon(x, z)$ be a maximizer of V at (x, z), that is,*

$$V(x, z) = u(x, y^*(x, z), z) + \beta \bar{A}\left[V(\phi(x, y^*(x, z), z'), z')|z\right].$$

Consider the plan $h \in H$ given by

$$h_t(x_t, z^t) = y^*(x, z_t).$$

Then $v^* = V$, and h defined above attains the supremum in (2.73).

Thus, the principle of optimality provides sufficient conditions for a solution v to the functional equation be the supremum function. Again, we refer the reader to de Castro and Galvao (2019) for details.

3.4 Examples

In this Section, we present some examples of families of operators $A = (A_x)_{x \in \mathcal{X}}$ and discuss their suitability to the results and techniques from the preceding Sections 3.2 and 3.3.

3.4.1 Expectation

The classical approach to Dynamic Economics is by means of expectation, as seen extensively in Stokey et al. (1989). In this context, with the aid of a continuous law of motion $\phi : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X}$, expectation gives raise to the following family of operators acting on bounded and continuous funtions $f(x, z)$:

$$(A_x f)(y, z) \equiv \mathrm{E}\left[f(\phi(x, y, z'), z')|z\right]. \tag{3.14}$$

Thus, one has the correspondent recursive problem

$$V(x, z) = \max_{y \in \Gamma(x,z)} \left\{u(x, y, z) + \beta \mathrm{E}\left[V(\phi(x, y, z'), z')|z\right]\right\}. \tag{3.15}$$

By taking all the assumptions concerning $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, u, \Gamma, \beta$ and ϕ from Sections 3.2 and 3.3, all the results there stated apply, as long as we make the following assumption:

Assumption 24. *The operator* $f(x, z) \mapsto \mathrm{E}[f(x, z')|z]$ *has the Feller property, that is,*

$$\mathrm{E}[f(x, z')|z] \in \mathcal{C}(x, z) \text{ \textit{whenever} } f \in \mathcal{C}(x, z).$$

The existence and uniqueness result of a continuous and bounded solution V to (3.15) needs Assumption 24. This is so since, in order to apply Theorem 3.2.2, we need to ensure that the family of operators (3.14) satisfies Assumption 18.

The Feller property from Assumption 24 directly implies the valid of Assumption 18-(vi), and the remaining items (vii) and $(viii)$ are well known properties of expectations, so Assumption 18 will hold and Theorem 3.2.2 follows.

In order to obtain the remaining results from Section 3.2, some hypotheses over the law of motion ϕ are also necessary. Linear properties of expectations assure the majority of hypotheses from Section 3.2 over the family $A = (A_x)_{x \in \mathcal{X}}$ given by (3.14) will hold. The first exception is Assumption 18-(vi), already treated with the aid of Assumption 24.

Another exception is Assumption 19-(iii), used to establish the strict increasingness of V in the first variable (Theorem 3.2.3). To ensure this Assumption, one only needs to impose $\phi(x, y, z)$ to be increasing in the first variable.

For the result concerning concavity in the first variable of V, it is necessary to assume that ϕ is concave in the first two variables. This will imply the validity of Assumption 20-(v), so Theorem 20 applies.

For differentiability, one just needs to assume that $\phi = \phi(y, z)$, that is, it has no dependence on x. Also, since concavity is used to establish differentiability, it is also necessary to assume that $\phi(y, z)$ is concave in y. Hence, differentiability will follow from Theorem 3.2.5.

The Euler equations (Theorem 3.2.7) require ϕ to be again independent of x and concave in y. This is enough to ensure Assumption 22, since the interchangeability of diferentiation and expectation is well known.

The results from Section 3.3, which deal with the formulation of the sequential problem and the principle of optimality, do not require nothing more than the linear properties of expectations and Assumption 24 in order to the family (3.14) satisfy Assumption 23.

3.4.2 Quantile

With the definition

$$A_x f(y, z) \equiv Q_\tau[f(\phi(x, y, z'), z')],$$

a detailed development of the results from Sections 3.2 and 3.3 were already carried on in he previous chapter. All the results could also be derived from our general theory, with suitable assumptions over the distribution Z of the shocks and the law of motion ϕ. For instance, the continuity property from Assumption 18-(νi) is a consequence of Lemma 2.3.1 which rely in some mild hypotheses (Assumption 1) over the distribution of the shocks Z. Assumptions 18-(νii) – (νiii) are due to standard properties of quantiles, and hence the existence and uniqueness result from Theorem 3.2.2 follows.

Monotonicity of the value function with respect to the first variable V needs the same increasing property of ϕ in the x variable as in the case of expectations.

To establish concavity, however, some difficulties arise when one tries to verify Assumption 20-(ν). This is due to the lack of additivity for quantiles, that is, $Q_\tau[X+Y] \neq Q_\tau[X]+Q_\tau[Y]$ in general. However, equality can be obtained under comonotonicity of X and Y. Hence, Assumption 20-(ν) would be true for increasing functions $f(x,z)$ with respect to both variables.

Indeed, in this case, $f(\phi(x,y,z'),z')$ would be an increasing function of z' as long as ϕ is also increasing in the last variable. Hence, if f is also concave in the first variable, and $\phi(x,y,z)$ is concave in the first two variables, then

$$\begin{aligned}
& (1-\theta)Q_\tau[f(\phi(x_0,y_0,z'),z')|z] + \theta Q_\tau[f(\phi(x_1,y_1,z'),z')|z] \\
= \quad & Q_\tau[(1-\theta)f(\phi(x_0,y_0,z'),z') + \theta f(\phi(x_1,y_1,z'),z')|z] \\
\leqslant \quad & Q_\tau[f(\phi(x_\theta,y_\theta,z')z')|z].
\end{aligned}$$

Since we need to have f increasing in both variables, this is the reason why, for quantiles, concavity of V in the first variable depends on increasingness of V in both variables in general. If ϕ does not depend on z, however, only increasingness in z of the value function is required. Thus, no isolated result for concavity as in Theorem 3.2.4 is available for quantiles: monotonicity in both variables must also be assured.

Differentiation depends on concavity, so the same observations apply when dealing with Theorem 3.2.5. To produce Euler equations for quantiles, one can interchange quantile with differentiation only under some hypotheses. One possibility is to make use of comonotonicity, and that is the reason why the result of Euler equations for quantiles stated in Theorem 2.3.12 demands strict increasingness of the function $z_t \mapsto \frac{\partial u}{\partial x}(x_t,y_t,z_t) \cdot \frac{\partial \phi}{\partial y_i}(y_{t-1},z_t)$. This enables the desired interchangeability between quantiles and differentiation, thus allowing the usual form for Euler equations.

The general theory for the sequential problem from Section 3.3 requires no extra adaption for quantiles then those already mentioned.

3.4.3 Expectile

Next, we discuss the expectiles, as introduced by Newey and Powell (1987). Expectiles are largely employed in finance and defined as the solutions to asymmetric least squares minimization. The expectile is also closely related to two commonly used measures, value at risk and expected shortfall (see, e.g., Satchell (2010) and Ziegel (2016)).

As introduced in Newey and Powell (1987), expectiles are a set of minimizers given by

$$\mu_\tau[X] \equiv \arg\min_{\mu} \left\{ (1-\tau)\int_{[X<\mu]} (X-\mu)^2 dF + \tau \int_{[X>\mu]} (X-\mu)^2 dF \right\},$$

where $\tau \in (0,1)$ and X is a random variable with cdf F. With no difficult, conditional expectiles can be defined, so we can form a family of operators $A = (A_x)_{x\in\mathcal{X}}$ with the aid of a law of motion $\phi(x,y,z)$ as

$$A_x f(y,z) \equiv \mu_\tau\left[f(\phi(x,y,z'),z')|z\right].$$

The introduction of a law of motion offers no other difficulties than those already treated for expectation and quantiles. Hence, to focus only in the particularities due to expectiles, we will make the simplifying assumption that $\phi(x,y,z) \equiv y$ in this subsection, so here we have $\mathcal{X} = \mathcal{Y}$. Thus, we are mainly interested in expectiles of the form

$$\begin{aligned}\mu_\tau[f(y,z')|z] = \arg\min_\mu \Big\{(1-\tau)\int_{[f(y,z')<\mu]} (f(y,z')-\mu)^2\, dF(z'|z) \,+ \\ \tau \int_{[f(y,z')>\mu]} (f(y,z')-\mu)^2\, dF(z'|z)\Big\},\end{aligned} \tag{3.16}$$

where $F(z'|z)$ denotes the conditional cdf of the shocks.

In this framework, we have the recursive problem

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta\mu_\tau\left[V(y,z')|z\right]\}. \tag{3.17}$$

Under suitable hypotheses, the functional equation problem (3.17) satisfies almost all the results from Section 3.2. To establish this, we need to ensure that the family of operators $A = (A_x)_{x\in\mathcal{X}}$ (which, in fact, does not depend on x) given by

$$A_x f(y,z) \equiv \mu_\tau\left[f(y,z')|z\right] \tag{3.18}$$

satisfy the necessary Assumptions from Section 3.2. We begin with some basic properties of expectiles concerning invariance under certain affine transformations:

Lemma 3.4.1. *Expectiles given by* (3.16) *satisfy the following:*

(i) For all $f \in \mathcal{C}(y,z)$ *and any function* $g : \mathcal{Z} \to \mathbb{R}$,

$$\mu_\tau\left[f(y,z') + g(z)|z\right] = \mu_\tau\left[f(y,z')\right] + g(z); \tag{3.19}$$

(ii) for all $f \in \mathcal{C}(y,z)$ *and any constant* $a \geqslant 0$,

$$\mu_\tau\left[af(y,z')|z\right] = a\mu_\tau\left[f(y,z')|z\right]. \tag{3.20}$$

Proof of Lemma 3.4.1: We will write $\mathbb{1}[X < b]$ for the characteristic function of the interval $(-\infty, b)$. Then, (3.16) can be written as

$$\mu_\tau\left[f(y,z')|z\right] = \arg\min_\mu \int_{\mathcal{Z}} |\tau - \mathbb{1}[f(y,z') < \mu]|\,(f(y,z')-\mu)^2\, dF(z'|z).$$

Let $f \in \mathcal{C}(y,z)$, $a > 0$ be a constant and $g : \mathcal{Z} \to \mathbb{R}$ be any function. We have

$$\mu_\tau[af(y,z') + g(z)|z] = \arg\min_\mu \int_{\mathcal{Z}} |\tau - \mathbb{1}[af(y,z') + g(z) < \mu]|\,(af(y,z') + g(z) - \mu)^2\, dF(z'|z)$$

$$
\begin{aligned}
&= \arg\min_{\mu} a^2 \int_{\mathcal{Z}} \left|\tau - \mathbb{1}\left[f(y,z') < \left(\frac{\mu - g(z)}{a}\right)\right]\right| \left(f(y,z') - \left(\frac{\mu - g(z)}{a}\right)\right)^2 dF(z'|z) \\
&= \arg\min_{\mu} \int_{\mathcal{Z}} \left|\tau - \mathbb{1}\left[f(y,z') < \left(\frac{\mu - g(z)}{a}\right)\right]\right| \left(f(y,z') - \left(\frac{\mu - g(z)}{a}\right)\right)^2 dF(z'|z) \\
&= g(z) + a \arg\min_{\mu'} \int_{\mathcal{Z}} |\tau - \mathbb{1}[f(y,z') < \mu']| \left(f(y,z') - \mu'\right)^2 dF(z'|z) \\
&= a\mu_\tau[f(y,z')|z] + g(z).
\end{aligned}
$$

For $a = 1$, this proves (3.19). For $g(z) \equiv 0$, this proves (3.20) when $a > 0$. For $a = 0$, we have

$$
\begin{aligned}
\mu_\tau[0|z] &= \arg\min_{\mu} \int_{\mathcal{Z}} |\tau - \mathbb{1}[0 < \mu]| \, \mu^2 dF(z'|z) \\
&= \arg\min_{\mu} |\tau - \mathbb{1}[0 < \mu]| \, \mu^2 \int_{\mathcal{Z}} dF(z'|z) \\
&= 0,
\end{aligned}
$$

so (3.20) also hold for $a = 0$.

□

Now we state another important result for expectiles, this time related to monotonicity:

Lemma 3.4.2. *Let* $f, g \in \mathcal{C}(y,z)$ *be such that* $f \leqslant g$. *Then*

$$
\mu_\tau[f(y,z')|z] \leqslant \mu_\tau[g(y,z')|z]. \tag{3.21}
$$

Proof of Lemma 3.4.2: For $f \in \mathcal{C}$ and fixed (y,z), let H_f denote the function to be minimized in (3.16). That is,

$$
H_f(\mu) = (1-\tau) \int_{[f(y,z')<\mu]} \left(f(y,z') - \mu\right)^2 dF(z'|z) + \tau \int_{[f(y,z')>\mu]} \left(f(y,z') - \mu\right)^2 dF(z'|z). \tag{3.22}
$$

Notice that the integrands in (3.22) are strictly convex functions of μ. Therefore, $H_f(\mu)$ is itself also strictly convex. Let $\mu_f = \mu_\tau[f(y,z')|z]$, and similarly define μ_g. If $f \leqslant g$, we want to show that $\mu_f \leqslant \mu_g$. Since $H_f(\mu)$ is a strictly convex differentiable function, all we need to do is verify that $H_f'(\mu_g) \geqslant 0$, and this will imply that $\mu_f \leqslant \mu_g$, since μ_f is the minimizer of H_f. We have

$$
\begin{aligned}
H_f'(\mu_g) &= -2(1-\tau) \int_{[f<\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) - 2\tau \int_{[f>\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) \\
&\geqslant -2(1-\tau) \int_{[g<\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) - 2\tau \int_{[f>\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) \\
&\geqslant -2(1-\tau) \int_{[g<\mu_g]} \left(g(y,z') - \mu_g\right) dF(z'|z) - 2\tau \int_{[f>\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) \\
&= 2\tau \int_{[g>\mu_g]} \left(g(y,z') - \mu_g\right) dF(z'|z) - 2\tau \int_{[f>\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z) \\
&\geqslant 2\tau \int_{[f>\mu_g]} \left(g(y,z') - \mu_g\right) dF(z'|z) - 2\tau \int_{[f>\mu_g]} \left(f(y,z') - \mu_g\right) dF(z'|z)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant 2\tau \int_{[f>\mu_g]} (f(y,z') - \mu_g)\, dF(z'|z) - 2\tau \int_{[f>\mu_g]} (f(y,z') - \mu_g)\, dF(z'|z) \\
&= 0,
\end{aligned}
$$

where we used that $f \leqslant g$ implies $-g \leqslant -f$, $[g < \mu] \subset [f < \mu]$ and $[f > \mu] \subset [g > \mu]$, as well as the first order condition $H'_g(\mu_g) = 0$:

$$
-2(1-\tau) \int_{[g<\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) = 2\tau \int_{[g>\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z). \tag{3.23}
$$

□

With similar techniques, we can establish a subadditivity property for expectiles:

Lemma 3.4.3. *Let* $\tau \geqslant 1/2$, *and let* $f, g \in \mathcal{C}$. *Then*

$$
\mu_\tau [f(y,z')|z] + \mu_\tau [g(y,z')|z] \leqslant \mu_\tau [f(y,z') + g(y,z')|z]. \tag{3.24}
$$

Proof of Lemma 3.4.3: Following the notation introduced in the proof of Lemma 3.4.2, we want to show that $\mu_f + \mu_g \leqslant \mu_{f+g}$. Since H_{f+g} is strictly convex with global minimum at μ_{f+g}, we just need to show that $H'_{f+g}(\mu_f + \mu_g) \leqslant 0$. We have

$$
\begin{aligned}
\frac{1}{2} H'_{f+g}(\mu_f + \mu_g) &= -(1-\tau) \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad -\tau \int_{[f+g>\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&= -(1-2\tau) \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad -\tau \int_{\mathcal{Z}} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&= -(1-2\tau) \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad -\tau \int_{\mathcal{Z}} (f(y,z') - \mu_f)\, dF(z'|z) - \tau \int_{\mathcal{Z}} (g(y,z') - \mu_g)\, dF(z'|z) \\
&= -(1-2\tau) \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad -\tau \int_{[f<\mu_f]} (f(y,z') - \mu_f)\, dF(z'|z) - \tau \int_{[f>\mu_f]} (f(y,z') - \mu_f)\, dF(z'|z) \\
&\quad -\tau \int_{[g<\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) - \tau \int_{[g>\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) \\
&= -(1-2\tau) \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad -\tau \int_{[f<\mu_f]} (f(y,z') - \mu_f)\, dF(z'|z) + (1-\tau) \int_{[f<\mu_f]} (f(y,z') - \mu_f)\, dF(z'|z) \\
&\quad -\tau \int_{[g<\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) + (1-\tau) \int_{[g<\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) \\
&= -(1-2\tau) \Big\{ \int_{[f+g<\mu_f+\mu_g]} (f(y,z') + g(y,z') - \mu_f - \mu_g)\, dF(z'|z) \\
&\quad - \int_{[f<\mu_f]} (f(y,z') - \mu_f)\, dF(z'|z) - \int_{[g<\mu_g]} (g(y,z') - \mu_g)\, dF(z'|z) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= (2\tau - 1) \int_{\mathcal{Z}} (f(y, z') + g(y, z') - \mu_f - \mu_g)^+ - (f(y, z') - \mu_f)^+ \\
&\qquad\qquad\qquad\qquad\qquad\qquad - (g(y, z') - \mu_g)^+ \, dF(z'|z) \\
&\leqslant 0,
\end{aligned}
$$

where in the second equality we added and subtracted the first integral multiplied by τ. In the fifth equality, we used the first order condition (3.23). In the last equality, we used the notation

$$
u^+ = \max\{u, 0\} = \frac{u + |u|}{2},
$$

which satisfies

$$
(u + v)^+ \leqslant u^+ + v^+.
$$

This, together with the assumption $\tau \geqslant 1/2$, justifies the final inequality.

□

Corollary 3.4.4. *Let $f \in \mathcal{C}(y, z)$ be concave in the first variable and $\tau \geqslant 1/2$. Then $\mu_\tau[f(y, z')|z]$ is concave in y.*

Proof of Corollary 3.4.4: Let $y_0, y_1 \in \mathcal{Y}$ and $\theta \in (0, 1)$. We have

$$
\begin{aligned}
\theta\mu_\tau [f(y_0, z')|z] + (1 - \theta)\mu_\tau [f(y_1, z')|z] &= \mu_\tau [\theta f(y_0, z')|z] + \mu_\tau [(1 - \theta)f(y_1, z')|z] \\
&\leqslant \mu_\tau [\theta f(y_0, z') + (1 - \theta)f(y_1, z')|z] \\
&\leqslant \mu_\tau [f(\theta y_0 + (1 - \theta)y_1, z')\,|z],
\end{aligned}
$$

where used (3.20) in the first line, Lemma 3.4.2 in the second and the concavity of f with respect o y in the third.

□

Soon, we will show how these results concerning expectiles satisfy the Assumptions from Sections 3.2 and 3.3. Before this, we will establish a continuity property for expectiles. Notice that, for this, we make again use of Assumption 24, which is known as the Feller property and were used before for expectation:

Theorem 3.4.5. *Under Assumption 24,*

$$
\mu_\tau [f(y, z')|z] \in \mathcal{C}(y, z) \text{ whenever } f \in \mathcal{C}(y, z). \tag{3.25}
$$

Proof of Theorem 3.4.5: For fixed $f \in \mathcal{C}(y, z)$, let

$$
H(y, z, \mu) \equiv H_f(\mu),
$$

where H_f were introduced in (3.22) and the new notation $H(y, z, \mu)$ is to emphasize the dependence on the variables (y, z, μ). Thus, the expectile can be seen as the optimal correspondence of the minimization problem

$$
\min_{-\|f\| \leqslant \mu \leqslant \|f\|} H(y, z, \mu),
$$

that is,

$$\mu_\tau\left[f(y,z')|z\right] = \underset{-\|f\|\leqslant\mu\leqslant\|f\|}{\arg\min}\ H(y,z,\mu). \tag{3.26}$$

Since f is bounded, $[\|f\|,\|f\|]$ is compact. Also, continuity of f together with Assumption 24 imply that $H(y,z,\mu)$ is continuous with respect to all variables. Therefore, as a consequence of Berge's Maximum Theorem, the correspondence $\arg\min_{\mu\in[-\|f\|,\|f\|]} H(y,z,\mu)$ is upper hemi-continuous. Since H is strictly convex with respect to μ (as seen in the proof of Lemma 3.4.2), the correspondence is single valued, that is, the expectile is unique. But this imply that the correspondence is continuous. In face of (3.26), this means that it is a continuous function of (y,z), that is,

$$\mu_\tau\left[f(y,z')|z\right] \in \mathcal{C}(y,z).$$

□

Now we have all the tools to show how the results from Sections 3.2 and 3.3 apply to expectiles. As in the cases of expectation and quantiles, we only need to verify how the Assumptions are satisfied by the family of operators (3.18).

For the existence and uniqueness result from Theorem 3.2.2, Assumptions 18-$(vi)-(viii)$ are assured by Theorem 3.4.5 and Lemmas 3.4.1 and 3.4.2. Therefore, there is a unique bounded and continuous function V satisfying the recursive relation (3.17).

Theorem 3.2.3 readly applies to our framework, since the family of operators (3.18) does not depend on x. This makes Assumption 19-(iii) trivially true, hence the value function $V(x,z)$ is strictly increasing under the hypothesis of Theorem 3.2.3.

To show strict concavity of V with respect to the first variable, however, me must make the restriction $\tau \geqslant 1/2$, since we want to make use of Corollary 3.4.4 to ensure Assumption 20-(v) and then apply Theorem 3.2.4. Thus, $V(x,z)$ will be strictly concave in x if $\tau \geqslant 1/2$.

This restriction also applies to show that V is differentiable with respect to x, as well as the validity of the expression

$$\frac{\partial V}{\partial x_i}(x,z) = \frac{\partial u}{\partial x_i}(x,y^*,z),$$

since concavity is required to establish Theorem 3.2.4.

The very definition (3.18), together with Lemma 3.4.1 and Assumption 24, are all one needs to ensure that expectiles satisfy Assumption 23, so all the results from Section 3.3 hold. That is, one can properly define a sequential problem with (3.18) and show that the principle of optimality hold. Since concavity plays no role here, we can again have these results for any $\tau \in (0,1)$.

Finally, the lack of a general criteria for the interchange of expectiles and differentiation offers difficulties in stating a result concerning Euler equations, as done in Theorem 3.2.7. This is due to the lack of comonotonic additivity for expectiles, so the techniques empolyed in the proof of Theorem 3.2.7 will not work, in general, with expectiles.

Indeed, Lemma 3.4.3 established a subadditive property for expectiles when $\tau \geqslant 1/2$. The proof also shows that, when $\tau \leqslant 1/2$, one has superadditivity. Therefore, when $\tau = 1/2$, expectiles become additive, hence Theorem 3.2.7 can be applyed to show the existence of Euler equations in the familiar form. This case, however, isn't new, since when $\tau = 1/2$, expectile is the same as expectation.

3.4.4 Mode

The definition of mode offers great difficult to the methods from Sections 3.2 and 3.3. As in the previous example, we will make the simplifying assumption that $\mathcal{X} = \mathcal{Y}$. Firstly, one must be aware that different definitions of mode apply when one deals with discrete or continuous shocks. For discrete $\mathcal{Z}$ and $g \in \mathcal{C}$, the mode is defined as

$$M\left[g(y,z')|z\right] = \arg\max_{m\in\mathbb{R}} P\left[g(y,z') = m|z\right], \tag{3.27}$$

while for continuous $\mathcal{Z} \subset \mathbb{R}$ with continuous joint probability density function $f(z',z)$,

$$M\left[g(y,z')|z\right] = \arg\max_{z'\in\mathcal{Z}} f_g^y(z',z), \tag{3.28}$$

where f_g^y denotes the joint pdf of $g(y,z')$. The first issue with these definitions is that $g(y,z')$ could be multimodal, even when z is unimodal. This alone makes it unlikely that the problem

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + M\left[V(y,z')|z\right]\} \tag{3.29}$$

can be properly defined over $\mathcal{C}$, so some restriction in the domain is needed. It is true that the mode can be interchanged with strictly increasing functions, that is,

$$g\left(y, M[z'|z]\right) = M\left[g(y,z')|z\right] \text{ if } g(y,z) \text{ is strictly increasing in } z.$$

Unlike with quantiles, however, this property does not extend to increasing only functions. Since the subset of strictly increasing functions is not closed, this represents an obstacle in establishing a fixed point satisfying (3.29) departing from z-strictly increasing functions. This is so because increasing only functions may have undefined modes, or even be multimodal.

For instance, let $\mathcal{Z} = [0,4]$ and consider an iid continuous distribution with pdf

$$f(z) = \begin{cases} \dfrac{z}{2}, & \text{if } z \in [0,1] \\ \dfrac{4-z}{6}, & \text{if } z \in [1,4] \end{cases}$$

Let $g \in \mathcal{C}$ be an increasing function given by

$$g(z) = \begin{cases} z, & \text{if } z \in [0,2] \\ 2, & \text{if } z \in [2,4] \end{cases}$$

Then, the cdf for $g(Z)$ is given by

$$G(x) = P\left[g(z) \leqslant x\right] = \begin{cases} P[Z \leqslant x], & \text{if } z \in [0,2] \\ 1, & \text{if } z \in [2,4] \end{cases}$$

Hence, G is discontinuous, so our definition of mode does not apply to $g(z)$. One could think of Dirac measures in order to extend the definition (3.28) and define the mode to be $M[g(z)] = 2$.

This, however, would represent a violation of Assumption 18-(vii), since for $h(z) = z$, we would have $g \leqslant h$ while $M[h(z)] = M[z] = 1 < 2 = M[g(z)]$.

A similar problem would occur for discrete z. Consider $\mathcal{Z} = \{1, 2, 3\}$ and let

$$P[Z = z] = \begin{cases} 0.4 & \text{if } z = 1 \\ 0.35 & \text{if } z = 2 \\ 0.25 & \text{if } z = 3 \end{cases}$$

Then, with the same functions g, h as above, and with no other definition than (3.27), we would have

$$M[h(z)] = 1 < 2 = M[g(z)],$$

which again violates Assumption 18-(vii) since $g \leqslant h$.

The problem with this is that Assumption 18 is needed to establish Theorem 3.2.2, which proves the existence and uniqueness of a solution to the Bellman equation. Therefore, these difficulties bring no hope that the general methods from Section 3.2 could work for the problem (3.29). It would be necessary to find some properly defined closed subset $\mathcal{C}' \in \mathcal{C}$ where no violations of, at least, Assumption 18-(vii) occurs. However, we could not yet find an interesting and nontrivial $\mathcal{C}'$ satisfying this requirement.

3.4.5 Prospect Theory

Prospect Theory was originally introduced in Kahneman and Tversky (1979) as an alternative model to expected utility. The mathematical model for discrete prospects $(x_1, p_1, ..., x_n, p_n)$, where the z_i represent the possible outcomes and the p_i stand for their respective probabilities, can be modeled with a decision function given by

$$PT[X] = \sum_{i=1}^{n} \pi(p_i) v(x_i),$$

where π is a probability weighting function that captures the idea that people tend to overreact to low probability events, and underreact to high probability events. v denotes a value function of the outcomes. One possibility is to have

$$v(x) = \begin{cases} x^{\alpha}, \text{ if } x \geqslant 0 \\ -\lambda(-x)^{\gamma}, \text{ if } x < 0 \end{cases} \tag{3.30}$$

with $\alpha, \gamma \in (0, 1)$ and $\lambda > 1$.

A natural way to bring prospect theory to our context is to define, for $f \in \mathcal{C}$,

$$PT[f(y, z)] = \sum_{i=1}^{n} \pi(p_i) v(f(y, z_i)), \tag{3.31}$$

where we are making the simplifying assumption that $\mathcal{X} = \mathcal{Y}$.

In a discrete context, continuity offers no difficulty, so the family of operators (independent of x) given by

$$A_x f(y, z) = PT\left[f(y, z)\right] \tag{3.32}$$

satisfy Assumption 18-(vi). Moreover, since the function v from (3.30) is increasing, Assumption 18-(vii) also hold (unlike the mode). Unfortunately, however, Assumption 18-($viii$) fail, and this makes Theorem 3.2.2 inapplicable.

Indeed, this failure is due to the fact that

$$(x+a)^{\alpha} \leqslant x^{\alpha} + a$$

does not hold in general. For instance, consider $x = a = 0.1$, $\alpha = 0.5$. Thus, defining $f(y,z) \equiv 0.1$ would imply that

$$PT[f(y,z)+a] = \sqrt{0.2} = 0.44... > 0.41... = \sqrt{0.1} + 0.1 = PT[f(y,z)] + a,$$

a violation of Assumption 18-($viii$).

Therefore, the general methods used in Section 3.2 would not work to establish the existence and uniqueness of a solution to

$$V(x,z) = \max_{y \in \Gamma(x,z)} \{u(x,y,z) + \beta PT[V(y,z')]\}$$

in the case of discrete prospect theory operator given by (3.31).

3.4.6 Cumulative Prospect Theory

An improvement of Prospect Theory was presented in Tversky and Kahneman (1992), known as Cumulative Prospect Theory (CPT). The main difference is that the weighting is now applied over cumulative probability distribution instead of individual probabilities. Let X be a random variable. Then its CPT value is calculated as

$$C[X] = \int_0^{\infty} \omega_+ (P[u_+(X_+) > t])\, dt - \int_0^{\infty} \omega_- (P[u_-(X_-) > t])\, dt,$$

where $u_+, u_- : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and increasing utility functions and $\omega_+, \omega_- : [0,1] \to [0,1]$ are continuous and increasing probability weighting functions satisfying $\omega_+(0) = \omega_-(0) = 0$, $\omega_+(1) = \omega_-(1) = 1$. Here, X_+ denotes $\max\{X, 0\}$ and X_- is $-\min\{X, 0\}$.

To bring CPT to our scheme, we are interested in expressions of the form

$$C[f(y,z')|z] = \int_0^{\infty} \omega_+ (P[u_+(f(y,z')_+) > t|z])\, dt - \int_0^{\infty} \omega_- (P[u_-(f(y,z')_-) > t|z])\, dt \tag{3.33}$$

for $f \in \mathcal{C}$, where the probability is taken over z' conditional on z and, as before, we are making the simplifying assumption that $\mathcal{X} = \mathcal{Y}$.

We now proceed to check if CPT suits the hypotheses of Sections 3.2 and 3.3. Since the utilities and probability weighting functions are continuous, if one assumes that $(y,z) \mapsto P[f(y,z')|z]$ is continuous for each $f(y,z) \in \mathcal{C}$, this implies that $C[f(y,z')|z] \in C$ as long as ω_+ and ω_- are both integrable, which is a standard assumption for CPT. Thus, the family of operators (independent of x) given by

$$A_x f(y,z) = C[f(y,z')|z] \tag{3.34}$$

satisfies Assumption 18-(vi). Moreover, the (weak) increasingness of utilities and weightings imply that monotonicity is preserved, that is, $f \leqslant g$ implies $C[f(y,z')|z] \leqslant C[g(y,z')|z]$, so Assumption 18-($vii$) is also satisfied.

Assumption 18-($viii$), however, cannot be establish in our current general context. Further restrictions over utilities are necessary in order to assure that $C[f(y,z')+a|z] \leqslant C[f(y,z')|z] + a$ for constant

$a \geqslant 0$. Sufficient conditions are provided in Kun Lin and Marcus (2018), which we reproduce with minor adaptions in the following Assumption:

Assumption 25. *The following hold:*

(i) u_+, u_- *are invertible and differentiable, with* $u_+(0) = u_-(0) = 0$*;*

(ii) u'_+, u'_- *are decreasing;*

(iii) for any non-negative random variable X *and any* $a > 0$*, we have*

$$\int_0^a \omega_+ \left(P[X < t|z]\right) u'_+(a - t)dt - \int_0^a \omega_- \left(P[X > t|z]\right) u'_-(t)dt \leqslant a.$$

This implies the following Theorem, whose proof is equivalent to that of Theorem 6 from Kun Lin and Marcus (2018), so we omit it here:

Theorem 3.4.6. *Under Assumption 25,*

$$C[f(y, z') + a|z] \leqslant C[f(y, z')|z] + a$$

for every $f \in \mathcal{C}$*, and every constant* $a \geqslant 0$*.*

Therefore, under Assumption 25, all the items from Assumption 18 concerning the family of operators (3.34) hold, so Theorem 3.2.2 can be used to assure the existence and uniqueness of a bounded and continuous value function $V(x, z)$ satisfying the Bellman equation

$$V(x, z) = \max_{y \in \Gamma(x,z)} \left\{u(x, y, z) + \beta C[V(y, z')|z]\right\}. \tag{3.35}$$

Furthermore, since the family of operators (3.34) does not depend on x, Assumption 19-(iii) is true, so Theorem 3.2.3 establishes that the value function $V(x, z)$ is strictly increasing.

The lack of an additive (or even a superadditive, like we used for expectiles) result for CPT make Assumption 20-(v) fail, hence the results from Section 3.2, concerning concavity, and consequentially differentiability, will not apply. Thus, we cannot assure that the value function (3.35) is concave nor differentiable.

Another difficult rises due to Assumption 23-(iii). Indeed, Theorem 3.4.6 may offer strictly inequality under Assumption 25, and this contradicts Assumption 23-(iii). Therefore, one cannot use the results from Section 3.3 to properly define a sequential problem with a principle of optimality connecting it to the functional equation (3.35).

However, this difficult can be overcome by a different definition of both the Bellman equation and the sequential problem, as presented in Kun Lin and Marcus (2018). Their idea was to consider a Bellman equation of the form

$$V(x, z) = \max_{y \in \Gamma(x,z)} C\left[u(x, y, z) + \beta V(y, z')|z\right]. \tag{3.36}$$

In this form, Assumption 25 also assures the existence and uniqueness of the value function V solving (3.36) by a fixed point argument, like the one from Theorem 3.2.2.

The definition of the sequential problem for (3.36) in Kun Lin and Marcus (2018) has also a nested structure. Their definition can be translated to our notation as follows. First, for $n \in \mathbb{N}$ and a plan $h \in H$, define C_n^h as

$$
\begin{aligned}
C_n^h[u](x,z) = C\Big[u(x_1^h, x_2^h, z_1) + \beta C\Big[u(x_2^h, x_3^h, z_2) + \beta C\big[\ldots \\
+\beta C\Big[u(x_n^h, x_{n+1}^h, z_n)\Big| z_n\Big]\ldots\big]\Big| z_2\Big]\Big| z_1\Big],
\end{aligned}
$$

where $x_1 = x$, $z_1 = z$ and $x_{k+1} = h_k(x_k, z_k)$. Then, one defines the sequential problem as

$$
v^*(x,z) = \sup_{h \in H} \lim_{n \to \infty} C_n^h[u](x,z). \tag{3.37}
$$

This definition resembles (2.73), which is based on (2.70), whose difference to (3.37) is that the $u(x_k^h, x_{k+1}^h, z_K)$ terms are not passed inside the next $C[\cdot]$ operator, that is, they are not grouped under a sum inside all the n occurrences of $C[\cdot]$. This is so because, for CPT, Assumption 23-(iii) does not hold, hence definitions (3.36) and (3.37) are a way to avoid its necessity. The sequential problem (3.37) connects to the Bellman equation (3.36) via a principle of optimality by arguments that do not differ much from the ones given at Section 3.3.

3.4.7 Variational Preferences

Variational Preferences were introduced in Maccheroni et al. (2006a) as

$$
f \succcurlyeq g \Leftrightarrow \min_{p \in \Delta}\left(\int u(f)dp + c(p)\right) \geqslant \min_{p \in \Delta}\left(\int u(g)dp + c(p)\right), \tag{3.38}
$$

where the minimum is taken over the set $\Delta = \Delta(\Sigma)$ of probability measures over the measurable space $(\mathcal{Z}, \Sigma)$, with Σ denoting the Borel σ-algebra. Here, $f, g : \mathcal{Z} \to \mathbb{R}$ are measurable, real valued functions, and $u : \mathbb{R} \to \mathbb{R}$ is an utility function. Variational preferences generalizes multiple priors preferences, axiomatized in Gilboa and Schmeidler (1989) (also known as maxmin expected utility preferences), and given by

$$
V(f) = \min_{p \in C} \int u(f)dp,
$$

for some convex subset $C \subset \Delta$. Another generalization encompassed by variational preferences are multiplier preferences from Hansen and Sargent (2001), represented as

$$
V(f) = \min_{p \in \Delta}\left(\int u(f)dp + \theta R(p \parallel q)\right),
$$

where $R(\cdot \parallel q) : \Delta \to [0, \infty]$ is the relative entropy with respect to some fixed countably additive and nonatomic measure $q \in \Delta$, and $\theta \in (0, \infty]$ is a parameter.

To study variational preferences in our framework, we will use the operator I, acting on bounded and continuous functions $f : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$, and defined as

$$
I[f|z] = \min_{K \in D}\left(\int f(x,z')K(z,dz') + c(K(z,\cdot))\right). \tag{3.39}
$$

Here, minimization is taken over $K \in D \subset C(\mathcal{Z}, \Delta)$, that is, a compact subset (whose topology we will specify later) D of the set of continuous functions from $\mathcal{Z}$ to the space of probability measures Δ, where Δ is endowed with the ω^* topology. This means that $\mu_\alpha \to \mu$ in Δ if and only if

$$\int f d\mu_\alpha \to \int f d\mu \text{ for all } f \in \mathcal{C},$$

where $\mathcal{C}$ stands for the space of bounded and continuous real-valued functions on $\mathcal{Z}$. The choice of $C(\mathcal{Z}, \Delta)$ is justified since it corresponds, for $\mathcal{Z} \subset R^k$ compact, to the Markov transition functions with the Feller property. Indeed, Markov transition functions $K : \mathcal{Z} \times \Sigma \to [0,1]$ can be thought as Borel measurable elements from the functional space $\Delta^{\mathcal{Z}}$, that is, the space of functions from $\mathcal{Z}$ to Δ, since, for each $z \in \mathcal{Z}$, K associates a probability measure $K(z, \cdot) \in \Delta$. The Feller property for K means that

$$\text{for all } f \in \mathcal{C} \Rightarrow \int f(z')K(z, dz') \in \mathcal{C}, \tag{3.40}$$

that is, the integral term is bounded and continuous in the conditional z. When Δ is endowed with the ω^* topology, the Feller property (3.40) is the same as having $K \in C(\mathcal{Z}, \Delta)$, that is, having $z \in \mathcal{Z} \mapsto K(z, \cdot) \in \Delta$ continuous.

For our proposes, it is necessary to impose a topology on the Feller processes $C(\mathcal{Z}, \Delta)$. By observing that, for $\mathcal{Z} \subset \mathbb{R}^k$ compact, Δ with the ω^* topology is metrizable (see Dudley (2002), Theorem 11.3.3), we can endow $C(\mathcal{Z}, \Delta)$ with the uniform convergence topology:

$$K_n \to K \Leftrightarrow \sup_{z \in \mathcal{Z}} \beta\left(K_n(z, \cdot), K(z, \cdot)\right) \to 0,$$

where $\beta : \Delta \times \Delta \to [0, \infty)$ is a metric representing the ω^* topology. In this topology, $C(\mathcal{Z}, \Delta)$ is closed. The minimization in (3.39) is taken over $D \subset C(\mathcal{Z}, \Delta)$ compact. Due to Ascoli's Theorem, it is only necessary to check that D is closed and equicontinuous.

We also consider some state space $\mathcal{X} \subset \mathbb{R}^p$. Under these definitions, we have the following result:

Theorem 3.4.7. *For $D \subset C(\mathcal{Z}, \Delta)$ compact in the uniform topology and $(z, K) \in \mathcal{Z} \times D \mapsto c(K(z, \cdot)) \in \mathbb{R}$ bounded and uniformly continuous, we have that I given by (3.39) is such that $I[f(x, \cdot)|z]$ is continuous and bounded in (x, z) whenever $f(x, z)$ is bounded and continuous.*

Proof of Theorem 3.4.7: Fix $f \in \mathcal{C}$ and let $H : \mathcal{X} \times \mathcal{Z} \times D \to \mathbb{R}$ be defined as

$$H(x, z, K) = \int f(x, z')K(z, dz') + c(K(z, \cdot)).$$

Our hypothesis ensures that H is continuous in all variables when D is endowed with the uniform topology. Since D is compact, Berge's Maximum Theorem implies that

$$I[f(x, \cdot)|z] = \min_{K \in D} H(x, z, K)$$

is continuous in (x, z). □

Theorem 3.4.7 implies that the family of operators (independent of x) given by

$$(A_x f)(y, z) = I[f(y, \cdot)|z]$$

satisfies Assumption 18-(vi) (we are assuming $\mathcal{X} = \mathcal{Y}$ for simplicity). Since monotonicity (vii) also holds and we have, for any function $g(z)$,

$$I[f(y,\cdot)+g(z)|z] = I[f(y,\cdot)|z] + g(z), \tag{3.41}$$

Assumption 18-($viii$) is also true, hence Theorem 3.2.2 applies to show that the Bellman equation

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta I[V(y,z')|z]\} \tag{3.42}$$

has a unique bounded and continuous solution $V(x,z)$ under Assumption 18.

Remark 3.4.8. *For discrete $\mathcal{Z}$, we can take $D = C(\mathcal{Z},\Delta)$ in* (3.39).

Another important property from (3.39) is its invariance with respect to dilations:

$$I[\alpha f|z] = \alpha I[f|z] \text{ for all } \alpha > 0 \text{ constant.} \tag{3.43}$$

Therefore, Assumptions 19, 20 and 21 hold, so Theorems 3.2.3, 3.2.4 and 3.2.5 can be used to assure that V is strictly increasing, strictly concave and differentiable in the first variable.

Moreover, (3.41) and (3.43) imply that Assumption 23 hold, hence a sequential problem can be properly defined for variational preferences in the basis of Section 3.3.

3.4.8 Confidence preferences

Confidence preferences were introduced by Chateauneuf and Faro (2009). We can adapt it to our framework defining

$$CP[f(y,\cdot)|z] = \min_{K\in L_{\alpha,D,z,\varphi}} \frac{1}{\varphi(K(z,.))} \int f(y,z')K(z,dz'), \tag{3.44}$$

where $\varphi : \Delta \to [0,1]$ is quasiconcave and ω^* upper semicontinuous, $\alpha \in (0,1)$ is a parameter, $D \subset C(\mathcal{Z},\Delta)$ is compact and $L_{\alpha,D,z,\varphi} = \{K \in D; \varphi(K(z,\cdot) \leqslant \alpha\}$. Here we are considering the topological assumptions from the Subsection 3.4.7, and also considering $\mathcal{X} = \mathcal{Y}$.

Although the continuity Theorem 3.4.7 could be adapted to suit the operator (3.44), there is a critical limitation concerning confidence preferences that offers difficulties to our approach to dynamic recursive problems. The reason is that Assumption 18-($viii$), as well as Assumption 23-(iii), will not hold in general. Defining

$$(A_x f)(y,z) = CP[f(y,z')|z],$$

we may have

$$A_x(f+a)(y,z) > A_x f(y,z) + a$$

for constant $a > 0$. This is precisely the case when $\varphi(K(z,\cdot)) \equiv \alpha$, for $\alpha \in (0,1)$. This issue impacts both the solution to the Bellman equation as well as the definition of the sequential problem.

3.4.9 Smooth ambiguity preferences

Klibanoff et al. (2009) introduced smooth ambiguity preferences, which can be adapted to our context as

$$S[f(y,z')|z] = \varphi^{-1}\left(\int_{C(\mathcal{Z},\Delta)} \varphi\left(\int f(y,z')K(z,dz')\right)\mu(K)\right), \tag{3.45}$$

where, as in Subsection 3.4.7, $\Delta = \Delta(\Sigma)$ denotes the space of probability measures (endowed with the $\omega^\star$ topology) over the Borel σ-algebra Σ of $\mathcal{Z}$, $C(\mathcal{Z},\Delta)$ denotes the set of transition functions with the Feller property (endowed with the uniform convergence topology discussed in Subsection 3.4.7), μ is a fixed Borel probability measure with respect to $C(\mathcal{Z},\Delta)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and concave. Once more, we consider $\mathcal{X} = \mathcal{Y}$ for simplicity.

The next result establishes the continuity property from Assumption 18-(vi):

Theorem 3.4.9. *For each $f \in \mathcal{C}$ fixed, $S[f(y,z')|z]$ given by (3.45) is a continuous function of (y,z).*

Proof of Theorem 3.4.9: Fix $f \in \mathcal{C}$, and let $\Psi : C(\mathcal{Z},\Delta) \times \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ be given by

$$\Psi(K,y,z) \equiv \varphi\left(\int f(y,z')K(z,dz')\right).$$

Since $K \in C(\mathcal{Z},\Delta)$, $(y,z) \mapsto \Psi(K,y,z)$ is continuous for each K fixed. Since φ is strictly increasing and continuous, the fact that f is bounded imply

$$|\Psi(K,y,z)| \leqslant \varphi(\|f\|) < \infty,$$

where $\|f\|$ denotes the sup norm of f. Therefore, since μ is a finite measure, Lebesgue Dominated Convergence Theorem implies that

$$\int_{C(\mathcal{Z},\Delta)} \Psi(K,y_n,z_n)\mu(K) \to \int_{C(\mathcal{Z},\Delta)} \Psi(K,y^*,z^*)\mu(K)$$

whenever $(y_n,z_n) \to (y^*,z^*)$. Finally, continuity and strict increasingness of φ imply that φ^{-1} is also continuous, thus concluding the proof of continuity of $(y,z) \mapsto S[f(y,z')|z]$. □

Besides continuity, the monotonicity from Assumption 18-(vii) also holds. However, condition $(viii)$ fails in general for concave φ. This can be seen taking $\varphi(u) = u^p$ for $0 < p < 1$. It is well known that, for $0 < p < 1$, L^p spaces satisfy the reverse Minkowski inequality:

$$\|f+g\|_p \geqslant \|f\|_p + \|g\|_p, \tag{3.46}$$

where

$$\|f\|_p = \varphi^{-1}\left(\int \varphi(f)d\lambda\right)$$

is the L^p norm. When strict, (3.46) contradicts Assumption 18-$(viii)$.

So, the definition of smooth ambiguity preferences with concave φ does not fit our methods. It is worth noting, however, that a convex φ, under some circumstances, would work. For instance, if $\varphi(u) = u^p$ for $p \geqslant 1$, then Minkowski inequality holds, and this, together with the easily checkable fact that $S[a|z] = a$ for constant a, imply Assumption 18-$(viii)$. Therefore, Theorem 3.2.2 can be applied for

$$\varphi(u) = u^p,\ p \geqslant 1, \tag{3.47}$$

and then a unique continuous and bounded V satisfies the Bellman equation

$$V(x,z) = \max_{y\in\Gamma(x,z)} \{u(x,y,z) + \beta S[V(y,z')|z]\}. \tag{3.48}$$

With φ defined by (3.47), (3.45) is invariant with respect to dilation, that is,

$$S[\alpha f|z] = \alpha S[f|z] \text{ for all } \alpha > 0 \text{ constant.}$$

Therefore, Assumptions 19, 20 and 21 hold, so Theorems 3.2.3, 3.2.4 and 3.2.5 can be used to assure that V is strictly increasing, strictly concave and differentiable in the first variable.

With respect to the sequential problem, however, smooth ambiguity preferences fail to fit Assumption 23-(iii). This is due to Minkowski inequality, due to which one could have

$$S[f + g(z)|z] < S[f|z] + g(z),$$

so our methods from Section 3.3 will not work. This difficulty could be overcome if one defines the Bellman equation as

$$V(x,z) = \max_{y\in\Gamma(x,z)} S\left[u(x,y,z) + \beta V(y,z')|z\right]. \tag{3.49}$$

In this form, existence and uniqueness of the value function V solving (3.48) is also assured by a fixed point argument, like the one from Theorem 3.2.2. This idea is the same as the one presented at the end of Subsection 3.4.6. Like there, we use here a nested structure to define the sequential problem for (3.48). For $n \in \mathbb{N}$ and a plan $h \in H$, define S_n^h as

$$\begin{aligned} S_n^h[u](x,z) = S\Big[u(x_1^h, x_2^h, z_1) + \beta S\Big[u(x_2^h, x_3^h, z_2) + \beta S\big[\ldots \\ +\beta S\Big[u(x_n^h, x_{n+1}^h, z_n)\Big|z_n\Big]\ldots\Big]\Big|z_2\Big]\Big|z_1\Big], \end{aligned}$$

where $x_1 = x$, $z_1 = z$ and $x_{k+1} = h_k(x_k, z_k)$. Then, the sequential problem is defined as

$$v^*(x,z) = \sup_{h\in H} \lim_{n\to\infty} S_n^h[u](x,z). \tag{3.50}$$

Like in Subsection 3.4.6, the sequential problem (3.50) connects to the Bellman equation (3.49) via a principle of optimality.

3.4.10 Choquet integral

Choquet integral was first introduced in Choquet (1954), and generalizes the notion of integrals to monotone measures, which may be non-additive, as well as not continuous.

Definition 3.4.10. *Let $(\mathcal{Z}, \Sigma)$ be a measurable space. A fuzzy measure μ on $(\mathcal{Z}, \Sigma)$ is a set function $\mu : \Sigma \to [0,1]$ such that*

(i) $\mu(\varnothing) = 0$;

(ii) $\mu(\mathcal{Z}) = 1$;

(iii) if $A, B \in \Sigma$ *and* $A \subset B$, *then* $\mu(A) \leqslant \mu(B)$.

Condition (ii) is not always required for a general fuzzy measure (which could be unbounded, for instance), but we will make this restriction in our framework, since it is closer to probabilities. The first main difference of a fuzzy measure to a probability measure is the lack of an additive property. In general, we may have $\mu(A \cup B) \neq \mu(A) + \mu(B)$ even when $A \cap B = \varnothing$. The second difference is the lack of continuity from below and above. Indeed, we may have

$$A_1 \subset A_2 \subset A_3 \subset \ldots \quad \text{with} \quad \mu\left(\bigcup A_n\right) \neq \lim_n \mu(A_n),$$

and also

$$B_1 \supset B_2 \supset B_3 \supset \ldots \quad \text{with} \quad \mu\left(\bigcap B_n\right) \neq \lim_n \mu(B_n).$$

The Choquet integral defines integrals for fuzzy measures:

Definition 3.4.11 (Choquet integral)**.** *Let $(\mathcal{Z}, \Sigma)$ be a measurable space, μ a fuzzy measure and $f : \mathcal{Z} \to \mathbb{R}$ a Σ-measurable function. Then the Choquet integral of f with respect to μ is defined as*

$$(C)\int f d\mu \equiv \int_0^\infty \mu([f > x]) dx + \int_{-\infty}^0 \{\mu([f > x]) - 1\} dx,$$

where $[f > x] = \{z \in \mathcal{Z}; f(z) > x\}$ and the right hand side is the ordinary Stieltjes integral.

Choquet integral has some useful properties, which we collect in the next proposition. The reader interested in the proof, which are straightforward for the first three properties, may check Wang and Klir (2009). For the last property, a proof can be found in Schmeidler (1986).

Proposition 3.4.12. *Let $(\mathcal{Z}, \Sigma)$ be a measurable space, μ a fuzzy measure, $f, g : \mathcal{Z} \to \mathbb{R}$ Σ-measurable functions. The following hold:*

(i) (Monotonicity) if $f \leqslant g$, then

$$(C)\int f d\mu \leqslant (C)\int g d\mu;$$

(ii) (Positive Dilation) for all $a \geqslant 0$ constant,

$$(C)\int a f d\mu = a (C)\int f d\mu;$$

(iii) (Translation) for all $a \in \mathbb{R}$,

$$(C)\int f + a d\mu = (C)\int f d\mu + a;$$

(iv) (Comonotone additivity) if f and g are comonotonic, that is,

$$(f(z) - f(z'))(g(z) - g(z')) \geqslant 0 \textit{ for all } z, z' \in \mathcal{Z},$$

then

$$(C)\int f + g d\mu = (C)\int f d\mu + (C)\int g d\mu.$$

To deal with conditional Choquet integrals in a Markov setting suitable to our framework, we use the following definition, inspired by the usual Markov transition for expectations:

Definition 3.4.13 (Fuzzy Markov transition)**.** *Let $(\mathcal{Z}, \Sigma)$ be a measurable space. A fuzzy Markov transition is a function $K : \mathcal{Z} \times \Sigma \to [0, 1]$ such that* *(i)* $K(z,\cdot):\Sigma\to[0,1]$ *is a fuzzy measure for all* $z\in\mathcal{Z}$*;*

(ii) $K(\cdot,A):\mathcal{Z}\to[0,1]$ *is a* Σ*-measurable function for all* $A\in\Sigma$.

Given a fuzzy Markov transition K and a Σ-measurable function $f:\mathcal{Z}\to\mathbb{R}$, we define the condition Choquet integral as

$$(C)\int f(w)K(z,dw)\equiv(C)\int f dK(z,\cdot),$$

where the right hand side is the Choquet integral of f with respect to the fuzzy measure $K(z,\cdot)$. Therefore, we are interested in operators A acting on bounded and continuous functions $f:\mathcal{X}\times\mathcal{Z}\to\mathbb{R}$ of the form

$$(Af)(y,z)=(C)\int f(y,w)K(z,dw).$$

To ensure that $(Af)(y,z)$ is continuous for $f\in\mathcal{C}$, we will make use of the next assumption:

Assumption 26. *The fuzzy Markov transition* K *has the following property: for each* $f:\mathcal{Z}\to\mathbb{R}$ *bounded and continuous,*

$$z\in\mathcal{Z}'\mapsto(C)\int f(w)K(z,dw)\in\mathbb{R}$$

is a continuous function of z*.*

This Assumption is the equivalent version for Choquet integral of the Feller property for expectations. It enables us to prove the following:

Theorem 3.4.14. *Let* $\mathcal{X}$ *and* $\mathcal{Z}$ *metric spaces, with* $\mathcal{Z}$ *compact. Let* $f:\mathcal{X}\times\mathcal{Z}\to\mathbb{R}$ *be continuous and bounded, and let* $K:\mathcal{Z}\times\Sigma\to[0,1]$ *be a fuzzy Markov transition. Then*

$$(x,z)\in\mathcal{X}\times\mathcal{Z}\mapsto(C)\int f(x,w)K(z,dw)\in\mathbb{R}$$

is bounded and continuous.

Proof of Theorem 3.4.14: Let $(\{x_n,z_n)\}$ be a sequence converging to (x^*,z^*). Let $D\subset\mathcal{X}$ be a compact subset containing $\{x_n\}_{n\in\mathbb{N}}$. Then f is uniformly continuous on the compact $D\times\mathcal{Z}$. So, given $\epsilon>0$, there exists some $N_0\in\mathbb{N}$ such that

$$-\epsilon/2+f(x_n,w)\leqslant f(x^*,w)\leqslant f(x_n,w)+\epsilon/2$$

for all $w\in\mathcal{Z}$ if $n\geqslant N_0$. Hence, Proposition 3.4.12-(i) and (iii) imply that

$$-\epsilon/2+(C)\int f(x_n,w)K(z_n,dw)\leqslant(C)\int f(x^*,w)K(z_n,dw)\leqslant(C)\int f(x_n,w)K(z_n,dw)+\epsilon/2$$

for all $n\geqslant N_0$. Thus,

$$\left|(C)\int f(x_n,w)K(z_n,dw)-(C)\int f(x^*,w)K(z_n,w)\right|<\epsilon/2\quad\text{if } n\geqslant N_0. \tag{3.51}$$

Assumption 26 implies that there exists some $N_1\in\mathbb{N}$, which we may suppose $N_1\geqslant N_0$, such that

$$\left|(C)\int f(x^*,w)K(z_n,dw)-(C)\int f(x^*,w)K(z^*,dw)\right|<\epsilon/2\quad\text{if } n\geqslant N_1. \tag{3.52}$$

Therefore, (3.51) and (3.52) imply, via triangular inequality, that, for all $n \geqslant N_1$,

$$\left|(C)\int f(x_n,w)K(z_n,dw) - (C)\int f(x^*,w)K(z,dw)\right| < \epsilon,$$

and thus continuity is established since $\epsilon > 0$ was arbitrary.

Boundedness is a direct consequence of Proposition 3.4.12-(i), the inequalities $-\|f\| \leqslant f(x,w) \leqslant \|f\|$ (for all (x,w)) and the fact that

$$(C)\int aK(z,dw) = a$$

for all $a \in \mathbb{R}$ constant. □

We can now consider the Bellman equation problem

$$V(x,z) = \max_{y\in\Gamma(x,z)}\left\{u(x,y,z) + \beta(C)\int f(y,w)K(z,dw)\right\}, \tag{3.53}$$

where K is a fuzzy Markov transition. Theorem 3.4.14, Proposition 3.4.12 and the trivial law of motion $\phi(x,y,z) = y$ in (3.53) imply that Assumptions 18-$(vi)-(viii)$ and 19-(iii) hold, so Theorems 3.2.2 and 3.2.3 ensure the existence of a unique bounded and continuous solution to (3.53), which is also strictly increasing in the first variable whenever $u(x,y,z)$ is.

For concavity and differentiability, the same issue concerning additivity with quantiles appears when dealing with Choquet integral. This is no surprise, since quantiles themselves can be expressed as Choquet integrals. Indeed, if P is a probability measure on Σ and $\tau \in (0,1)$, define the fuzzy measure $\mu_\tau^P : \Sigma \to \{0,1\}$ by

$$\mu_\tau^P(A) = \begin{cases} 1, \text{ if } P[A^c] < \tau \\ 0, \text{ if } P[A^c] \geqslant \tau \end{cases} \quad \text{for } A \in \Sigma.$$

We then have

$$Q_\tau[f(x,w)] = (C)\int f(x,w)d\mu_\tau^P.$$

For quantiles, we could only ensure Assumption 20-(v) for $\mathcal{Z} \subset \mathbb{R}$. The same restriction is need for Choquet integrals, and the reason is Proposition 3.4.12-(iv). Additivity is required at this point because one wants to write

$$\begin{aligned}(C)\int \theta f(x,w)d\mu_z + (C)\int (1-\theta)f(x',w)d\mu_z &= (C)\int \theta f(x,w) + (1-\theta)f(x',w)d\mu_z \\ &\leqslant (C)\int f(\theta x + (1-\theta)x', w)d\mu_z\end{aligned}$$

for f concave in the first variable, where $d\mu_z = K(z,dw)$. But additivity holds only under comonotonicity. In dimension 1, however, monotonicity implies comonotonicity, and is easy to prove that V is monotonic with respect to z. Thus, this is the reason why we have to restrict $\mathcal{Z}$ to belong to $\mathbb{R}$. Likewise for quantiles, concavity will hold for (3.53) only after increasingness with respect to z is established.

The same observations for quantiles apply to the Choquet integral when dealing with differentiation and Euler equations. We refer the reader to Subsection 3.4.2 to avoid repetition, since it is all related to additivity for increasing functions under unidimensional shocks. Extension of these results to multidimensional shocks can be done in cases where comonotonicity (hence additivity) is assured.

3.5 Summary and Open Questions

This chapter develops a general dynamic model of rational behavior under uncertainty for an agent maximizing a general family of operators $\mathsf{A} = (\mathsf{A}_{\mathsf{x}})_{\mathsf{x}\in\mathcal{X}}$, which may vary according to the current state. We showed how to properly define a sequential problem related to the Bellman equation. This leads to dynamic consistent recursive general preferences. Also, this general dynamic problem yields a value function, using a fixed-point argument. Desirable properties of the value function are also provided. In addition, we derive the corresponding Euler equation.

We employ this general theory and show how it is able to derive known theories for expectation and quantiles. Morever, we also develop, as an example of application, dynamic programming for expectiles and cumulative prospect theory. Finally, we show how our general methods fail when applied to the mode and prospect theory.

Many issues remain to be investigated. Finding a closed subset of continuous and bounded functions that is well behaved under the mode would enable our methods to work with it. Another interesting avenue would be to investigate what general operator's methods could say about classical dynamic models, as well as how specific operators that fit our model - such as expectiles, CPT, variational preferences and Choquet integral - would modify the known insights of classical models already given by expectation and quantiles.

Appendix A

Appendix

This appendix collects useful results concerning quantiles and also the majority of proofs of the results in the first chapter.

A.1 Preliminaries

Before we proceed to the proofs we review a few useful properties of quantiles.

Given a random variable (r.v.) X, let F_X (or simply F) denote its cumulative distribution function (c.d.f.), that is, $F_X(\alpha) \equiv \Pr[X \leqslant \alpha]$. If X is clear from the context and we can omit it, the quantile function $Q : [0,1] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the generalized inverse of F:

$$Q(\tau) \equiv \begin{cases} \inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant \tau\}, & \text{if } \tau \in (0,1] \\ \sup\{\alpha \in \mathbb{R} : F(\alpha) = 0\}, & \text{if } \tau = 0. \end{cases}$$

The definition is special for $\tau = 0$ so that the quantile assumes a value in the support of X.[1] It is clear that if F is invertible, that is, if F is strictly increasing, its generalized inverse coincide with the inverse, that is, $Q(\tau) = F^{-1}(\tau)$.

A well-known and useful property of quantiles is "invariance" with respect to monotonic transformations, that is, if $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, then[2]

$$Q_\tau[g(X)] = g\left(Q_\tau[X]\right). \tag{A.1}$$

Quantiles are monotonic in the following sense: if X first-order stochastically dominates Y then $Q_\tau[X] \geqslant Q_\tau[Y]$. If X is risk-free, that is, $X = x$ with probability one for some x, then $Q_\tau[X] = x$. Quantiles are also translation-invariant, that is, $Q_\tau[\alpha + X] = \alpha + Q_\tau[X]$, $\forall \alpha \in \mathbb{R}$; and scale-invariant, that is, $Q_\tau[\alpha X] = \alpha Q_\tau[X]$, $\forall \alpha \in \mathbb{R}_+$. Indeed, $Q_\tau[-X] = -Q_{1-\tau}[X]$. On the other hand, quantiles do not share many of the convenient properties of expectations. We highlight three properties that fail for quantiles and would be important for our results. First, in general, quantiles are not linear: $Q_\tau[X + Y] \neq Q_\tau[X] + Q_\tau[Y]$, although Proposition A.1.4 below provides a comonotonicity condition under which this additivity holds. Second, quantiles do not satisfy an analogue of the law of iterated

[1] Indeed, $\inf\{\alpha \in \mathbb{R} : F(\alpha) \geqslant 0\} = -\infty$, no matter what is the distribution.

[2] In fact, equation (A.1) is also valid for a left-continuous and non-decreasing function; see Lemma A.1.1.

expectations: if $\Sigma_0 \subset \Sigma_1$ are two σ-algebras, then, in general, $Q_\tau[Q_\tau[X|\Sigma_1]|\Sigma_0] \neq Q_\tau[X|\Sigma_0]$. Third, in general, it is not possible to interchange a differentiation and a quantile operator, as it is for expectations, that is, $\frac{\partial Q_\tau}{\partial x}[h(x,Z)] \neq Q_\tau\left[\frac{\partial h}{\partial x}(x,Z)\right]$.

Many of the proofs here will appeal to results in de Castro and Galvao (2019). We suggest the reader to consult that paper for results below. We begin with a generalization of de Castro and Galvao (2019, Lemma A.2) to include the case in which $\mathcal{Z}$ can be discrete.

Let Θ be a set (of parameters) and $g : \Theta \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$ be a measurable function. We denote by $Q_\tau[g(\theta,\cdot)|z]$ the quantile function associated with g, that is:

$$Q_\tau[g(\theta,\cdot)|z] \equiv \inf\{\alpha \in \mathbb{R} : \Pr\left[(g(\theta,W) \leqslant \alpha)|Z = z\right] \geqslant \tau\}. \tag{A.2}$$

The following Lemma generalizes equation (A.1) to conditional quantiles.

Lemma A.1.1. *Assume that $\mathcal{Z} \subset \mathbb{R}$ is closed and $g : \Theta \times \mathcal{Z} \to \mathbb{R}$ is non-decreasing and left-continuous in $\mathcal{Z}$, where closedness and left-continuity are relative to the usual topology on $\mathbb{R}$. Then,*

$$Q_\tau[g(\theta,\cdot)|z] = g\left(\theta, Q_\tau[w|z]\right). \tag{A.3}$$

Our model allows $\mathcal{Z}$ to be countable or even finite. In this setup, we usually endow $\mathcal{Z}$ with the discrete topology. Since this topology is trivial, every function is continuous with respect to it. But for the purpose of this lemma, some structure is needed for the proof. Assuming continuity with respect to the usual topology of $\mathbb{R}$ allows us to prove the result when $\mathcal{Z}$ is discrete.

To see this assumption is needed, we provide the following counterexample. Let $\mathcal{Z} = \{1 - 1/n; n \in \mathbb{N}\} \cup \{1, 2\}$. Then $\mathcal{Z}$ is discrete and closed in the usual $\mathbb{R}$-topology. Consider the probabilities

$$\Pr\left[Z = 1 - \frac{1}{n}\right] = \frac{1}{2^{n+1}}; \quad n \in \mathbb{N}, \quad \text{and} \quad \Pr[Z = 1] = \Pr[Z = 2] = \frac{1}{4}.$$

Instead of considering functions continuous with respect to the usual topology, assume only continuity with respect to the discrete topology on $\mathcal{Z}$. Let g be given by

$$g\left(1 - 1/n\right) = \left(1 - 1/n\right); \quad g(1) = 2 \quad \text{and} \quad g(2) = 3.$$

For $\tau = 1/2$, one has $Q_\tau\left[g(Z)\right] = 1$ while $g\left(Q_\tau[Z]\right) = g(1) = 2$.

Proof of Lemma A.1.1: With the above assumption on $\mathcal{Z}$, when it is discrete, the proof of this Lemma is identical to the proof of Lemma A.2 of de Castro and Galvao (2019). ☐

The following result relates Q_τ and $Q^*_{1-\tau}$, where the τ-quantile* (or right quantile) is defined by

$$Q^*_\tau[X] = \sup\{\alpha \in \mathbb{R} : \Pr[X \leqslant \alpha] \leqslant \tau\} :$$

Lemma A.1.2. *Let X be a random variable and $\tau \in [0,1]$. Then*

$$Q_\tau[X] = -Q^*_{1-\tau}[-X] \tag{A.4}$$

Proof of Lemma A.1.2: Recall that, whenever $A \subset \mathbb{R}$, $\inf A = -\sup(-A)$. Hence,

$$\begin{aligned} -Q^*_{1-\tau}[-X] &= -\sup\{\alpha \in \mathbb{R}; P[-X \leqslant \alpha] \leqslant 1-\tau\} \\ &= \inf\{-\alpha \in \mathbb{R}; P[X \geqslant -\alpha] \leqslant 1-\tau\} \\ &= \inf\{\alpha \in \mathbb{R}; P[X \geqslant \alpha] \leqslant 1-\tau\} \\ &= \inf\{\alpha \in \mathbb{R}; 1 - P[X \geqslant \alpha] \geqslant \tau\} \\ &= \inf\{\alpha \in \mathbb{R}; P[X < \alpha] \geqslant \tau\}. \end{aligned}$$

So, it suffices to prove that

$$\inf\{\alpha \in \mathbb{R}; P[X < \alpha] \geqslant \tau\} = \inf\{\alpha \in \mathbb{R}; P[X \leqslant \alpha] \geqslant \tau\}, \tag{A.5}$$

since the right-hand side equals $Q_\tau[X]$ by definition.

Let $A = \{\alpha \in \mathbb{R}; P[X < \alpha] \geqslant \tau\}$, $B = \{\alpha \in \mathbb{R}; P[X \leqslant \alpha] \geqslant \tau\}$. Since $A \subset B$, we have $\inf B \leqslant \inf A$. For a contradiction, suppose that $\inf B < \inf A$. Then, there would be some $b \in B$ and $y \in \mathbb{R}$ such that

$$\inf B < b < y < \inf A.$$

Therefore,

$$\tau \leqslant P[X \leqslant b] \leqslant P[X < y]. \tag{A.6}$$

On the other hand, $y < \inf A$ implies that $y \notin A$, so

$$P[X < y] < \tau,$$

which contradicts (A.6). This establishes (A.5), thus completing the proof. □

We have the following result concerning interchangeability between quantiles and monotone functions:

Lemma A.1.3. *Let* $\tau \in [0,1]$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be increasing. Then*

$$Q_\tau[g(X)] = g(Q_\tau[X]) \text{ if } g \text{ is left-continuous} \tag{A.7}$$

and

$$Q^*_\tau[g(X)] = g(Q^*_\tau[X]) \text{ if } g \text{ is right-continuous.} \tag{A.8}$$

If, instead, $g : \mathbb{R} \to \mathbb{R}$ *is decreasing, then*

$$Q_\tau[g(X)] = g(Q^*_{1-\tau}[X]) \text{ if } g \text{ is right-continuous} \tag{A.9}$$

and

$$Q^*_{1-\tau}[g(X)] = g(Q_\tau[X]) \text{ if } g \text{ is left-continuous.} \tag{A.10}$$

Proof of Lemma A.1.3: Equation (A.7) is exactly Lemma A.2 from de Castro and Galvao (2019).

Now assume that g is increasing and right-continuous. To prove (A.8), we show that $g\left(Q_\tau^*[X]\right)$ is the supremum of $\{\alpha \in \mathbb{R}; P[g(X) \leqslant \alpha] \geqslant \tau\}$.

For this, let $y < g\left(Q_\tau^*[X]\right)$. Then

$$P[g(X) \leqslant y] \leqslant P\left[g(X) < g\left(Q_\tau^*[X]\right)\right] \leqslant P\left[X < Q_\tau^*[X]\right] \leqslant \tau,$$

that is,

$$y < g\left(Q_\tau^*[X]\right) \text{ implies } P[g(X) \leqslant y] \leqslant \tau. \tag{A.11}$$

Now, let $y > g\left(Q_\tau^*[X]\right)$. We want to show that $Q_\tau^*[X] < \inf\{x; g(x) \geqslant y\} = \hat{\alpha}$, since it implies that $P[g(X) \leqslant y] \geqslant P[X \leqslant \hat{\alpha}] > \tau$, that is, it proves that

$$y > g\left(Q_\tau^*[X]\right) \text{ implies } P[g(X) \leqslant y] > \tau. \tag{A.12}$$

Let x_n be a strictly decreasing sequence converging to $\hat{\alpha}$. Since $x_n > \hat{\alpha}$, then $g(x_n) \geqslant y$. Hence, $g\left(Q_\tau^*[X]\right) < y \leqslant \lim_{n\to\infty} g(x_n) = g(\hat{\alpha})$, since g is right-continuous. As g is increasing, this implies that $Q_\tau^*[X] < \hat{\alpha}$, thus establishing (A.12).

Since (A.11) and (A.12) together characterize the supremum of $\{\alpha \in \mathbb{R}; P[g(X) \leqslant \alpha] \geqslant \tau\}$, this proves (A.8).

Now, if g is decreasing and right-continuous, then

$$\begin{aligned} Q_\tau[g(X)] &= -Q_{1-\tau}^*\left[-g(X)\right] \\ &= g\left(Q_{1-\tau}^*[X]\right), \end{aligned}$$

where we used Lemma A.1.2 in the first equality and (A.8) in the second, since $-g$ is increasing and right-continuous. This proves (A.9).

Finally, if g is decreasing and left-continuous, then

$$\begin{aligned} Q_{1-\tau}^*[g(X)] &= -Q_\tau\left[-g(X)\right] \\ &= g\left(Q_\tau[X]\right), \end{aligned}$$

where we used Lemma A.1.2 in the first equality and (A.7) in the second, since $-g$ is increasing and left-continuous. This proves (A.10) and concludes the proof. □

The next result is an extension of Proposition A.4 from de Castro and Galvao (2019), this time also for decreasing functions and right-quantiles:

Proposition A.1.4. *Given random variables X and Y, assume that there are continuous and both increasing or both decreasing functions h and g such that $X = h(Z)$ and $Y = g(Z)$. Then*

$$Q_\tau[X+Y] = Q_\tau[X] + Q_\tau[Y] \tag{A.13}$$

and

$$Q_{1-\tau}^*[X+Y] = Q_{1-\tau}^*[X] + Q_{1-\tau}^*[Y] \tag{A.14}$$

Proof of Proposition A.1.4: Assume that h, g are both decreasing. By successive applying Lemma A.1.3, we have

$$\begin{aligned}
Q^*_{1-\tau}[X+Y] &= Q^*_{1-\tau}[h(Z)+g(Z)] \\
&= Q^*_{1-\tau}[(h+g)(Z)] \\
&= (h+g)\left(Q_\tau[Z]\right) \\
&= h\left(Q_\tau[Z]\right)+g\left(Q_\tau[Z]\right) \\
&= Q^*_{1-\tau}[h(Z)]+Q^*_{1-\tau}[g(Z)] \\
&= Q^*_{1-\tau}[X]+Q^*_{1-\tau}[Y].
\end{aligned}$$

An entirely analogous proof applies to each of the other cases, with Lemma A.1.3 being again used successively. □

We conclude with a useful property of convergence:

Lemma A.1.5. *Left $f_n : \mathcal{X} \subset \mathbb{R}^p \to \mathbb{R}$ be a sequence of functions converging uniformly to a function $f : \mathcal{X} \to \mathbb{R}$. Then*

$$\lim_{n\to\infty} Q_\tau[f_n(X)] = Q_\tau[f(X)].$$

Proof of Lemma A.1.5: Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists some $N \in \mathbb{N}$ such that

$$-\frac{\epsilon}{2} + f(x) < f_n(x) < f(x) + \frac{\epsilon}{2}$$

for all $x \in \mathcal{X}$ whenever $n \geqslant N$. Taking quantiles imply

$$\begin{aligned}
-\epsilon + Q_\tau[f(X)] &< -\frac{\epsilon}{2} + Q_\tau[f(X)] = Q_\tau\left[-\frac{\epsilon}{2} + f(X)\right] \\
&\leqslant Q_\tau[f_n(X)] \\
&\leqslant Q_\tau\left[f(X) + \frac{\epsilon}{2}\right] = Q_\tau[f(X)] + \frac{\epsilon}{2} \\
&< Q_\tau[f(X)] + \epsilon,
\end{aligned}$$

so

$$|Q_\tau[f_n(X)] - Q_\tau[f(X)]| < \epsilon$$

if $n \geqslant N$. Thus, the result follows. □

A.2 Proofs from Chapter 2

Here we collect the proofs of the results from Sections 2.2 and 2.3.

A.2.1 Proofs of Section 2.2

Proof of Theorem 2.2.1: One must only check that Assumption 1 hold, then apply Theorem 2.3.2. This is straightforward when $0 < \gamma < 1$.

When $\gamma > 1$, the function $u(x, y, z) = \frac{(xz-y)^{1-\gamma}}{1-\gamma}$ is no longer bounded, so we need another approach. In this case, we directly exhibit a solution. This is done in the proof of Theorem 2.2.3.

Notice that, although Assumption 2 is mentioned in the statement of Theorem 2.2.3, the calculations involved in the proof that the explicit expression (2.17) satisfies the functional equation do not require this Assumption.

□

Proof of Theorem 2.2.2: One must only check that Assumptions 1, 2, 14, 15 and 17 hold, then apply Theorems 2.3.7 and 2.3.10. This is straightforward when $0 < \gamma < 1$.

When $\gamma > 1$, the function $u(x, y, z) = \frac{(xz-y)^{1-\gamma}}{1-\gamma}$ is not bounded anymore, thus the above mentioned Theorems won't generally hold. Therefore, we proceed in the same way as we did in the proof of Theorem 2.2.1, by directly showing that the explicit solution (2.17) satisfies the stated properties. Since this explicit solution also holds for $0 < \gamma < 1$, the fact that the stated properties hold when $0 < \gamma < 1$ will imply that they also hold for $\gamma > 1$.

□

Proof of Theorem 2.2.3: Let's assume first that $0 < \gamma < 1$. As in Theorems 2.2.1 and 2.2.2, Assumptions 1, 2, 14, 15 and 17 hold, so the results from theorems 2.3.2, 2.3.7 and 2.3.10 are at our disposal. Hence, in addition to other properties, the unique solution V of equation (2.10) is continuous and increasing in the z variable. Moreover, as seen in the proof of Theorem 2.2.1, $\mathbb{M}^\tau$ is a contraction, so its fixed point V can be calculated iterating the operator

$$\mathbb{M}^\tau v(x,z) = \max_{0\leqslant y\leqslant xz} \left\{ \frac{(xz-y)^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[v(y,w) | z \right] \right\} \tag{A.15}$$

starting, for example, at $v_0 \equiv 0$.

To deduce V, it is useful to study $\mathbb{M}^\tau$ acting on $v(x,z) = x^{1-\gamma} L(z)/(1-\gamma)$, where L is continuous and increasing. One has

$$\begin{aligned}
\mathbb{M}^\tau v(x,z) &= \max_{0\leqslant y\leqslant xz} \left\{ \frac{(xz-y)^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[\frac{y^{1-\gamma}}{1-\gamma} L(w) \middle| z \right] \right\} && \text{(A.16)}\\
&= \max_{0\leqslant y\leqslant xz} \left\{ \frac{(xz-y)^{1-\gamma}}{1-\gamma} + \frac{y^{1-\gamma}}{1-\gamma} \beta L\left(Q_\tau[w|z] \right) \right\} && \text{(A.17)}\\
&= \frac{(xz)^{1-\gamma}}{1-\gamma} \max_{0\leqslant t\leqslant 1} \left\{ (1-t)^{1-\gamma} + \beta t^{1-\gamma} L(Q_\tau[w|z]) \right\}, && \text{(A.18)}
\end{aligned}$$

where in the second equality we used Lemma A.1.1. The FOC is

$$(1-t)^{-\gamma} = \beta L(Q_\tau[w|z]) t^{-\gamma} \iff t = \beta^{\frac{1}{\gamma}} \left(L(Q_\tau[w|z]) \right)^{\frac{1}{\gamma}} (1-t), \tag{A.19}$$

hence the maximum is achieved at

$$t^* = \frac{\beta^{\frac{1}{\gamma}} \left(L(Q_\tau[w|z]) \right)^{\frac{1}{\gamma}}}{1 + \beta^{\frac{1}{\gamma}} \left(L(Q_\tau[w|z]) \right)^{\frac{1}{\gamma}}}. \tag{A.20}$$

Multiplying by $t(1-t)$ on both sides of the equation on the left of the equivalence in (A.19) produces

$$t^*(1-t^*)^{1-\gamma} = \beta L(Q_\tau[w|z]) t^{*1-\gamma} (1-t^*) \quad \Rightarrow \quad \frac{t^*}{1-t^*} (1-t^*)^{1-\gamma} = \beta L(Q_\tau[w|z]) t^{*1-\gamma}. \tag{A.21}$$

Therefore, (A.18) becomes

$$\mathbb{M}^\tau v(x,z) = \frac{(xz)^{1-\gamma}}{1-\gamma}(1-t^*)^{1-\gamma}\left(1+\frac{t^*}{1-t^*}\right) \tag{A.22}$$

$$= \frac{(xz)^{1-\gamma}}{1-\gamma}(1-t^*)^{-\gamma} \tag{A.23}$$

$$= \frac{(xz)^{1-\gamma}}{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(L(Q_\tau[w|z])\right)^{\frac{1}{\gamma}}\right\}^\gamma, \tag{A.24}$$

where the last equality comes from (A.20).

Thus, since we aim to calculate the limit of the sequence $v_{n+1}(x,z) = \mathbb{M}^\tau v_n(x,z)$, with $v_n(x,z) = \frac{x^{1-\gamma}}{1-\gamma}L_n(z)$, we are interested in studying iterations of the operator

$$L(z) \mapsto z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(L(Q_\tau[w|z])\right)^{\frac{1}{\gamma}}\right\}^\gamma, \tag{A.25}$$

that is, define recursively the functions L_s by:

$$L_{s+1}(z) \equiv z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(L_s(Q_\tau[w|z])\right)^{\frac{1}{\gamma}}\right\}^\gamma, \tag{A.26}$$

with $L_0(z) \equiv 0$. We have

$$\begin{aligned}
L_1(z) &= z^{1-\gamma}\\
L_2(z) &= z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(Q_\tau[w_1|z]\right)^{\frac{1-\gamma}{\gamma}}\right\}^\gamma\\
L_3(z) &= z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(Q_\tau[w_1|z]\right)^{\frac{1-\gamma}{\gamma}}\left[1+\beta^{\frac{1}{\gamma}}\left(Q_\tau\left[w_2|\,Q[w_1|z]\right]\right)^{\frac{1-\gamma}{\gamma}}\right]\right\}^\gamma\\
&= z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left(Q_\tau[w_1|z]\right)^{\frac{1-\gamma}{\gamma}}+\beta^{\frac{2}{\gamma}}\left(Q_\tau\left[w_2|\,Q[w_1|z]\right]\right)^{\frac{1-\gamma}{\gamma}}\left(Q_\tau[w_1|z]\right)^{\frac{1-\gamma}{\gamma}}\right\}^\gamma\\
&= z^{1-\gamma}\left\{1+\beta^{\frac{1}{\gamma}}\left[r_1(z)\right]^{\frac{1-\gamma}{\gamma}}+\beta^{\frac{2}{\gamma}}\left[r_2(z)\right]^{\frac{1-\gamma}{\gamma}}\right\}^\gamma,
\end{aligned}$$

where we used the notation from (2.15) and, for simplicity, we wrote r_s instead of $r_{\tau,s}$.

Induction leads to

$$L_{n+1}(z) = z^{1-\gamma}\left\{1+\sum_{s=1}^{n}\beta^{\frac{s}{\gamma}}\left(r_{\tau,s}(z)\right)^{\frac{1-\gamma}{\gamma}}\right\}^\gamma.$$

By Assumption 3(iii), there exists some constant b such that $0 < \beta^{\frac{1}{\gamma}}z^{\frac{1-\gamma}{\gamma}} < b < 1$ for all z. This implies that $0 < \beta^{\frac{1}{\gamma}}Q_\tau[w|z]^{\frac{1-\gamma}{\gamma}} < b < 1$ for all z, hence $0 < \beta^{\frac{1}{\gamma}}\left(r_{\tau,s}(z)\right)^{\frac{1-\gamma}{s\gamma}} < b < 1$ for all z, s. Then, L_n is a continuous bounded sequence dominated by $\frac{z^{1-\gamma}}{1-b}$ that converges uniformly to the continuous function

$$L^*(z) = z^{1-\gamma}\left\{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z)\right)^{\frac{1-\gamma}{\gamma}}\right\}^\gamma. \tag{A.27}$$

Therefore, $v_n(x,z) = \frac{x^{1-\gamma}}{1-\gamma}L_n(z) = (\mathbb{M}^\tau)^n v_0(x,z)$ converges to

$$V(x,z) = \frac{x^{1-\gamma}}{1-\gamma}L^*(z) = \frac{(xz)^{1-\gamma}}{1-\gamma}\cdot\left\{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z)\right)^{\frac{1-\gamma}{\gamma}}\right\}^\gamma,$$

establishing (2.17).

Now assume that $\gamma > 1$. We will show directly that V satisfies $\mathbb{M}^\tau V = V$, where V is given by (2.17). By substituting the maximization on $y \in [0, xz]$ by the maximization of $t = \frac{y}{xz} \in [0,1]$, we have:

$$\begin{aligned}
\mathbb{M}^\tau V(x,z) &= \max_{0\leqslant y\leqslant xz} \left\{ \frac{(xz-y)^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[\frac{(yw)^{1-\gamma}}{1-\gamma} \left\{ 1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(w))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma \middle| z \right] \right\} \\
&= (xz)^{1-\gamma} \max_{t\in[0,1]} \left\{ \frac{(1-t)^{1-\gamma}}{1-\gamma} + \beta Q_\tau \left[\frac{(tw)^{1-\gamma}}{1-\gamma} \left\{ 1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(w))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma \middle| z \right] \right\} \\
&= (xz)^{1-\gamma} \max_{t\in[0,1]} \left\{ \frac{(1-t)^{1-\gamma}}{1-\gamma} + \beta \frac{t^{1-\gamma}}{1-\gamma} (Q_\tau[w|z])^{1-\gamma} \left\{ 1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma \right\} \\
&= (xz)^{1-\gamma} \max_{t\in[0,1]} \left\{ \frac{(1-t)^{1-\gamma}}{1-\gamma} + \frac{t^{1-\gamma}}{1-\gamma} \left\{ \beta^{\frac{1}{\gamma}} (Q_\tau[w|z])^{\frac{1-\gamma}{\gamma}} + \sum_{s=1}^{\infty} \beta^{\frac{s+1}{\gamma}} (r_{s+1}(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma \right\} \\
&= (xz)^{1-\gamma} \max_{t\in[0,1]} \left\{ \frac{(1-t)^{1-\gamma}}{1-\gamma} + \frac{t^{1-\gamma}}{1-\gamma} \left\{ \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma \right\},
\end{aligned} \tag{A.28}$$

where in the third equality we made use of the fact that the function

$$z \mapsto \frac{z^{1-\gamma}}{1-\gamma} \cdot \left\{ 1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(z))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma$$

is increasing, together with Lemma A.1.1, to interchange the function with the quantile. This claimed increasingness follows from the fact that $\gamma > 1$ and each $r_s(z)$ is increasing in z, in face of an inductive use of Lemma A.2.6, starting with the increasingness of $r_1(z) = Q[w|z]$ as a direct application of this Lemma. Moreover, in the fourth equality above we used the definition (2.15) of $r_s(z)$.

The first-order condition of the maximization problem (A.28) is

$$(1-t)^{-\gamma} = \left\{ \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma t^{-\gamma} \iff t \left\{ \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^{-1} = (1-t), \tag{A.29}$$

hence the maximum is achieved at

$$t^* = \frac{\sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}}}{1 + \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}}}. \tag{A.30}$$

Multiplying by $t(1-t)$ both sides of the equation on the left of the equivalence in (A.29), we obtain

$$t^*(1-t^*)^{1-\gamma} = \left\{ \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma (t^*)^{1-\gamma}(1-t^*), \tag{A.31}$$

which leads to:

$$\frac{(t^*)^{1-\gamma}}{1-\gamma} \left\{ \sum_{s=1}^{\infty} \beta^{\frac{s}{\gamma}} (r_s(Q_\tau[w|z]))^{\frac{1-\gamma}{\gamma}} \right\}^\gamma = \frac{t^*(1-t^*)^{-\gamma}}{1-\gamma}. \tag{A.32}$$

Substituting the above into (A.28), we obtain:

$$\mathbb{M}^\tau V(x,z) = (xz)^{1-\gamma} \frac{(1-t^*)^{1-\gamma}}{1-\gamma} (1 - t^* + t^*) = \frac{(xz)^{1-\gamma}}{1-\gamma} (1-t^*)^{-\gamma}.$$

Using (A.30), we obtain:

$$\mathbb{M}^{\tau}V(x,z) = \frac{(xz)^{1-\gamma}}{1-\gamma}\left\{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s\left(Q_{\tau}[w|z]\right)\right)^{\frac{1-\gamma}{\gamma}}\right\}^{\gamma} = V(x,z).$$

This establishes that (2.17) satisfies (2.10) also for $\gamma > 1$.

Now, let $0 < \gamma < 1$ again. Lemma A.2.6 implies that $z \mapsto Q[w|z]$ is increasing. This, together with the definition (2.15) of $r_s(z)$, results that $L^*(z)/(1-\gamma)$ is increasing for all $\gamma > 0$, $\gamma \neq 1$, where $L^*(z)$ is given by (A.27). Since V has the form $V(x,z) = \frac{x^{1-\gamma}}{1-\gamma}L^*(z)$, above calculations (see equation (A.19)) imply that the maximum y^* which realizes (A.15) is given by

$$y^*(x,z) = xzt^* = xz\cdot\frac{\beta^{\frac{1}{\gamma}}\left(L^*(Q_{\tau}[w|z])\right)^{\frac{1}{\gamma}}}{1+\beta^{\frac{1}{\gamma}}\left(L^*(Q_{\tau}[w|z])\right)^{\frac{1}{\gamma}}}. \tag{A.33}$$

By noticing that

$$\begin{aligned}
\beta^{\frac{1}{\gamma}}\left(L^*(Q_{\tau}[w|z])\right)^{\frac{1}{\gamma}} &= \beta^{\frac{1}{\gamma}}\left(Q_{\tau}[w|z]\right)^{\frac{1-\gamma}{\gamma}}\left\{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(Q_{\tau}[w|z])\right)^{\frac{1-\gamma}{\gamma}}\right\} \\
&= \beta^{\frac{1}{\gamma}}\left(Q_{\tau}[w|z]\right)^{\frac{1-\gamma}{\gamma}} + \sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\left(r_s(Q_{\tau}[w|z])Q[w|z]\right)^{\frac{1-\gamma}{\gamma}} \\
&= \beta^{\frac{1}{\gamma}}\left(Q_{\tau}[w|z]\right)^{\frac{1-\gamma}{\gamma}} + \sum_{s=1}^{\infty}\beta^{\frac{s+1}{\gamma}}\left(r_{s+1}(z)\right)^{\frac{1-\gamma}{\gamma}} \\
&= \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z)\right)^{\frac{1-\gamma}{\gamma}},
\end{aligned}$$

one obtains (2.18) for $0 < \gamma < 1$. When $\gamma > 1$, this was already proved in (A.30). In either case, since the optimal consumption is $c^* = xz - y^*$, (2.19) follows easily.

If we consider the optimal consumption path $\{c_t\}_{t=1}^{\infty}$, this gives us

$$\begin{aligned}
c_{t+1} &= \frac{1}{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z_{t+1})\right)^{\frac{1-\gamma}{\gamma}}}\cdot x_{t+1}z_{t+1}, \text{ and} \\
c_t &= \frac{1}{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z_t)\right)^{\frac{1-\gamma}{\gamma}}}\cdot x_t z_t,
\end{aligned}$$

where $x_{t+1} = y_t = \frac{\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}(r_s(z_t))^{\frac{1-\gamma}{\gamma}}}{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}(r_s(z_t))^{\frac{1-\gamma}{\gamma}}}\cdot x_t z_t = \sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z_t)\right)^{\frac{1-\gamma}{\gamma}}\cdot c_t$. Therefore,

$$\begin{aligned}
c_{t+1} &= \frac{\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z_t)\right)^{\frac{1-\gamma}{\gamma}}}{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z_{t+1})\right)^{\frac{1-\gamma}{\gamma}}}\cdot z_{t+1}c_t \\
&= m_{\tau}(z_t, z_{t+1})\cdot c_t,
\end{aligned}$$

for

$$m_{\tau}(z,w) \equiv \frac{\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(z)\right)^{\frac{1-\gamma}{\gamma}}}{1+\sum_{s=1}^{\infty}\beta^{\frac{s}{\gamma}}\left(r_s(w)\right)^{\frac{1-\gamma}{\gamma}}}\cdot w.$$

That is, (2.20) holds. □

Proof of Theorem 2.2.6: Since Assumptions 1, 2, 14, 15 and 17 hold, Theorems 2.3.2, 2.3.7 and 2.3.10 are valid, and the stated results follow.

□

Proof of Theorem 2.2.8: From Theorem 2.2.6, V is strictly increasing in both variables and strictly concave in the first variable. Therefore, Lemma A.1.3 imply that (2.26) can be rewritten as

$$V(x,z) = \max_{y\in[0,zx]} \left\{-\frac{1}{\gamma}e^{-\gamma(zx-y)} + \beta V(y,q)\right\}, \tag{A.34}$$

where

$$q \equiv Q_\tau[z'|z] = Q_\tau[z']$$

is a constant since z is iid.

The first order condition from (A.34) is

$$e^{-\gamma c^*(x,z)} = e^{-\gamma(zx-y^*(x,z))} = \beta\partial_x V(y^*(x,z),q). \tag{A.35}$$

First we show that $y^*(x,z)$ is strictly increasing in x. If not, there would be $x_0 < x_1$ such that $y^*(x_1,z) \leqslant y^*(x_0,z)$. Therefore,

$$c^*(x_0,z) = zx_0 - y^*(x_0,z) < zx_1 - y^*(x_1,z) = c^*(x_1,z),$$

so (A.35) would imply that

$$\partial_x V(y^*(x_1,z),q) = \frac{1}{\beta}e^{-\gamma c^*(x_1,z)} < \frac{1}{\beta}e^{-\gamma c^*(x_0,z)} = \partial_x V(y^*(x_0,z),q).$$

Since $y^*(x_1,z) \leqslant y^*(x_0,z)$, this contradicts the fact that V is strictly concave in the first variable. This contradiction proves that y^* is strictly increasing in x.

An entirely analogous argument can be taken to prove that y^* is strictly increasing in z.

To prove that $c^*(x,z)$ is increasing in x, assume by absurd that it is not. Then, there would be $x_0 < x_1$ such that $c^*(x_1,z) < c^*(x_0,z)$. Since

$$y^*(x,z) = zx - c^*(x,z),$$

we would also have $y^*(x_0,z) < y^*(x_1,z)$. Thus, we are essentially in the same case as before, and the same reasoning establishes a contradiction. This proves that c^* must be increasing in x.

Again, an entirely analogous argument proves that c^* is also increasing in z, thus concluding the proof.

□

Proof of Theorem 2.2.9: From Theorem 2.2.6, V is strictly increasing in both variables and strictly concave in the first variable. Therefore, Lemma A.1.3 imply that (2.26) can be rewritten as

$$V(x,z) = \max_{y\in[0,zx]} \left\{-\frac{1}{\gamma}e^{-\gamma(zx-y)} + \beta V(y,Q_\tau[z'|z])\right\}. \tag{A.36}$$

The first order condition from (A.34) is

$$e^{-\gamma c^*(x,z)} = e^{-\gamma(zx - y^*(x,z))} = \beta \partial_x V(y^*(x,z), Q_\tau[z'|z]). \tag{A.37}$$

The envelope condition stated in Theorem 2.2.6 implies that

$$\partial_x V(x,z) = ze^{-\gamma(zx-y^*(x,z))} = ze^{-\gamma c^*(x,z)}. \tag{A.38}$$

Substituting (A.38) in (A.37) yields

$$\begin{aligned} e^{-\gamma c^*(x,z)} &= \beta Q_\tau[z'|z] e^{-\gamma c^*(x, Q_\tau[z'|z])} \\ &= \beta Q_\tau[z'|z] e^{-\gamma Q_\tau[c^*(x,z')|z]}, \end{aligned} \tag{A.39}$$

where we used Lemma A.1.3 in the second equality since $c(x,z)$ is assumed to be increasing in z.

In sequential notation, (A.39) can be rewritten as

$$e^{-\gamma c_t} = \beta Q_\tau[z_{t+1}|z_t] e^{-\gamma Q_\tau[c_{t+1}|z_t]}. \tag{A.40}$$

Taking log in both sides of (A.40) will produce (2.28) after a rearrangement.

□

Proof of Theorem 2.2.10: It is straightforward to verify that this model satisfies Assumption 1, so the result is a direct application of Theorem 2.3.2.

□

Proof of Theorem 2.2.11: It is easy to verify that this model satisfies Assumptions 1, 2 and 14, hence Theorems 2.3.2 and 2.3.7 apply. From Theorem 2.3.12, the Euler equation has the following representation:

$$\begin{aligned} &- U'(g(k_t, z_t) + (1-\delta)k_t - k_{t+1}) \\ &+ \beta Q_\tau\left[U'(g(k_{t+1}, z_{t+1}) + (1-\delta)k_{t+1} - k_{t+2})(g_k(k_{t+1}, z_{t+1}) + (1-\delta))|z_t\right] = 0. \end{aligned} \tag{A.41}$$

By noting that $c_t = g(k_t, z_t) + (1-\delta)k_t - k_{t+1}$ and rearranging, one can express the above equation as

$$Q_\tau\left[\beta \frac{U'(c_{t+1})}{U'(c_t)}(g_k(k_{t+1}, z_{t+1}) + (1-\delta))\Big| z_t\right] = 1,$$

thus establishing (2.32).

Now let $y^*(k,z)$ be the optimal policy. Since $V(k,z)$ is increasing in the z variable, Lemma A.2 from de Castro and Galvao (2019) implies

$$Q_\tau\left[V(k', w)|z\right] = V\left(k', Q_\tau[w|z]\right),$$

so (2.35) can be written as

$$V(k,z) = \max_{k' \in [0, g(k,z) + (1-\delta)k]} \left\{U(g(k,z) + (1-\delta)k - k') + \beta V\left(k', Q_\tau[w|z]\right)\right\}. \tag{A.42}$$

Taking the first order condition at (A.42), we obtain

$$U'(g(k,z)+(1-\delta)k-k') = \beta\frac{\partial V}{\partial k}(k', Q_\tau[w|z]) \tag{A.43}$$

For a contradiction, assume that, for some $\bar{k} > k$, we have $\bar{k}' = y^*(\bar{k},z) \leqslant y^*(k,z) = k'$. Then $g(k,z)+(1-\delta)k-k' < g(\bar{k},z)+(1-\delta)\bar{k}-\bar{k}'$, which implies that

$$U'(g(k,z)+(1-\delta)k-k') > U'\left(g(\bar{k},z)+(1-\delta)\bar{k}-\bar{k}'\right)$$

since u is strictly concave. Hence, (A.43) implies that

$$\frac{\partial V}{\partial k}(k', Q_\tau[w|z]) > \frac{\partial V}{\partial k}\left(\bar{k}', Q_\tau[w|z]\right),$$

a contradiction since $\bar{k}' \leqslant k'$ and v is strictly concave in the first variable. This shows that $y^*(k,z)$ is strictly increasing in k.

□

Proof of Theorem 2.2.12: For v continuous, let $\mathbb{M}^\tau$ be defined by

$$\mathbb{M}^\tau v(x,z) = \sup_{y\in[0,zx^\alpha)}\left\{\log(zx^\alpha - y) + \beta Q_\tau\left[v(y,z')|z\right]\right\}.$$

Notice that, since $\log(c)$ is unbounded, $\mathbb{M}^\tau$ is not bounded anymore even if v is bounded (for example, $\mathbb{M}^\tau 0 = \log(zx^\alpha)$ is unbounded). Nevertheless, a direct calculation will show that $V(x,z)$ given by (2.38) is a fixed point of $\mathbb{M}^\tau$. Let

$$C = \frac{\log\left[(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\beta)(1-\alpha\beta)}.$$

Then,

$$\begin{aligned}
\mathbb{M}^\tau V(x,z) &= \sup_{y\in[0,zx^\alpha)}\left\{\log(zx^\alpha - y) + \beta Q_\tau\left[\sum_{s=0}^\infty \frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z') + \frac{\log y^\alpha}{1-\alpha\beta} + C\Big| z\right]\right\} \\
&= \sup_{y\in[0,zx^\alpha)}\left\{\log(zx^\alpha - y) + \beta\left[\sum_{s=0}^\infty \frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(Q_\tau[z'|z]) + \frac{\log y^\alpha}{1-\alpha\beta} + C\right]\right\} \\
&= \sum_{s=1}^\infty \frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z) + \beta C + \sup_{y\in[0,zx^\alpha)}\left\{\log(zx^\alpha - y) + \frac{\alpha\beta}{1-\alpha\beta}\log y\right\},
\end{aligned} \tag{A.44}$$

where we used Lemma A.2 from de Castro and Galvao (2019) in the second equality, since the $q_{\tau,s}(z)$ are increasing by a successive application of Lemma A.2.6. In the third equality, we used the recursive relation (2.39).

The first order condition for the expression in brackets from (A.44) for optimal y is

$$\frac{1}{zx^\alpha - y} = \frac{\alpha\beta}{1-\alpha\beta}\frac{1}{y},$$

hence, the optimal policy is given by

$$y = y^*(x,z) = \alpha\beta z x^\alpha.$$

Substituting this expression in (A.44) yields

$$\begin{aligned}
\mathbb{M}^\tau V(x,z) &= \sum_{s=1}^{\infty}\frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z)+\beta C+\log\left[(zx^\alpha)(1-\alpha\beta)\right]+\frac{\alpha\beta}{1-\alpha\beta}\log\left[(zx^\alpha)(\alpha\beta)\right]\\
&= \sum_{s=1}^{\infty}\frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z)+\beta C+\left(1+\frac{\alpha\beta}{1-\alpha\beta}\right)\log(zx^\alpha)+\frac{(1-\alpha\beta)\log(1-\alpha\beta)+\alpha\beta\log(\alpha\beta)}{1-\alpha\beta}\\
&= \sum_{s=1}^{\infty}\frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z)+\frac{\log(zx^\alpha)}{1-\alpha\beta}+\beta C+\frac{\log\left[(\alpha\beta)^{\alpha\beta}(1-\alpha\beta)^{1-\alpha\beta}\right]}{(1-\alpha\beta)}\\
&= \left[\sum_{s=1}^{\infty}\frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z)+\frac{\log(z)}{1-\alpha\beta}\right]+\frac{\log(x^\alpha)}{1-\alpha\beta}+\beta C+(1-\beta)C\\
&= \sum_{s=0}^{\infty}\frac{\beta^s}{1-\alpha\beta}\log q_{\tau,s}(z)+\frac{\log(x^\alpha)}{1-\alpha\beta}+C\\
&= V(x,z).
\end{aligned}$$

□

Discussion on Remark 2.2.13: Consider $\alpha,\gamma\neq 1$, $U(c)=c^{(1-\gamma)}/(1-\gamma)$, $c=zx^\alpha-y$. We then ask if we could have the optimal path of the form

$$y^*(x,z)=f(z)x^\alpha?$$

Since, for $u(x,y,z)=U(zx^\alpha-y)$, we have

$$\partial_1 v(x,z)=\partial_1 u(x,y^*(x,z),z)=\alpha z x^{\alpha-1}(zx^\alpha-y^*(x,z))^{-\gamma},$$

then

$$\partial_1 v(y^*(x,z),w)=\alpha w(y^*(x,z))^{\alpha-1}(w(y^*(x,z))^\alpha-y^*(y^*(x,z),w))^{-\gamma},$$

thus the substitution $y^*(x,z)=f(z)x^\alpha$ produces

$$\partial_1 v(y^*(x,z),w)=\alpha w(f(z)x^\alpha)^{\alpha-1}(w(f(z)x^\alpha)^\alpha-f(w)\,(f(z)x^\alpha)^\alpha)^{-\gamma}.$$

Therefore, the FOC

$$(zx^\alpha-y^*(x,z))^{-\gamma}=\beta Q_\tau\left[\partial_1 v(y^*(x,z),w)|z\right]$$

becomes

$$\begin{aligned}
(zx^\alpha-f(z)x^\alpha)^{-\gamma} &= \beta Q_\tau\left[\alpha w f(z)^{\alpha-1}x^{\alpha(\alpha-1)}\left(wf(z)^\alpha x^{\alpha^2}-f(w)f(z)^\alpha x^{\alpha^2}\right)^{-\gamma}|z\right]\\
&= \alpha\beta f(z)^{\alpha-1-\alpha\gamma}x^{\alpha(\alpha-1)-\gamma\alpha^2}Q_\tau\left[w(w-f(w))^{-\gamma}|z\right].
\end{aligned}$$

Thus,

$$(z-f(z))^{-\gamma}=\alpha\beta f(z)^{\alpha-1-\alpha\gamma}x^{\alpha(\alpha-1)(1-\gamma)}Q_\tau\left[w(w-f(w))^{-\gamma}|z\right].$$

We can rearrange the terms to get

$$x^{\alpha(1-\alpha)(1-\gamma)} = \alpha\beta(z - f(z))^{\gamma} f(z)^{\alpha-1-\alpha\gamma} Q_{\tau}\left[w(w - f(w))^{-\gamma}|z\right]. \tag{A.45}$$

The LHS is a non-constant function of x, while the RHS is a function of z. Thus, for $\alpha \neq 1$, $\gamma \neq 1$, it is not possible to have $y^*(x,z) = f(z)x^{\alpha}$. In the special case $\alpha = 1$ (this is the intertemporal consumption model), we can solve for f, like we did above in (2.18), because the LHS will be equal to 1. Although we cannot automaticaly say that (A.45) is valid for $\gamma = 1$, taking this value in (A.45) makes the LHS again be 1, making it again possible to solve for f. This could make us understand why in the log utility we have $y^*(x,z) = f(z)x^{\alpha}$ as seen in (2.40).

Thus, the conclusion is that, even in the iid case, for $\alpha \neq 1$, $\gamma \neq 1$, it is not possible to have $y^*(x,z) = f(z)x^{\alpha}$.

Proof of Theorem 2.2.16: Assumption 1 hold, hence Theorem 2.3.2 establishes the result. □

Proof of Theorem 2.2.17: Assumptions 1, 2, 14, 15 and 17 hold, so Theorems 2.3.7, 2.3.10 and 2.3.12 establish the result.

□

Proof of Theorem 2.2.18: Under Assumption 7, Assumption 1 holds, so Theorem 2.3.2 assures the existence of a solution to the functional equation (2.43). To see that $V(K,A) = \varphi(A)K$, consider the action of the operator $\mathbb{M}^{\tau}$ over functions of the form $v(K,A) = h(A)K$, where h is continuous and $\mathbb{M}^{\tau}$ is given by

$$\mathbb{M}^{\tau}v(K,A) = \max_{K'}\left\{AK - \frac{\gamma}{2}\left(\frac{K' - (1-\delta)K}{K}\right)^2 K - p(K' - (1-\delta)K) + \beta Q_{\tau}\left[v(A',K')\middle|A\right]\right\}.$$

We have

$$\begin{aligned}
\mathbb{M}^{\tau}v(K,A) &= \max_{K'}\left\{AK - \frac{\gamma}{2}\left(\frac{K' - (1-\delta)K}{K}\right)^2 K - p(K' - (1-\delta)K) + \beta Q_{\tau}\left[h(A')K'\middle|A\right]\right\} \\
&= \max_{K'}\left\{AK - \frac{\gamma}{2}\left(\frac{K' - (1-\delta)K}{K}\right)^2 K - p(K' - (1-\delta)K) + \beta K' Q_{\tau}\left[h(A')\middle|A\right]\right\}.
\end{aligned}$$

The first order condition over K' is

$$\gamma\left(\frac{K' - (1-\delta)K}{K}\right) + p = \beta Q_{\tau}\left[h(A')\middle|A\right].$$

In particular, this implies that investment satisfies

$$I = K' - (1-\delta)K = \frac{K}{\gamma}\left(\beta Q_{\tau}\left[h(A')\middle|A\right] - p\right). \tag{A.46}$$

Hence, once we conclude the proof that $v(K,A) = \varphi(A)K$, (A.46) will establish that the investment rate

$$i \equiv \frac{I}{K} = \frac{1}{\gamma}\left(\beta Q_\tau\left[\varphi(A')\middle|A\right] - p\right)$$

does not depend on the current capital level K. Continuing our computation substituting the first order condition, we obtain

$$\begin{aligned}
\mathbb{M}^\tau v(K,A) &= AK - \frac{1}{2\gamma}K\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right)^2 - \frac{p}{\gamma}K\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right) \\
&\quad + \beta Q_\tau\left[h(A')\middle|A\right]K\left[\frac{1}{\gamma}\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right) + 1 - \delta\right] \\
&= \left\{A - \frac{1}{2\gamma}\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right)^2 - \frac{p}{\gamma}\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right)\right. \\
&\quad \left. + \beta Q_\tau\left[h(A')\middle|A\right]\left[\frac{1}{\gamma}\left(\beta Q_\tau\left[h(A')\middle|A\right] - p\right) + 1 - \delta\right]\right\}K. \qquad \text{(A.47)}
\end{aligned}$$

This proves that $\mathbb{M}^\tau v(K,A) = \hat{h}(A)K$ for some $\hat{h}$. Since we can arrive at the value function $V(K,A)$ by iterations of $\mathbb{M}^\tau$ starting at functions of the type $v(K,A) = h(A)K$, this establishes that $V(K,A) = \varphi(A)K$ for some φ.

Finally, to see that φ is increasing, let

$$u(K,K',A) \equiv AK - \frac{\gamma}{2}\left(\frac{K' - (1-\delta)K}{K}\right)^2 K - p(K' - (1-\delta)K),$$

which is strictly increasing in A. With this notation, the operator $\mathbb{M}^\tau$ takes the form

$$\mathbb{M}^\tau v(K,A) = \max_{K'}\left\{u(K,K',A) + \beta Q_\tau\left[v(K',A')\middle|A\right]\right\}. \qquad \text{(A.48)}$$

Let $v(K,A)$ be a continuous function strictly increasing in both K and A. Therefore, Lemma A.1.1 implies that

$$Q_\tau\left[v(K',A')\middle|A\right] = v\left(K', Q_\tau\left[A'|A\right]\right).$$

Let $A_1 < A_2$ and let K_j', $j = 1,2$, be the values that realize the maxima in the corresponding functional equations (A.48) for $\mathbb{M}^\tau v(K,A_j)$.

Then, we have

$$\begin{aligned}
\mathbb{M}^\tau v(K,A_1) &= u(K,K_1',A_1) + \beta v\left(K_1', Q_\tau\left[A'|A_1\right]\right) \\
&< u(K,K_1',A_2) + \beta v\left(K_1', Q_\tau\left[A'|A_2\right]\right) \\
&\leqslant u(K,K_2',A_2) + \beta v\left(K_2', Q_\tau\left[A'|A_2\right]\right) \\
&= \mathbb{M}^\tau v(K,A_2),
\end{aligned}$$

where the first inequality uses the fact that both u and v are strictly increasing in A, together with Lemma A.2.6, which assures that $Q_\tau[A'|A_1] \leqslant Q_\tau[A'|A_2]$. The second inequality uses the fact that K_2' is a maximum on $[(1-\delta)K, M]$.

Since the set $\mathcal{C}'$ of continuous increasing functions on K and A is closed, and we proved that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, where $\mathcal{C}'' \subset \mathcal{C}'$ is the subset of strictly increasing functions on A, this shows that the fixed point V of $\mathbb{M}^\tau$ is strictly increasing in A. Since $V(K,A) = \varphi(A)K$, φ must be strictly increasing. Therefore, Lemma A.1.1 can be used, and (A.47) can be written for V as

$$\begin{aligned}\varphi(A)K &= \left\{A - \frac{1}{2\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right]^2 - \frac{p}{\gamma}\left[\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right]\right. \\ &\left. +\beta\varphi\left(Q_\tau\left[A'|A\right]\right)\left[\frac{1}{\gamma}\left(\beta\varphi\left(Q_\tau\left[A'|A\right]\right) - p\right) + 1 - \delta\right]\right\}K,\end{aligned}$$

thus establishing (2.46).

□

Proof of Theorem 2.2.21: We cannot use directly Theorem 2.3.2 to prove Theorem 2.2.21 because it requires maximization in a compact set, but in (2.49) y is maximized over $\mathbb{R}_{++}$, which is not compact. In order to circumvent this problem, we show that (2.49) is equivalent to the following

$$V(x,z) = \sup_{y\in\Gamma(x)}\left\{U(x,z) - xc\left(\frac{y}{x}\right) + \beta Q_\tau\left[V(y,z')\Big|z\right]\right\}, \tag{A.49}$$

where $\Gamma : \mathbb{R}_{++} \to R_{++}$ is defined by

$$\Gamma(x) = \begin{cases}[(1-\delta')x, M], & \text{if } x \in (0,M] \\ [(1-\delta')x, x], & \text{if } x \in (M,+\infty)\end{cases} \tag{A.50}$$

for some $\delta' \in (\delta, 1)$ and $M > 0$ suitably chosen below.

Observe that Γ is a non-empty, compact-valued and continuous correspondence (since its endpoints are continuous functions of x). Therefore, Assumption 1 hold, except for item (iv). Although $u(x,y,z) = U(x,z) - xc(y/x)$ is continuous, it is not bounded. A closer look at the proof of Theorem 2.3.2 shows that, indeed, all we really need is to guarantee that $\mathbb{M}^\tau$ maps $\mathcal{C}$ to $\mathcal{C}$, where $\mathcal{C}$ denotes the Banach space of bounded and continuous functions, and, for $v \in \mathcal{C}$,

$$\mathbb{M}^\tau v(x,z) = \sup_{y\in\Gamma(x)}\left\{U(x,z) - xc\left(\frac{y}{x}\right) + \beta Q_\tau\left[v(y,z')\Big|z\right]\right\}.$$

The only problem to deal with is to show that $\mathbb{M}^\tau v$ is bounded whenever $v \in \mathcal{C}$. The rest of the result will follow exactly as in the proof of Theorem 2.3.2.

We have

$$\begin{aligned}\mathbb{M}^\tau v(x,z) &= \sup_{y\in\Gamma(x)}\left\{U(x,z) - xc\left(\frac{y}{x}\right) + \beta Q_\tau\left[v(y,z')\Big|z\right]\right\} \\ &\leqslant \sup_{y\in\Gamma(x)}\left\{U(x,z) - xc\left(\frac{y}{x}\right)\right\} + \beta \sup_{y\in\Gamma(x)}\left\{Q_\tau\left[v(y,z')\Big|z\right]\right\} \\ &\leqslant B + \beta\|v\|,\end{aligned}$$

where $\|v\| = \sup_{x,z}|v(x,z)|$. On the other hand, making $y = (1-\delta')x < (1-\delta)x$, $c\left(\frac{y}{x}\right) = 0$ and

$$\begin{aligned}\mathbb{M}^\tau v(x,z) &\geqslant U(x,z) - xc(1-\delta') + \beta Q_\tau\left[v\left((1-\delta')x, z'\right)\Big|z\right] \\ &= U(x,z) + \beta Q_\tau\left[v\left((1-\delta')x, z'\right)\Big|z\right]\end{aligned}$$

$$\begin{aligned} &\geqslant 0 - \beta\|\nu\| \\ &\geqslant -(B + \beta\|\nu\|). \end{aligned}$$

Therefore, $\|\mathbb{M}^\tau \nu\| \leqslant B + \beta\|\nu\|$, which proves the required boundedness. Then, the proof of Theorem 2.3.2 shows that there exists an unique bounded and continuous solution V to (A.49). It remains to show that V also satisfies (2.49).

For this, we must show that for each x, the solution $\bar{y}$ of the maximization problem (2.49) is such that $\bar{y} \in \Gamma(x)$. Choose M such that, for all $y \geqslant M$,

$$B - Mc\left(\frac{y}{M}\right) + B\frac{\beta}{1-\beta} < 0 \tag{A.51}$$

and, for all $x \leqslant M$,

$$B - xc\left(\frac{M}{x}\right) + B\frac{\beta}{1-\beta} < 0. \tag{A.52}$$

In our analysis, we will consider the sequential problem for (2.49):

$$\sup_{\{x_t\}_{t=0}^{\infty}} Q_\tau^\infty\left[\sum_{t=0}^{\infty} \beta^t \left(U(x_t, z_t) - x_t c\left(\frac{x_{t+1}}{x_t}\right)\right)\right], \tag{A.53}$$

where the notation $Q_\tau^\infty[\cdot]$ is explained in de Castro and Galvao (2019). By the Principle of Optimality, an optimal policy of a solution to (2.49) also attains the supremum in (A.53).

Now let $x > M$. This implies that $\bar{y} \leqslant x$, otherwise (A.51) would imply that (A.53) would be negative for any $x_0 \geqslant M$. This would be true because, by the Principle of Optimality, an optimal policy of a solution to (2.49) also attains the supremum in (A.53). However, by taking, for example, $y = (1-\delta')x$, one sees that the solution to (A.53) cannot be negative. This contradiction establishes that $\bar{y} \leqslant x$ if $x > M$.

Now let $x \leqslant M$. The same argument, this time using (A.52), shows that $\bar{y} < M$ when $x \leqslant M$. Since $c(a) = 0$ if $a \leqslant (1-\delta)$, we see that $\bar{y} > (1-\delta')x$ in both cases. This establishes that $\bar{y} \in \Gamma(x)$ for all x, completing the proof. □

Proof of Theorem 2.2.22 Once Theorem 2.2.21 establishes the existence of the value function V, we begin by considering the optimal policy. Let $u(x, y, z) = U(x, z) - xc(y/x)$. Since c is strictly convex by assumption and U is strictly concave (a direct consequence of the strictly decreasingness of D in the first variable and equation (2.47)), one has

$$\begin{aligned} u_{xx} &= U_{xx} - \frac{y^2}{x^3}c''\left(\frac{y}{x}\right) < 0 \\ u_{yy} &= -\frac{1}{x}c''\left(\frac{y}{x}\right) < 0 \\ u_{xy} &= \frac{y}{x^2}c''\left(\frac{y}{x}\right) > 0. \end{aligned}$$

Let H be the Hessian matrix of u. Then, for any vector (ξ, η),

$$(\xi, \eta) \cdot H(\xi, \eta) = u_{xx}\xi^2 + u_{yy}\eta^2 + 2u_{xy}\xi\eta$$

$$
\begin{aligned}
&= \xi^2 U_{xx} - \frac{1}{x} c''\left(\frac{y}{x}\right)\left[\left(\xi\frac{y}{x}\right)^2 + \eta^2 - 2\left(\xi\frac{y}{x}\right)\eta\right] \\
&= \xi^2 U_{xx} - \frac{1}{x} c''\left(\frac{y}{x}\right)\left(\xi\frac{y}{x} - \eta\right)^2 < 0,
\end{aligned}
$$

so H is negative definite. This implies that u is strictly concave in the first two variables. Then, Lemma A.2.5 proves that $V(x,z)$ is strictly concave in x. As a consequence, the optimal policy Υ is single-valued, and its continuity follows from this and the Maximum Theorem for correspondences. Also, Theorem 2.3.10 shows that $V(x,z)$ is differentiable in the first variable.

On the other hand, since $U(x,z)$ is strictly increasing in z, Lemma A.2.7 can be applied to show that $V(x,z)$ is strictly increasing in the second variable. Then, by Lemma A.2 from de Castro and Galvao (2019), we have

$$Q_\tau[V(y,w)|z] = V\left(y, Q_\tau[w|z]\right). \tag{A.54}$$

Therefore, the first order condition for (2.49) can be written as

$$c'\left(\frac{y^*(x,z)}{x}\right) = \beta\frac{\partial V}{\partial x}\left(y^*(x,z), Q_\tau[w|z]\right). \tag{A.55}$$

For a contradiction, suppose that $y^*(x,z) > y^*(x',z)$ for some $x < x'$. Since V is strictly concave in the first variable, it follows that $\frac{\partial V}{\partial x}$ is decreasing in x. Hence,

$$\frac{\partial V}{\partial x}\left(y^*(x,z), Q_\tau[w|z]\right) \leqslant \frac{\partial V}{\partial x}\left(y^*(x',z), Q_\tau[w|z]\right).$$

Then, (A.55) implies

$$c'\left[\frac{y^*(x,z)}{x}\right] \leqslant c'\left[\frac{y^*(x',z)}{x'}\right],$$

contradicting the strictly convexity of c. This shows that $y^*(x,z)$ must be strictly increasing in the first variable.

Now, for another contradiction, assume that $y^*(x,z)/x$ is not strictly decreasing. That is, there exist $x < x'$ such that $y^*(x,z)/x \leqslant y^*(x',z)/x'$. Since c is convex, it follows that

$$c'\left[\frac{y^*(x,z)}{x}\right] \leqslant c'\left[\frac{y^*(x',z)}{x'}\right].$$

Therefore, (A.55) implies

$$\frac{\partial v}{\partial x}\left(y^*(x,z), Q_\tau[w|z]\right) \leqslant \frac{\partial v}{\partial x}\left(y^*(x',z), Q_\tau[w|z]\right),$$

a contradiction since $y^*(x',z) > y^*(x,z)$ (from the last argument) and we know that v is strictly concave, so $\frac{\partial v}{\partial x}$ is decreasing in x. This shows that $y^*(x,z)/x$ is strictly decreasing in x.

Since we showed that the results from Theorem 2.3.10 are valid, the proof of Theorem 2.3.12 can be reproduced to show the existence of the Euler Equations. Then, a direct computation based on this result establishes (2.50), and the proof is complete. □

Proof of Theorem 2.2.23: We cannot apply directly Theorem 2.3.2 because the shocks are neither connected nor finite. However, this is a limitation only with respect to the validity of Lemma 2.3.1, which is used in Theorem 2.3.2 to assure that the value function is continuous. Since, however, we are not worried about continuity, the rest of the proof of Theorem 2.3.2 can be used to establish existence of a unique bounded solution, since we still have a contraction in the Banach space of bounded real-valued functions under the sup norm. Hence, the stated result will hold. □

Proof of Theorem 2.2.24: Notice first that, although Assumption 10-(v) states that U is unbounded over $\mathbb{R}_+$, it is indeed bounded over its domain $\mathcal{X}$, since $\mathcal{X} \subseteq [0, \bar{w}]$ by Assumption 10-(vi). Then, Theorem 2.2.23 assures the existence of a unique fixed point v for the operator $\mathbb{M}^\tau : \mathcal{B} \to \mathcal{B}$ (where $\mathcal{B}$ denotes the Banach space of bounded real-valued functions) given by

$$\mathbb{M}^\tau g(x) = \max\left\{\beta Q_\tau\left[g(w')\right], U(x) + \beta Q_\tau\left[g(e'x)\right]\right\}.$$

Since $\mathbb{M}^\tau$ is a contraction by Theorem 2.3.2 (applied to $\mathcal{B}$ instead of $\mathcal{C}$, see the proof of Theorem 2.2.23), its fixed point v can be reached by iteration. Since the subset $\mathcal{B}'$ of bounded and increasing functions is closed, if we prove that $\mathbb{M}^\tau(\mathcal{B}') \subset \mathcal{B}'$, this establishes that v is increasing.

Let $g \in \mathcal{B}'$, $x < x'$. Then $g(e'x) \leqslant g(e'x')$, so Lemma A.1-(vi) from de Castro and Galvao (2019) implies that

$$Q_\tau[g(e'x)] \leqslant Q_\tau[g(e'x')].$$

Since U is increasing, this is sufficient to show that $\mathbb{M}^\tau g$ is increasing if one notices that $\beta Q_\tau[g(w')]$ is a constant independent of x. As a consequence, the fixed point v is increasing.

Hence, by Lemma A.1.1, we can rewrite the functional equation (2.54) as

$$v(x) = \max\left\{\beta v\left(Q_\tau[w']\right), U(x) + \beta v\left(Q_\tau[e']x\right)\right\} \tag{A.56}$$

Let $A \equiv \beta v\left(Q_\tau[w']\right)$. We will show latter that A satisfies indeed (2.56).

Substituting $x = 0$ and A in (A.56) and recalling that $U(0) = 0$ yields

$$v(0) = \max\left\{A, \beta v(0)\right\}$$

Hence, if $v(0) \neq 0$, then $v(0) = A$ since $0 < \beta < 1$ makes it impossible to have $v(0) = \beta v(0)$. Otherwise, if $v(0) = 0$, then $0 = v(0) \geqslant A = \beta v(Q_\tau[w']) \geqslant \beta v(0) = v(0) = 0$, since v is increasing and $Q_\tau[w'] > 0$ by assumption. This implies that $A = 0$. So, indeed,

$$v(0) = A \tag{A.57}$$

is always true. We will latter show that A cannot be zero.

Notice that the discreteness of e implies that $Q_\tau[e']$ is either 0 or 1. Assume first that $Q_\tau[e'] = 0$. Then (A.56) becomes

$$v(x) = \max\left\{A, U(x) + \beta A\right\} \tag{A.58}$$

by (A.57). Since U is continuous, strictly increasing, $U(0) = 0$ and $\lim_{x\to\infty} U(x) = +\infty$, there exists a unique x^* such that

$$U(x^*) + \beta A = A,$$

so

$$U(x^*) = (1 - \beta)A \tag{A.59}$$

and

$$A = \frac{U(x^*)}{1-\beta}.$$

We can guarantee that $x^* \in \mathcal{X}$ by taking $\bar{w}$ large enough so that $U(\bar{w}) \geqslant A(1-\beta)$, where A is given by (2.56) (as will be indeed the case, as we will show below). Therefore, (A.58) implies that

$$v(x) = \begin{cases} A, & \text{if } x \leqslant x^* \\ U(x) + \beta A, & \text{if } x > x^* \end{cases}. \tag{A.60}$$

Notice that (A.60) and (A.59) will agree, respectively, with (2.55) and (2.57) in the case where $Q_\tau[e'] = 0$ as long as we prove that

$$A = \frac{\beta}{1-\beta^2} U\left(Q_\tau[w']\right). \tag{A.61}$$

Also, (A.61) would also agree with (2.56) when $Q_\tau[e'] = 0$, so now we aim to establish (A.61).

If $A = 0$, then (A.60) would imply that $v(x) = U(x)$. Therefore, we would have

$$0 = \beta v\left(Q_\tau[w']\right) = U\left(Q_\tau[w']\right),$$

a contradiction with the assumptions, since $U(Q_\tau[w']) > U(0) = 0$ as $Q_\tau[w'] > 0$ and U is strictly increasing. So A cannot be zero.

Since $A = \beta v\left(Q_\tau[w']\right) \neq 0$ and $0 < \beta < 1$, we have $v\left(Q_\tau[w']\right) > A$, so $Q_\tau[w'] > x^*$ and, by (A.60), we have

$$A/\beta = v\left(Q_\tau[w']\right) = U\left(Q_\tau[w']\right) + \beta A,$$

which establishes (A.61). This completes the case where $Q_\tau[e'] = 0$.

Now assume that $Q_\tau[e'] = 1$. Then (A.56) becomes

$$v(x) = \max\left\{A, U(x) + \beta v(x)\right\}. \tag{A.62}$$

Since U is continuous, strictly increasing, $U(0) = 0$ and $\lim_{x\to\infty} U(x) = +\infty$, there exists a unique x^* such that

$$v(x^*) = U(x^*) + \beta v(x^*) = A,$$

so

$$U(x^*) = (1-\beta)A \tag{A.63}$$

as before and

$$A = \frac{U(x^*)}{1-\beta}$$

also as before. Therefore, (A.62) implies that

$$v(x) = \begin{cases} A, & \text{if } x \leqslant x^* \\ \frac{U(x)}{1-\beta}, & \text{if } x > x^* \end{cases}. \tag{A.64}$$

Notice that (A.64) and (A.63) will agree, respectively, with (2.55) and (2.57) in the case where $Q_\tau[e'] = 1$ as long as we prove that

$$A = \frac{\beta}{1-\beta} U\left(Q_\tau[w']\right). \tag{A.65}$$

Also, just as in the previous case, (A.65) would also agree with (2.56) when $Q_\tau[e'] = 1$, so now we aim to establish (A.65).

If $A = 0$, then (A.64) would imply that $v(x) = U(x)/(1-\beta)$. Therefore, we would have

$$0 = A/\beta = v\left(Q_\tau[w']\right) = U\left(Q_\tau[w']\right)/(1-\beta),$$

a contradiction with the assumptions, since $U(Q_\tau[w']) > U(0) = 0$ as $Q_\tau[w'] > 0$ and U is strictly increasing. So A cannot be zero.

Since $A = \beta v\left(Q_\tau[w']\right) \neq 0$ and $0 < \beta < 1$, we have $v\left(Q_\tau[w']\right) > A$, so $Q_\tau[w'] > x^*$ and, by (A.64), we have

$$A/\beta = v\left(Q_\tau[w']\right) = U\left(Q_\tau[w']\right)/(1-\beta),$$

which establishes (A.65). This completes the case where $Q_\tau[e'] = 1$, and the whole proof is done. □

A.2.2 Proofs of Section 2.3

Proof of Lemma 2.3.1: Since ϕ is continuous, by setting $y' = (x, y)$ and $v'(y', w) = v(\phi(x, y, w), w)$, it suffices to prove that $(y', z) \mapsto Q_\tau[v'(y', w)|z]$ is continuous. We proceed in this direction, simply writing y and v instead of y' and v', respectively.

Consider a sequence $(y^n, z^n) \to (y^*, z^*)$. Let $K : \mathcal{Z}\times\Sigma \to [0,1]$ be the transition function representing the Markov process of the shocks $\mathcal{Z}$, where Σ is the Borel σ-algebra. Let

$$m^n(\alpha) \equiv \Pr\left(\{w : v(y^n, w) \leqslant \alpha\}|z^n\right) = K\left(z^n, \{w : v(y^n, w) \leqslant \alpha\}\right)$$

and

$$m^*(\alpha) \equiv \Pr\left(\{w : v(y^*, w) \leqslant \alpha\}|z^*\right) = K\left(z^*, \{w : v(y^*, w) \leqslant \alpha\}\right)$$

Let $\alpha^n \equiv \inf\{\alpha \in \mathbb{R} : m^n(\alpha) \geqslant \tau\} = Q_\tau[v(y^n, \cdot)|z^n]$ and $\alpha^* \equiv \inf\{\alpha \in \mathbb{R} : m^*(\alpha) \geqslant \tau\} = Q_\tau[v(y^*, \cdot)|z^*]$. We want to show that $\alpha^n \to \alpha^*$. We will proceed in two main parts, first showing that $\liminf_n \alpha^n \geqslant \alpha^*$ and then showing that $\alpha^* \leqslant \limsup_n \alpha^n$. This second part is more delicate and will require different proofs depending on whether $\mathcal{Z}$ is connected or finite, and on the value of $K(z^*, \{v(y^*, w) \leqslant \alpha^*\})$.

Step 1. ($\liminf_n \alpha^n \geqslant \alpha^*$)

Let $\underline{\alpha} \equiv \liminf_n \alpha^n$. We will show that $\underline{\alpha} \geqslant \alpha^*$ by contradiction. So, assume that $\underline{\alpha} < \alpha^*$. This means that there exists $\epsilon > 0$ and a subsequence n_j such that $\alpha^{n_j} \to \underline{\alpha}$, with $\underline{\alpha} + \epsilon < \alpha^*$.

Since $m^*(\underline{\alpha} + \epsilon) < \tau$ because $\underline{\alpha} + \epsilon < \alpha^* = Q_\tau[v(y^*, w)|z^*]$, we can take $\eta \in (0, 1)$ sufficiently small such that

$$m^*(\underline{\alpha} + \epsilon) + \eta < \tau. \tag{A.66}$$

By Assumption 1-(i), there exists $\mathcal{Z}' \subset \mathcal{Z}$ compact such that

$$K(z^*, \mathcal{Z}\backslash\mathcal{Z}') < \eta/4. \tag{A.67}$$

Let D be a compact set containing the sequence (y^n) (and, of course, its limit y^*). Then, since v is continuous, it is uniformly continuous in the compact $D \times \mathcal{Z}'$. Hence, there exists $j_1 \in \mathbb{N}$ such that if $j \geqslant j_1$ then

$$|v(y^{n_j}, w) - v(y^*, w)| < \epsilon/2 \quad \forall w \in \mathcal{Z}' \quad \text{and} \quad |\underline{\alpha} - \alpha^{n_j}| < \epsilon/2.$$

Thus, if $j \geqslant j_1$ and $w \in \mathcal{Z}'$ is such that $\nu(y^{n_j}, w) \leqslant \alpha^{n_j}$, then

$$\nu(y^*, w) \leqslant \nu(y^{n_j}, w) + \epsilon/2 \leqslant \alpha^{n_j} + \epsilon/2 < \underline{\alpha} + \epsilon < \alpha^*,$$

so

$$\{w \in \mathcal{Z}'; \nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \subset \{w \in \mathcal{Z}'; \nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\}. \tag{A.68}$$

Moreover, by Assumption 1-(ii), we can also assume that $j \geqslant j_1$ implies

$$K(z^{n_j}, \mathcal{Z} \backslash \mathcal{Z}') < K(z^*, \mathcal{Z} \backslash \mathcal{Z}') + \eta/4 \tag{A.69}$$

and

$$K(z^*, \{\nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\} \cap \mathcal{Z}') > K(z^{n_j}, \{\nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\} \cap \mathcal{Z}') - \eta/2. \tag{A.70}$$

Then, we have, for $j \geqslant j_1$,

$$\begin{aligned}
\tau &\leqslant K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\}) \\
&\leqslant K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \cap \mathcal{Z}') + K(z^{n_j}, \mathcal{Z} \backslash \mathcal{Z}') \\
&\leqslant K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \cap \mathcal{Z}') + K(z^*, \mathcal{Z} \backslash \mathcal{Z}') + \eta/4 && \text{by (A.69)} \\
&< K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \cap \mathcal{Z}') + \eta/4 + \eta/4 && \text{by (A.67)} \\
&= K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \cap \mathcal{Z}') + \eta/2.
\end{aligned}$$

Hence,

$$\begin{aligned}
m^*(\underline{\alpha} + \epsilon) &= K(z^*, \{\nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\}) \\
&\geqslant K(z^*, \{\nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\} \cap \mathcal{Z}') \\
&> K(z^{n_j}, \{\nu(y^*, w) \leqslant \underline{\alpha} + \epsilon\} \cap \mathcal{Z}') - \eta/2 && \text{by (A.70)} \\
&\geqslant K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \cap \mathcal{Z}') - \eta/2 && \text{by (A.68)} \\
&> \tau - \eta/2 - \eta/2 = \tau - \eta.
\end{aligned}$$

Thus, $m^*(\underline{\alpha} + \epsilon) + \eta > \tau$. This, however, is contradiction against the choice of η in (A.66). The contradiction shows that $\alpha^* \leqslant \underline{\alpha}$.

Step 2. ($\limsup_n \alpha^n \leqslant \alpha^*$ for $\mathcal{Z}$ connected and $K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) < 1$)

Now it remains to show that $\bar{\alpha} \equiv \limsup_n \alpha^n \leqslant \alpha^*$. This time, we will proceed in different ways according to whether $\mathcal{Z}$ is connected or finite. In this step, we deal with the connected case under the assumption that

$$K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) < 1. \tag{A.71}$$

We will again proceed by contradiction. Assume, for an absurd, that $\bar{\alpha} > \alpha^*$. There exists a subsequence n_j such that $\alpha^{n_j} \to \bar{\alpha}$.

By (A.71), there exists some $w_0 \in \mathcal{Z}$ such that $\nu(y^*, w_0) > \alpha^*$. By the definition of α^*, we also have $K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) \geqslant \tau > 0$. The fact that $\nu(y^*, \cdot)$ is continuous and $\mathcal{Z}$ is connected implies that its range is also connected. Therefore, for sufficiently small $\epsilon > 0$ such that $\alpha^* + \epsilon < \nu(y^*, w_0)$ and also

$$\alpha^* + \epsilon < \bar{\alpha} - \epsilon, \tag{A.72}$$

$\{\alpha^* + \epsilon < \nu(y^*, w) < \bar{\alpha} - \epsilon\} \subset \mathcal{Z}$ is nonmepty. Since ν is continuous, this set is also open. Therefore, by Assumption 1-(iii),

$$K(z^*, \{\alpha^* + \epsilon < \nu(y^*, w) < \bar{\alpha} - \epsilon\}) > 0. \tag{A.73}$$

We now want to show that

$$\limsup_{j \to \infty} m^{n_j}(\alpha^* + \epsilon/2) < \tau. \tag{A.74}$$

In face of (A.73), Assumption 1-(i) enables us to find a compact $\mathcal{Z}' \subset \mathcal{Z}$ such that

$$K(z^*, \{\alpha^* + \epsilon < \nu(y^*, w) < \bar{\alpha} - \epsilon\} \cap \mathcal{Z}') > 0.$$

In order to prove (A.74), it suffices to show that

$$\{\alpha^* + \epsilon < \nu(y^*, w) < \bar{\alpha} - \epsilon\} \cap \mathcal{Z}' \subset \{\alpha^* + \epsilon/2 < \nu(y^{n_j}, w) \leqslant \alpha^{n_j}\} \quad \text{for all } j \text{ large enough.} \tag{A.75}$$

Write $A \equiv \{\alpha^* + \epsilon < \nu(y^*, w) < \bar{\alpha} - \epsilon\} \cap \mathcal{Z}'$ for simplicity. Then, if (A.75) holds, Assumption 1-(ii) will imply that, since $K(z^{n_j}, A) \to K(z^*, A) > 0$, there exists some $M > 0$ such that

$$K(z^{n_j}, \{\alpha^* + \epsilon/2 < \nu(y^{n_j}, w) \leqslant \alpha^{n_j}\}) \geqslant K(z^{n_j}, A) \geqslant M$$

for all j sufficiently large. This fact is sufficient to establish (A.74).

Let $D \subset \mathcal{X}$ be a compact containing the sequence (y^n). Thus, ν is uniformly continuous in $D \times \mathcal{Z}'$. So, there exists some $j_1 \in \mathbb{N}$, such that

$$|\nu(y^{n_j}, w) - \nu(y^*, w)| < \epsilon/2 \text{ for all } w \in \mathcal{Z}',\ j \geqslant j_1. \tag{A.76}$$

We may assume that j_1 is large enough so we also have

$$|\bar{\alpha} - \alpha^{n_j}| < \epsilon/2.$$

Now notice that, if $w \in \mathcal{Z}'$ and $\alpha^* + \epsilon \leqslant \nu(y^*, w)$, (A.76) implies that

$$\alpha^* + \epsilon < \nu(y^{n_j}, w) + \epsilon/2,$$

so

$$\alpha^* + \epsilon/2 < \nu(y^{n_j}, w) \text{ for all } j \geqslant j_1. \tag{A.77}$$

Notice also that, if $w \in \mathcal{Z}'$ and $\nu(y^*, w) \leqslant \bar{\alpha} - \epsilon$, (A.76) implies that

$$\nu(y^{n_j}, w) - \epsilon/2 < \bar{\alpha} - \epsilon,$$

so

$$\nu(y^{n_j}, w) < \bar{\alpha} - \epsilon/2 \text{ for all } j \geqslant j_1. \tag{A.78}$$

Hence, (A.77), (A.78) and the fact that $\bar{\alpha} - \epsilon/2 < \alpha^{n_j}$ for $j \geqslant j_1$ imply that

$$\{\alpha^* + \epsilon \leqslant \nu(y^*, w) \leqslant \bar{\alpha} - \epsilon\} \cap \mathcal{Z}' \subset \{\alpha^* + \epsilon/2 < \nu(y^{n_j}, w) \leqslant \alpha^{n_j}\}$$

for all $j \geqslant j_1$, which is exactly (A.75). Thus, (A.74) hold.

Let $\eta \in (0,1)$ be sufficiently small such that

$$\limsup_{j\to\infty} m^{n_j}(\alpha^* + \epsilon/2) + \eta < \tau. \tag{A.79}$$

Again, by Assumption 1-(i), there exists $\mathcal{Z}'' \subset \mathcal{Z}$ compact such that

$$K(z^*, \mathcal{Z} \backslash \mathcal{Z}'') < \eta/2. \tag{A.80}$$

Once more, let D be a compact set containing the sequence (y^n). Then, since ν is continuous, it is uniformly continuous in the compact $D \times \mathcal{Z}''$. Hence, there exists $j_2 \in \mathbb{N}$ such that, if $j \geqslant j_2$, then

$$|\nu(y^{n_j}, w) - \nu(y^*, w)| < \epsilon/2 \quad \forall w \in \mathcal{Z}'' \quad \text{and} \quad |\bar{\alpha} - \alpha^{n_j}| < \epsilon/2.$$

Thus, if $j \geqslant j_2$ and $w \in \mathcal{Z}''$ is such that $\nu(y^*, w) \leqslant \alpha^*$, then (recalling (A.72))

$$\nu(y^{n_j}, w) \leqslant \nu(y^*, w) + \epsilon/2 \leqslant \alpha^* + \epsilon/2 < \alpha^* + \epsilon < \bar{\alpha} - \epsilon < \bar{\alpha} - \epsilon/2 < \alpha^{n_j},$$

so

$$\{w \in \mathcal{Z}''; \nu(y^*, w) \leqslant \alpha^*\} \subset \{w \in \mathcal{Z}''; \nu(y^{n_j}, w) \leqslant \alpha^* + \epsilon/2\}. \tag{A.81}$$

We have, for $j \geqslant j_2$,

$$\begin{aligned}
\tau &\leqslant K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) \\
&\leqslant K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') + K(z^*, \mathcal{Z} \backslash \mathcal{Z}'') \\
&< K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') + \eta/2
\end{aligned}$$

by (A.80).

Moreover, by Assumption 1-(ii), we can also assume that $j \geqslant j_2$ implies

$$K(z^{n_j}, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') > K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') - \eta/2. \tag{A.82}$$

Hence, for each $j \geqslant j_2$,

$$\begin{aligned}
m^{n_j}(\alpha^* + \epsilon/2) &= K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^* + \epsilon/2\}) \\
&\geqslant K(z^{n_j}, \{\nu(y^{n_j}, w) \leqslant \alpha^* + \epsilon/2\} \cap \mathcal{Z}'') \\
&\geqslant K(z^{n_j}, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') && \text{by (A.81)} \\
&\geqslant K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}'') - \eta/2 && \text{by (A.82)} \\
&> \tau - \eta/2 - \eta/2 = \tau - \eta.
\end{aligned}$$

This, however, contradicts the choice of η made in (A.79), thus proving that $\bar{\alpha} \leqslant \alpha^*$ and concluding the proof for $\mathcal{Z}$ connected when $K(z^*, \{v(y^*, w) \leqslant \alpha^*\}) < 1$.

For the case where

$$K(z^*, \{v(y^*, w) \leqslant \alpha^*\}) = 1, \tag{A.83}$$

we will show that $\bar{\alpha} \leqslant \alpha^*$ in the next step.

Step 3. ($\limsup_n \alpha^n \leqslant \alpha^*$ for $K(z^*, \{v(y^*, w) \leqslant \alpha^*\}) > \tau$)

In this step, we work simultaneously with connected and finite $\mathcal{Z}$. Since it will be necessary for the case where $\mathcal{Z}$ is finite, we will prove the result under a more general hypothesis than (A.83), namely, that

$$K(z^*, \{v(y^*, w) \leqslant \alpha^*\}) = \tau + \sigma \tag{A.84}$$

for some $\sigma > 0$.

Assumption 1-(ii) assures that there exists $j_3 \in \mathbb{N}$ such that

$$K(z^{n_j}, \{v(y^*, w) \leqslant \alpha^*\}) > \tau + \sigma/2 \text{ for all } j \geqslant j_3.$$

Assumption 1-(i) allows us to take some compact $\mathcal{Z}''' \subset \mathcal{Z}$ such that

$$K(z^*, \mathcal{Z} \backslash \mathcal{Z}''') < \sigma/4.$$

Recalling that D is a compact containing the sequence (y^n), then, given $\delta > 0$, uniform continuity of v on $D \times \mathcal{Z}'''$ implies that there exists $j_4 \geqslant j_3$ such that

$$v(y^{n_j}, w) - \delta < v(y^*, w) \text{ for all } w \in \mathcal{Z}''',\ j \geqslant j_4.$$

Hence, if $w \in \mathcal{Z}'''$ is such that $v(y^*, w) \leqslant \alpha^*$, then

$$v(y^{n_j}, w) \leqslant \alpha^* + \delta \text{ for all } j \geqslant j_4.$$

Thus,

$$\{v(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}''' \subset \{v(y^{n_j}, w) \leqslant \alpha^* + \delta\} \cap \mathcal{Z}'''.$$

Moreover, Assumption 1-(ii) implies that we may also assume that j_4 is large enough so

$$K(z^{n_j}, \mathcal{Z} \backslash \mathcal{Z}''') < K(z^*, \mathcal{Z} \backslash \mathcal{Z}''') + \sigma/4 < \sigma/2 \text{ for all } j \geqslant j_4.$$

Then, we have, for all $j \geqslant j_4$,

$$\begin{aligned} K(z^{n_j}, \{v(y^{n_j}, w) \leqslant \alpha^* + \delta\}) &\geqslant K(z^{n_j}, \{v(y^{n_j}, w) \leqslant \alpha^* + \delta\} \cap \mathcal{Z}''') \\ &\geqslant K(z^{n_j}, \{v(y^*, w) \leqslant \alpha^*\} \cap \mathcal{Z}''') \\ &\geqslant K(z^{n_j}, \{v(y^*, w) \leqslant \alpha^*\}) - K(z^{n_j}, \mathcal{Z} \backslash \mathcal{Z}''') \\ &> \tau + \sigma/2 - \sigma/2 = \tau. \end{aligned}$$

This implies that

$$K(z^{n_j}, \{v(y^{n_j}, w) \leqslant \alpha^* + \delta\}) > \tau \text{ for all } j \geqslant j_4.$$

Thus,

$$\alpha^{n_j} \leqslant \alpha^* + \delta \text{ for all } j \geqslant j_4.$$

Since $\delta > 0$ is arbitrary and $\alpha^{n_j} \to \bar{\alpha}$, it follows that $\bar{\alpha} \leqslant \alpha^*$ whenever (A.84) holds. This concludes the proof in the case where $\mathcal{Z}$ is connected.

Step 4. ($\limsup_n \alpha^n \leqslant \alpha^*$ for $\mathcal{Z}$ finite and $K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) = \tau$)

Now assume that $\mathcal{Z}$ is finite. In this case, $\mathcal{Z}$ has the discrete topology. Notice that the last step above, under hypothesis (A.84), does not rely on connectedness, so it is entirely valid for $\mathcal{Z}$ finite. Hence, to conclude the proof of the Lemma for $\mathcal{Z}$ finite, we only need to prove that $\bar{\alpha} \leqslant \alpha^*$ when

$$K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) = \tau. \tag{A.85}$$

We will prove that $\bar{\alpha} \leqslant \alpha^*$ in this case again by contradiction. So, assume that, for some $\epsilon > 0$,

$$\alpha^* + \epsilon < \bar{\alpha}. \tag{A.86}$$

Since $\mathcal{Z}$ is a finite metric space, it is endowed with the discrete topology, so we may assume, without loss of generality, that $z^{n_j} \equiv z^*$ for all $j \in \mathbb{N}$.

There exists $j_5 \in \mathbb{N}$ such that

$$|\alpha^{n_j} - \bar{\alpha}| < \epsilon/2 \text{ for all } j \geqslant j_5.$$

Thus we have, also by (A.86),

$$\alpha^* < \alpha^* + \epsilon/2 < \bar{\alpha} - \epsilon/2 < \alpha^{n_j} \text{ for all } j \leqslant j_5. \tag{A.87}$$

Therefore,

$$K(z^*, \{\nu(y^{n_j}, w) \leqslant \bar{\alpha} - \epsilon/2\}) < \tau \text{ for all } j \geqslant j_5. \tag{A.88}$$

Again, ν continuous and $D \supset (y^n)$ compact imply that, since $\mathcal{Z}$ is finite, ν is uniformly continuous on $D \times \mathcal{Z}$. Hence, there exists some $j_6 \geqslant j_5$ such that

$$|\nu(y^*, w) - \nu(y^{n_j}, w)| < \epsilon/2 \text{ for all } w \in \mathcal{Z},\ j \geqslant j_6.$$

Thus, if $j \geqslant j_6$ and $w \in \mathcal{Z}$ is such that $\nu(y^*, w) \leqslant \alpha^*$, we have

$$\nu(y^{n_j}, w) - \epsilon/2 < \nu(y^*, w) \leqslant \alpha^*,$$

so, by (A.87),

$$\nu(y^{n_j}, w) \leqslant \alpha^* + \epsilon/2 < \bar{\alpha} - \epsilon/2 < \alpha^{n_j}$$

for all $j \geqslant j_6$.

Therefore,

$$\{\nu(y^*, w) \leqslant \alpha^*\} \subset \{\nu(y^{n_j}, w) \leqslant \bar{\alpha} - \epsilon/2\} \text{ for all } j \geqslant j_6.$$

Combining this to (A.85) and (A.88) implies that, for $j \geqslant j_6$,

$$\tau = K(z^*, \{\nu(y^*, w) \leqslant \alpha^*\}) \leqslant K(z^*, \{\nu(y^{n_j}, w) \leqslant \bar{\alpha} - \epsilon/2\}) < \tau,$$

a contradiction. This concludes the proof also in the case where $\mathcal{Z}$ is finite, and the Lemma is complete. $\square$

Remark A.2.1. *The argument carried in the proof of Lemma A.5 from de Castro and Galvao (2019), which is the analogous to our Lemma 2.3.1, had an issue, since it claimed that (in the notation of the proof above for Lemma 2.3.1),* $m^n(\alpha) \to m^*(\alpha)$. *However, this convergence is not necessarily true for all* α, *as can be seen by considering* $\mathcal{X} = \mathcal{Z} = [0,1]$, $(x_n, z_n) = (1/n, 1)$ *and* $v(x,z) = 1 + xz$. *The distribution* $f(z'|1)$ *can be considered to be uniform (any distribution such that* $P[z' = 0 \,|\, z = 1] < 1$ *would also work). For* $\alpha = 1$, *we then have* $m^n(1) = P[1 + z'/n \leqslant 1 \,|\, 1] = P[z' = 0 \,|\, z = 1] = 0$ *for all* $n \in \mathbb{N}$, *while* $m^*(\alpha) = P[1 \leqslant 1 | z = 1] = 1$.

Proof of Theorem 2.3.2: The most delicate part in the proof of Theorem 2.3.2 is Lemma 2.3.1. The remaining part of the proof requires other Lemmas, which we present after the following remark:

Remark A.2.2. *When* Z *follows a* τ*-quantile martingale process (see Definition 2.1.1), an adaption in the result from Lemma 2.3.1 is needed. Instead of proving in Theorem 2.3.2 that* $\mathbb{M}^\tau$ *takes* $\mathcal{C}$ *to* $\mathcal{C}$, *where* $\mathcal{C}$ *stands for bounded and continuous real-valued functions over* $\mathcal{X} \times \mathcal{Z}$, *it suffices to prove that* $\mathbb{M}^\tau$ *maps* $\mathcal{C}'$ *to* $\mathcal{C}'$, *where* $\mathcal{C}' \subset \mathcal{C}$ *denotes the subset of bounded, continuous and increasing functions. To do this, we need also to assume that the utility function* u *is increasing in all variables in Theorem 2.3.2. Therefore, Lemma A.1.3 implies that, for* $v \in \mathcal{C}'$,

$$Q_\tau[v(y,w)|z] = v(y, Q_\tau[w|z]) = v(y,z),$$

so continuity of $(y,z) \mapsto Q_\tau[v(y,w)|z]$ *follows from the continuity of* v. *This is the result contained in Lemma 2.3.1.*

Of course, to fully establish that $\mathbb{M}^\tau : \mathcal{C}' \to \mathcal{C}'$, *it remains to show that* $\mathbb{M}^\tau v$ *is also increasing if* $v \in \mathcal{C}'$. *To establish this, we can employ almost the same argument concerning increasingness as in Lemmas A.2.5 and A.2.7, the only difference being that we can no longer directly use Lemma A.2.6, since its proof rely in Assumption 14-(iii), which is not always satisfied by* τ*-quantile martingales. To contour this and achieve the same conclusion, notice that, for* h *increasing and continuous,*

$$Q_\tau[h(w)|z] = h(Q_\tau[w|z]) = h(z) \leqslant h(z') = h(Q_\tau[w|z']) = Q_\tau[h(w)|z']$$

by Lemma A.1.3. Hence, the result from Lemma A.2.6 for τ*-quantile martingale follows, and the rest of Lemmas A.2.5 and A.2.7 can be combined to finally show that* $\mathbb{M}^\tau$ *maps* $\mathcal{C}'$ *to* $\mathcal{C}'$.

Indeed, these adaptions serve to establish the existence of the value function for τ*-quantile martingales (which is the true objective of Theorem 2.3.2), with the stronger assumption of having* u *also increasing, and the further properties of the value function listed in Theorems 2.3.7 and 2.3.9.*

Lemma A.2.3. *For each* $v \in \mathcal{C}$ *the supremum in (2.58) is attained and* $\mathbb{M}^\tau(v) \in \mathcal{C}$. *Moreover, the optimal correspondence* $\Upsilon : \mathcal{X} \times \mathcal{Z} \rightrightarrows \mathcal{Y}$ *defined by*

$$\Upsilon(x,z) \equiv \arg\max_{y \in \Gamma(x,z)} Q_\tau[u(x,y,z) + \beta V(\phi(x,y,w), w) | z] \tag{A.89}$$

is nonempty and upper semi-continuous.

Proof. The proof repeats the proof of de Castro and Galvao (2019, Lemma A.6). □

We conclude the proof of Theorem 2.3.2 by showing that $\mathbb{M}^{\tau}$ satisfies Blackwell's sufficient conditions for a contraction.

Lemma A.2.4. *$\mathbb{M}^{\tau}$ satisfies the following conditions:*

(a) For any $v, v' \in \mathcal{C}$, $v \leqslant v'$ implies $\mathbb{M}^{\tau}(v) \leqslant \mathbb{M}^{\tau}(v')$.

(b) For any $a \geqslant 0$ and $x \in X$, $\mathbb{M}(v+a)(x) \leqslant \mathbb{M}(v)(x) + \beta a$, with $\beta \in (0,1)$.

Then, $\|\mathbb{M}(v) - \mathbb{M}(v')\| \leqslant \beta \|v - v'\|$, that is, $\mathbb{M}$ is a contraction with modulus β. Therefore, $\mathbb{M}^{\tau}$ has a unique fixed-point $V \in \mathcal{C}$.

Proof. To see (a), let $v, v' \in \mathcal{C}$, $v \leqslant v'$ and define g as

$$g(x,y,z,w) = u(x,y,z) + \beta v(\phi(x,y,w),w) \tag{A.90}$$

and analogously for g'.It is clear that $g \leqslant g'$. Then, by de Castro and Galvao (2019, Lemma A.1(vi)), $Q_{\tau}[g(\cdot)|z] \leqslant Q_{\tau}[g'(\cdot)|z]$, which implies (a).

To verify (b), since a is a constant,:

$$Q_{\tau}[v(\phi(x,y,w),w) + a|z] \quad = \quad Q_{\tau}[v(\phi(x,y,w),w)|z] + a.$$

Thus, $\mathbb{M}^{\tau}(v+a) = \mathbb{M}^{\tau}(v) + \beta a$, that is, (b) is satisfied with equality. □

Proof of Theorem 2.3.5: Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ which are nondecreasing in x. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly increasing functions x. If we show that $\mathbb{M}^{\tau}(\mathcal{C}') \subset \mathcal{C}''$, then the fixed-point of $\mathbb{M}^{\tau}$ will be strictly increasing in x.

Let $v \in \mathcal{C}'$ and consider $x_0, x_1 \in \mathcal{X}$, $x_0 < x_1$. For $i = 0,1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, that is,

$$\mathbb{M}^{\tau}(v)\,(x_i, z) = u(x_i, y_i, z) + \beta Q_{\tau}[v(\phi(x_i, y_i, w), w)|z],$$

By Assumption 12, $\Gamma(x_0,z) \subset \Gamma(x_1,z)$, so $y_0 \in \Gamma(x_1,z)$. Therefore,

$$\begin{aligned}
\mathbb{M}^{\tau}v(x_0,z) &= u(x_0,y_0,z) + \beta Q_{\tau}[v(\phi(x_0,y_0,w),w)|z] \\
&< u(x_1,y,z) + \beta Q_{\tau}[v(\phi(x_1,y_0,w),w)|z] \\
&\leqslant \mathbb{M}^{\tau}v(x_1,z),
\end{aligned}$$

where in the first inequality we used that u is strictly increasing in x, both v and ϕ are increasing in x, and de Castro and Galvao (2019, Lemma A.1(vi)). This shows that $\mathbb{M}^{\tau}v$ is strictly increasing in x when $v \in \mathcal{C}'$, that is, $\mathbb{M}^{\tau}(\mathcal{C}') \subset \mathcal{C}''$, since $v \in \mathcal{C}'$ was arbitrary. □

Proof of Theorem 2.3.6: The strict increasingness of V with respect to x was already proved in Theorem 2.3.5. The remaining part, concerning strict increasingness in z, is the content of Lemma A.2.7, which rely on Lemma A.2.6. Both Lemmas will be used to establish Theorems 2.3.7 and 2.3.9, but, in reality, these Lemmas do not use Assumptions 15 nor 16, so they hold with the weaker hypotheses of Assumptions 2 and 13. In particular, we may have $\mathcal{Z} \subset \mathbb{R}^k$ for $k > 1$. □

Proof of Theorem 2.3.7: We organize the proof in a series of lemmas. It is convenient to introduce the following notation. Let $\mathcal{C}' \subset \mathcal{C}$ be the set of the bounded and continuous functions $v : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ which are concave in x and nondecreasing in both x and z. It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly concave functions in x and strictly increasing in both x and z. If we show that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, then the fixed-point of $\mathbb{M}^\tau$ will be strictly concave in x, as well as strictly increasing in both x and z . (See, for instance, Stokey et al. (1989, Corollary 1, p. 52).)

Lemma A.2.5. *Let Assumptions 1, 2, 11, 14 and 15 hold. Then $\mathbb{M}^\tau(\mathcal{C}') \subseteq \mathcal{C}''$. Therefore, $V \in \mathcal{C}''$.*

Proof. Let $\alpha \in (0,1)$, $v \in \mathcal{C}'$ and consider $x_0, x_1 \in \mathcal{X}$, $x_0 \neq x_1$. For $i = 0,1$, let $y_i \in \Gamma(x_i, z)$ attain the maximum, that is,

$$\mathbb{M}^\tau(v)\,(x_i, z) = u(x_i, y_i, z) + \beta Q_\tau[v(\phi(x_i, y_i, w), w)|z],$$

Let $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $y_\alpha \equiv \alpha y_0 + (1-\alpha)y_1$. Hence,

$$\begin{aligned}
\alpha\mathbb{M}^\tau v(x_0,z) + (1-\alpha)\mathbb{M}^\tau v(x_1,z) &= \alpha\,\{u(x_0,y_0,z) + \beta Q_\tau[v(\phi(x_0,y_0),w),w)|z]\} + \\
&\quad (1-\alpha)\,\{u(x_1,y_1,z) + \beta Q_\tau[v(\phi(x_1,y_1,w),w)|z]\} \\
&< u(x_\alpha,y_\alpha,z) + \beta\{Q_\tau[\alpha v(\phi(x_0,y_0,w),w)|z] + \\
&\quad Q_\tau[(1-\alpha)v(\phi(x_1,y_1,w),w)|z]\} \\
&= u(x_\alpha,y_\alpha,z) + \beta Q_\tau[\alpha v(\phi(x_0,y_0,w),w) + \qquad \text{(A.91)} \\
&\quad (1-\alpha)v(\phi(x_1,y_1,w),w)|z] \\
&\leqslant u(x_\alpha,y_\alpha,z) + \beta Q_\tau[v(\phi(x_\alpha,y_\alpha,w),w)|z] \\
&\leqslant \mathbb{M}^\tau v(x_\alpha,z),
\end{aligned}$$

where the first inequality is due to the strict concavity of u in the first two variables. The equality (A.91) that follows it is justified by Proposition A.1.4: since v is increasing in both variables and ϕ is increasing in the last variable, $v(\phi(x,y,w),w)$ is both increasing and continuous on w, so comonotonicity applies. The second inequality follows from concavity in x of v and in (x,y) of ϕ, as well as de Castro and Galvao (2019, Lemma A.1(vi)). The last inequality follows from Assumption 14. This proves that $\mathbb{M}^\tau v$ is strictly concave in x when $v \in \mathcal{C}'$.

Now assume that $x_0 < x_1$. By Assumption 14, $\Gamma(x_0,z) \subset \Gamma(x_1,z)$, so $y_0 \in \Gamma(x_1,z)$. Therefore,

$$\begin{aligned}
\mathbb{M}^\tau v(x_0,z) &= u(x_0,y_0,z) + \beta Q_\tau[v(\phi(x_0,y_0,w),w)|z] \\
&< u(x_1,y,z) + \beta Q_\tau[v(\phi(x_1,y_0,w),w)|z] \\
&\leqslant \mathbb{M}^\tau v(x_1,z),
\end{aligned}$$

where in the first inequality we used that u is strictly increasing in x, both v and ϕ are increasing in x, and de Castro and Galvao (2019, Lemma A.1(vi)). This shows that $\mathbb{M}^\tau v$ is strictly increasing in x when $v \in \mathcal{C}'$.

To conclude that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathbb{M}^\tau(C'')$, it remains to show that $\mathbb{M}^\tau v$ is strictly increasing in z. This is the content of Lemma A.2.7. □

Lemma A.2.6. *Let Assumptions 1 and 2 hold. If $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant z'$, then $Q_\tau[h(w)|z] \leqslant Q_\tau[h(w)|z']$.*

Proof. From Assumption 2, if $h : \mathcal{Z} \to \mathbb{R}$ is weakly increasing and $z \leqslant z'$:

$$E\left[-1_{\{W \in \mathcal{Z}: h(W) \leqslant \alpha\}}|z\right] \leqslant E\left[-1_{\{W \in \mathcal{Z}: h(W) \leqslant \alpha\}}|z'\right].$$

Thus,

$$\Pr\left([h(W) \leqslant \alpha] \,|\, z\right) = E\left[1_{\{W \in \mathcal{Z}: h(W) \leqslant \alpha\}}|z\right] \geqslant E\left[1_{\{W \in \mathcal{Z}: h(W) \leqslant \alpha\}}|z'\right] = \Pr\left([h(W) \leqslant w] \,|\, z'\right). \tag{A.92}$$

If we define $H(w|z) = \Pr\left([h(W) \leqslant w] \,|\, Z = z\right)$, then (A.92) can be written as:

$$H(w|z) \geqslant H(w|z').$$

Observe that $Q_\tau[h(w)|z] = \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\}$ and, whenever $z \leqslant z'$, $H(w|z') \leqslant H(w|z)$, for all w. Therefore, if $z \leqslant z'$, then

$$\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \supset \{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\},$$

which implies that

$$Q_\tau[h(w)|z] \quad = \quad \inf\{\alpha \in \mathbb{R} : H(\alpha|z) \geqslant \tau\} \leqslant \inf\{\alpha \in \mathbb{R} : H(\alpha|z') \geqslant \tau\} = Q_\tau[h(w)|z'],$$

as we wanted to show. □

Lemma A.2.7. *Let Assumptions 1, 2 and 14 hold. If $v \in \mathcal{C}'$, so it is increasing in z, then $\mathbb{M}^\tau(v)$ is strictly increasing in z.*

Proof. Let $z_1, z_2 \in \mathcal{Z}$, with $z_1 < z_2$. For $i = 1, 2$, let $y_i \in \Gamma(x, z_i)$ realize the maximum, that is,

$$\mathbb{M}^\tau(v)\,(x_i, z) = u(x, y_i, z_i) + \beta Q_\tau[v(\phi(x, y_i, w), w)|z_i].$$

Since u is strictly increasing in z, we have:

$$\begin{aligned} \mathbb{M}^\tau(v)\,(x, z_1) \quad &= \quad u(x, y_1, z_1) + \beta Q_\tau[v(\phi(x, y_1, w), w)|z_1] \\ &< \quad u(x, y_1, z_2) + \beta Q_\tau[v(\phi(x, y_1, w), w)|z_1]. \end{aligned}$$

As $v \in \mathcal{C}'$, it is increasing in both variables, and so is ϕ with respect to the last variable. Hence, their composition is increasing in w, and Lemma A.2.6 implies that

$$Q_\tau[v(\phi(x, y_1, w), w)|z_1] \leqslant Q_\tau[v(\phi(x, y_1, w), w)|z_2],$$

which gives:

$$\mathbb{M}^\tau(v)\,(x, z_1) < u(x, y_1, z_2) + \beta Q_\tau[v(\phi(x, y_1, w), w)|z_2].$$

From Assumption 14, $\Gamma(x,z_1) \subseteq \Gamma(x,z_2)$, that is, $y_1 \in \Gamma(x,z_2)$. Optimality thus implies that:

$$u(x,y_1,z_2) + \beta Q_\tau[v(\phi(x,y_1,w),w)|z_2] \leqslant u(x,y_2,z_2) + \beta Q_\tau[v(\phi(x,y_2,w),w)|z_2] = \mathbb{M}^\tau(v)(x,z_2).$$

Therefore, $\mathbb{M}^\tau(v)(x,z_1) < \mathbb{M}^\tau(v)(x,z_2)$, which shows strict increasingness in z. □

We conclude the proof of Theorem 2.3.7 by showing that the policy correspondence (A.89) is single-valued and continuous. Let V be the unique fixed point of $\mathbb{M}^\tau$ from Theorem 2.3.2. For an absurd, assume that there were $y \neq y'$ in $\Gamma(x,z)$ such that

$$V(x,z) = u(x,y,z) + \beta Q_\tau[V(\phi(x,y,w),w)|z] = u(x,y',z) + \beta Q_\tau[V(\phi(x,y',w),w)|z].$$

By Lemma A.2.5, V is strictly concave in x.

Let $y_\alpha \equiv \alpha y + (1-\alpha)y'$. By Assumption 14, $y_\alpha \in \Gamma(x,z)$. Hence,

$$\begin{aligned}
V(x,z) &= \alpha V(x,z) + (1-\alpha)V(x,z) \\
&= \alpha\{u(x,y,z) + \beta Q_\tau[V(\phi(x,y,w),w)|z]\} + \\
&\quad (1-\alpha)\{u(x,y',z) + \beta Q_\tau[V(\phi(x,y',w),w)|z]\} \\
&< u(x,y_\alpha,z) + \beta\{Q_\tau[\alpha V(\phi(x,y,w),w)|z] + \\
&\quad Q_\tau[(1-\alpha)V(\phi(x,y',w),w)|z]\} \\
&= u(x,y_\alpha,z) + \beta Q_\tau[\alpha V(\phi(x,y,w),w) + \\
&\quad (1-\alpha)v(\phi(x,y',w),w)|z] \qquad \text{(A.93)} \\
&\leqslant u(x,y_\alpha,z) + \beta Q_\tau[v(\phi(x,y_\alpha,w),w)|z] \\
&\leqslant V(x,z),
\end{aligned}$$

where the first inequality is due to the strict concavity of u in the first two variables. The equality (A.93) that follows it is justified by Proposition A.1.4: since V is increasing in both variables by Lemmas A.2.5 and A.2.7 and ϕ is increasing in the last variable, $V(\phi(x,y,w),w)$ is both increasing and continuous on w, so comonotonicity applies. The second inequality follows from concavity in x of V (by Lemma A.2.5) and in (x,y) of ϕ, as well as de Castro and Galvao (2019, Lemma A.1(vi)). The last inequality follows from Assumption 14 and the definition of V. This contradiction proves that the policy correspondence given by (A.89) is single-valued.

Lemma A.2.3 shows that the correspondence is upper semi-continuous. Then single-valuedness implies continuity. □

Remark A.2.8. *To see how Theorem 2.3.7 applies to a τ-quantile martingale process (see Definition 2.1.1), check Remark A.2.2.*

Proof of Lemma 2.3.8: One important point is that, for countable and discrete $\mathcal{Z}$, Theorem 2.3.2 does not fully hold, since Lemma 2.3.1 does not contemplate countable $\mathcal{Z}$. However, Lemma A.2.4 can be applied to show that, in this case, $\mathbb{M}^\tau$ is a contraction on $\mathcal{B}$, the Banach space of bounded functions in the sup norm, of which $\mathcal{C}$ is a subspace. Hence, a unique solution V on $\mathcal{B}$ exists.

However, Lemma A.1.1 can be used to show that, since, for ν continuous, bounded and increasing in both variables, as well as $\mathbb{R}$-continuous in z,

$$Q_\tau[\nu(\phi(x,y,w),w)|z] = \nu(\phi(x,y,Q_\tau[w|z]),Q_\tau[w|z]) \tag{A.94}$$

under our Assumptions (which requires, for instance, ϕ to be increasing in the first and last variables), so we have, indeed, $\mathbb{M}^\tau\nu(x,z)$ also continuous and increasing in both variables, and $\mathbb{R}$-continuous in z, by an entirely analogous argument than the one from the proof of Theorem 2.3.7, as long as we keep in mind the identity (A.94) and Berge's Maximum Theorem to ensure continuity in x and $\mathbb{R}$-continuity in z of $\mathbb{M}^\tau\nu(x,z)$ for $\nu \in \mathcal{C}'$, the subspace of continuous functions in x, $\mathbb{R}$-continuous in z and increasing in both variables. Indeed, in this case, we have

$$\mathbb{M}^\tau\nu(x,z) = \max_{y\in\Gamma(x,z)} u(x,y,z) + \beta\nu\left(\phi(x,y,Q_\tau[w|z]),Q_\tau[w|z]\right).$$

Thus, the same reasoning from Theorem 2.3.7 shows that $\mathbb{M}^\tau\mathcal{C}' \subset \mathcal{C}''$, where $\mathcal{C}'' \subset \mathcal{C}'$ denotes the subset of strictly increasing ν (in both variables). This is enough to show that the value function is continuous. □

Proof of Theorem 2.3.9: The proof is very similar to the one of Theorem 2.3.7, as long as one considers Lemma 2.3.8 (which holds by noticing that Assumption 14 encompasses 13) in place of Theorem 2.3.2 for countable $\mathcal{Z}$.

Another difference is the justification of equations (A.91) and (A.93), which use Proposition A.1.4. In order to apply it in the discrete case, one must be careful about continuity in the z variable with respect to the usual topology on $\mathbb{R}$.

Similarly as before, let $\mathcal{C}' \subset \mathcal{C}$ be the set of the continuous functions $\nu : \mathcal{X}\times\mathcal{Z} \to \mathbb{R}$ which are concave in x, nondecreasing in both x and z, and also $\mathbb{R}$-continuous in z. This means that an element $\nu \in \mathcal{C}'$ is required to be continuous in z with respect to the usual topology on $\mathbb{R}$. Note that a general function in $\mathcal{C}$ is assumed to be continuous in z with respect just to the discrete topology, which is always the case, since this topology is trivial. On the other hand, $\mathcal{C}'$ imposes more structure, as not every function will satisfy $\mathbb{R}$-continuity.

It is easy to see that $\mathcal{C}'$ is a closed subset of $\mathcal{C}$. Let $\mathcal{C}'' \subset \mathcal{C}'$ be the set of strictly concave functions in x and strictly increasing in both x and z. As before, if we show that $\mathbb{M}^\tau(\mathcal{C}') \subset \mathcal{C}''$, then the fixed-point of $\mathbb{M}^\tau$ will be strictly concave in x, as well as strictly increasing in both x and z.

We begin by proving the discrete analogous of Lemma A.2.5:

Lemma A.2.9. *Let Assumptions 1, 2, 11, 14 and 16 hold.* $\mathbb{M}^\tau(\mathcal{C}') \subseteq \mathcal{C}''$*. Therefore,* $V \in \mathcal{C}''$*.*

Proof. As already mentioned, the proof is almost the same, the difference being the new concern about $\mathbb{R}$-continuity in the z variable. One proves exactly as before that if $\nu \in \mathcal{C}'$ then $\mathbb{M}^\tau\nu$ is strictly concave and strictly increasing in x. The justification for the usage of Proposition A.1.4 to legitimate equations (A.91) and (A.93) lies in the $\mathbb{R}$-continuity in w of $\nu(\phi(x,y,w),w)$, since $\nu \in \mathcal{C}'$ and ϕ is $\mathbb{R}$-continuous in the last variable by Assumption 16. With this in mind, one can follows the proof of Lemma A.2.5 entirely and establish that $\mathbb{M}^\tau\nu(x,z)$ satisfies almost all conditions to belong to $\mathcal{C}''$ whenever $\nu \in \mathcal{C}'$, the only exceptions being $\mathbb{R}$-continuity and increasingness with respect to the z variable.

Let's prove first that $\mathbb{M}^\tau v(x,z)$ is $\mathbb{R}$-continuous in z. Notice that this is not assured by Theorem 2.3.2, since its proof was carried only with respect to the discrete topology in $\mathcal{Z}$.

By means of equation (2.58) and the Berge's maximum theorem, since $u(x,y,z)$ is already $\mathbb{R}$-continuous in z by Assumption 16, one need only to show that if $v \in \mathcal{C}'$ then $Q_\tau[v(\phi(x,y,w),w)|z]$ is $\mathbb{R}$-continuous in z.

Fix x and y. By assumption, $v(\phi(x,y,w),w)$ is $\mathbb{R}$-continuous and increasing with respect to the w variable. So, one needs only to verify that if $h : \mathcal{Z} \to \mathbb{R}$ if $\mathbb{R}$-continuous and increasing then $z \mapsto Q_\tau[h(w)|z]$ is $\mathbb{R}$-continuous.

But, by Lemma A.1.1,

$$Q_\tau[h(w)|z] = h(Q_\tau[w|z]).$$

Then, Assumption 16(ii) and the $\mathbb{R}$-continuity of h imply that the right-hand side is continuous in the usual topology of $\mathbb{R}$, and so $\mathbb{R}$-continuity of $Q_\tau[v(\phi(x,y,w),w)|z]$ in the z variable is established, concluding the proof of the Lemma.

□

Now the last missing part to conclude the proof of Theorem 2.3.9 is to show that $\mathbb{M}^\tau v(x,z)$ is strictly increasing in z. This is done by directly applying Lemmas A.2.6 and A.2.7, since they only depend on the common Assumptions 1, 2 and 14, so they remain valid in the discrete shock case. Thus, the proof is complete. □

Remark A.2.10. *To see how Theorem 2.3.9 applies to a τ-quantile martingale process (see Definition 2.1.1), check Remark A.2.2.*

Proof of Theorem 2.3.10: The proof follows from an easy adaptation of Benveniste and Scheinkman (1979)'s argument. For completeness and reader's convenience, we reproduce it here. Since the needed assumptions are valid, Theorems 2.3.7 or 2.3.9 apply, depending on whether $\mathcal{Z}$ is continuous or discrete. Then, the value function $V(x,z)$ is strictly concave in the first variable and the correspondence policy $\Upsilon(x,z) \in \Gamma(x,z)$ is single valued.

Thus, for all (x,z), we have, recording that ϕ does not depend on x:

$$V(x,z) = u(x,y^*(x,z),z) + \beta Q_\tau[V(\phi(y^*(x,z),w),w)|z].$$

Fix $z \in \mathcal{Z}$ and x_0 in the interior of X and define:

$$\bar{w}(x) = u(x,y^*(x_0,z),z) + \beta Q_\tau[V(\phi(y^*(x_0,z),w),w)|z].$$

Since Γ is continuous and $y^*(x_0,z) \in \operatorname{int}\Gamma(x_0,z)$, there exists a neighborhood D of x_0 such that $y^*(x_0,z) \in \Gamma(x,z)$ for all $x \in D$. Thus, we have $\bar{w}(x) \leqslant V(x,z)$ whenever $x \in D$, with equality at $x = x_0$, which implies $\bar{w}(x) - \bar{w}(x_0) \leqslant V(x,z) - V(x_0,z)$. Note that $\bar{w}$ is concave and differentiable in x because u is. Thus, any subgradient p of $V(\cdot,z)$ at x_0 must satisfy

$$p \cdot (x - x_0) \geqslant V(x,z) - V(x_0,z) \geqslant \bar{w}(x) - \bar{w}(x_0).$$

Thus, p is also a subgradient of $\bar{w}$. But since $\bar{w}$ is differentiable, p is unique. Therefore, $V(\cdot,z)$ is a concave function with a unique subgradient. Therefore, it is differentiable in x (cf. Rockafellar (1970, Theorem 25.1, p. 242)) and its derivative with respect to x is the same as that of $\bar{w}$, that is,

$$\frac{\partial V}{\partial x_i}(x,z) = \frac{\partial \bar{w}}{\partial x_i}(x) = \frac{\partial u}{\partial x_i}(x, y^*(x,z), z),$$

as we wanted to show. □

Proof of Theorem 2.3.11: As seen in Lemma A.2.3, the optimal correspondence Υ is uhc with compact and non-empty values. Let $\{h_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence of real numbers converging to 0. Let $e_i = (0, ..., 0, 1, 0, ..., 0) \subset \mathbb{R}^p$ be the ith canonical basis vector. Since $x \in \mathcal{X}$ is assumed to be interior, we can suppose that the h_n are small enough so that $x_n \equiv x + h_n e_i \in \mathcal{X}$ for all $n \in \mathbb{N}$. Clearly, we have $x_n \to x$.

Now, since we assumed that $\Upsilon(x,z)$ is lhc at (x,z), we claim that there exists some $N \in \mathbb{N}$ such that $\Upsilon(x,z) \subset \Upsilon(x_n, z)$ for $n \geqslant N$. To see this, assume, for a contradiction, that the statement is false. Then, it is possible to find some subsequence n_k and elements $y_{n_k} \in \Upsilon(x,z)$ such that $y_{n_k} \notin \Upsilon(x_{n_k}, z)$. Since $\Upsilon(x,z)$ is compact (as proved in Lemma A.2.3), there exists some subsequence $y_{n_{k_j}} \to y$ for some $y \in \Upsilon(x,z)$. Since $\mathcal{Y}$ is discrete, this means that there exists some $J \in \mathbb{N}$ such that

$$y_{n_{k_j}} \equiv y \quad \text{for all } j \geqslant J. \tag{A.95}$$

On the other hand, since $\Upsilon(x,z)$ is assumed to be lhc at (x,z), there exists some sequence $\tilde{y}_{n_{k_j}} \in \Upsilon(x_{n_{k_j}}, z)$ such that $\tilde{y}_{n_{k_j}} \to y$. Since $\mathcal{Y}$ is discrete, this means that $\tilde{y}_{n_{k_j}} \equiv y$ for $j \geqslant J_1$, where $J_1 \in \mathbb{N}$ is sufficiently large.

Therefore, by (A.95), if $j \geqslant \max\{J, J_1\}$, then

$$y_{n_{k_j}} = y = \tilde{y}_{n_{k_j}} \in \Upsilon(x_{n_{k_j}}, z),$$

a contradiction against the choice of the y_{n_k}. This contradiction proves that there must exist some $N \in \mathbb{N}$ such that $\Upsilon(x,z) \subset \Upsilon(x_n, z)$ for $n \geqslant N$.

Therefore, we can find a fixed $y^* \in \Upsilon(x,z)$ such that $y^* \in \Upsilon(x_n, z)$ for $n \geqslant N$. Then, (2.58) implies that, for large values of n,

$$v(x + h_n e_i, z) = u(x + h_n e_i, y^*, z) + \beta Q_\tau \left[v\left(\phi(y^*, z'), z'\right) | z \right],$$

so

$$\lim_{n\to\infty} \frac{v(x + h_n e_i, z) - v(x,z)}{h_n} = \lim_{n\to\infty} \frac{u(x + h_n e_i, y^*, z) - u(x, y^*, z)}{h_n} = \frac{\partial u}{\partial x_i}(x, y^*, z).$$

Since $\{h_n\}$ is an arbitrary sequence converging to 0, the proof is complete.

□

Proof of Theorem 2.3.12: Let $g(x,y,z) \equiv u(x,y,z) + \beta Q_\tau[V(\phi(y,w),w)|z]$ and $y^*(x,z)$ be an interior solution of the problem (2.59). Let $\tilde{v}(y,w) = V(\phi(y,w),w)$. Observe that $\tilde{v}$ is increasing in w, differentiable in its first variable and for $0 < y_i' - y_i < \epsilon$, for some small $\epsilon > 0$,

$$\begin{aligned}
\tilde{v}(y_i', y_{-i}, w) - \tilde{v}(y_i, y_{-i}, w) &= \int_{y_i}^{y_i'} \frac{\partial \tilde{v}}{\partial y_i}(\alpha, y_{-i}, w) d\alpha \\
&= \int_{y_i}^{y_i'} \frac{\partial V}{\partial x}(\phi(\alpha, y_{-i}, w), w) \cdot \frac{\partial \phi}{\partial y_i}(\alpha, y_{-i}, w) d\alpha
\end{aligned}$$

$$= \int_{y_i}^{y_i'} \frac{\partial u}{\partial x}(\phi(\alpha, y_{-i}, w), y^*(\phi(\alpha, y_{-i}, w), w), w) \cdot \frac{\partial \phi}{\partial y_i}(\alpha, y_{-i}, w) d\alpha$$

is increasing in w because, by hypothesis, the integrand is. Also, we applied the chain rule in the second equality, and Theorem 2.3.10 in the third. Therefore, the assumptions of Proposition 3.19 from de Castro and Galvao (2019) are satisfied and we conclude that $\frac{\partial Q_\tau}{\partial y_i}[\tilde{v}(y, w)] = Q_\tau\left[\frac{\partial \tilde{v}}{\partial y_i}(y, w)\right]$. Since u is differentiable in y, so is g. Since $y^*(x, z)$ is interior, the following first order condition holds:

$$\frac{\partial g}{\partial y_i}(x, y^*(x, z), z) = \frac{\partial u}{\partial y_i}(x, y^*(x, z), z) + \beta Q_\tau\left[\frac{\partial \tilde{v}}{\partial y_i}(y^*(x, z), w) | z\right] = 0.$$

Now we apply Theorem 2.3.10 and its expression: $\frac{\partial V}{\partial x}(x, z) = \frac{\partial u}{\partial x}(x, y^*(x, z), z)$, together with the chain rule, to conclude that

$$\frac{\partial u}{\partial y_i}(x, y^*(x, z), z) + \beta Q_\tau\left[\frac{\partial u}{\partial x}(\phi(y^*(x, z), w), y^*(\phi(y^*(x, z), w), w), w) \cdot \frac{\partial \phi}{\partial y_i}(y^*(x, z), w) \bigg| z\right] = 0 \tag{A.96}$$

Now, we have just to put the notation of a sequence. For this, let $h = (x_t)$ denote an optimal path beginning at (x_0, z_0), (A.96) can be rewritten, substituting x for x_t^h, $y^*(x, z)$ for y_t^h, $\phi(y^*(x, z), w)$ for x_{t+1}, $y^*(\phi(y^*(x, z), w), w)$ for y_{t+1}^h, z for z_t and w for z_{t+1}, as:

$$\frac{\partial u}{\partial y_i}\left(x_t^h, y_t^h, z_t\right) + \beta Q_\tau\left[\frac{\partial u}{\partial x}\left(x_{t+1}^h, y_{t+1}^h, z_{t+1}\right) \cdot \frac{\partial \phi}{\partial y_i}\left(y_t^h, z_{t+1}\right) \bigg| z_t\right] = 0, \tag{A.97}$$

which we wanted to establish. □

Bibliography

Acemoglu, D. (2009): *Introduction to Modern Economic Growth*, Princeton University Press.

Adda, J. and R. Cooper (2003): *Dynamic Economics: Quantitative Methods and Applications*, Cambridge, MA: The MIT Press.

Albrecht, J. and B. Axell (1984): "An Equilibrium Model of Search Unemployment," *Journal of Political Economy*, 92, 824–840.

Barunik, J. and F. Cech (2021): "Measurement of Common Risks in Tails: A Panel Quantile Regression Model for Financial Returns," *Journal of Financial Markets*, 52, 100562.

Benveniste, L. M. and J. A. Scheinkman (1979): "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica*, 47, 727–732.

Bhattacharya, D. (2009): "Inferring Optimal Peer Assignment From Experimental Data," *Journal of the American Statistical Association*, 104, 486–500.

Blume, L., D. Easley, and M. O'Hara (1982): "Characterization of Optimal Plans for Stochastic Dynamic Programs," *Journal of Economic Theory*, 28, 221–234.

Bommier, A., A. Kochov, and F. Le Grand (2017): "On Monotone Recursive Preferences," *Econometrica*, 85, 1433–1466.

Brock, W. A. and L. J. Mirman (1972): "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory*, 4, 479–513.

Chambers, C. P. (2009): "An Axiomatization of Quantiles on the Domain of Distribution Functions," *Mathematical Finance*, 19, 335–342.

Chateauneuf, A. and J. H. Faro (2009): "Ambiguity through confidence functions," *Journal of Mathematical Economics*, 45, 535–558.

Chen, L., J. J. Dolado, and J. Gonzalo (2021): "Quantile Factor Models," *Econometrica*, 89, 875–910.

Choquet, G. (1954): "Theory of Capacities," *Annales de l'Institut Fourier*, 5, 131–295.

de Castro, L. and A. F. Galvao (2019): "Dynamic Quantile Models of Rational Behavior," *Econometrica*, 87, 1893–1939.

——— (2021): "Static and Dynamic Quantile Preferences," *Economic Theory*, forthcoming.

de Castro, L., A. F. Galvao, G. Montes-Rojas, and J. Olmo (2021): "Portfolio Selection in Quantile Decision Models," *Annals of Finance*, forthcoming.

de Castro, L., A. F. Galvao, C. N. Noussair, and L. Qiao (2022): "Do People Maximize Quantiles?" *Games and Economic Behavior*, 132, 22–40.

Deaton, A. (1992): *Understanding Consumption*, Oxford Press.

Dejarnette, P., D. Dillenberger, D. Gottlieb, and P. Ortoleva (2020): "Time Lotteries and Stochastic Impatience," *Econometrica*, 88, 619–656.

Dudley, R. M. (2002): *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2 ed.

Epstein, L. and M. Schneider (2003): "Recursive Multiple-Priors," *Journal of Economic Theory*, 113, 1–31.

Epstein, L. G. and M. Le Breton (1993): "Dynamically Consistent Beliefs Must Be Bayesian," *Journal of Economic Theory*, 61, 1–22.

Epstein, L. G. and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

——— (1991): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Investigation," *Journal of Political Economy*, 99, 263–286.

Flavin, M. A. (1981): "The Adjustment of Consumption to Changing Expectations About Future Income," *Journal of Political Economy*, 89, 974–1009.

Gilboa, I. and D. Schmeidler (1989): "Maxmin Expected Utility with a Non-unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

Giovannetti, B. C. (2013): "Asset Pricing under Quantile Utility Maximization," *Review of Financial Economics*, 22, 169–179.

Grant, S., A. Kajii, and B. Polak (2000): "Temporal Resolution of Uncertainty and Recursive Non-Expected Utility Models," *Econometrica*, 68, 425–434.

Hall, R. E. (1978): "The Stochastic Implications of the Life Cycle – Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy*, 86, 971–987.

——— (1988): "Intertemporal Substitution in Consumption," *Journal of Political Economy*, 96, 339–357.

Hansen, L. P. and T. J. Sargent (2001): "Robust Control and Model Uncertainty," *The American Economic Review*, 91, 60–66.

——— (2004): "Misspecification in Recursive Macroeconomic Theory," Manuscript.

KAHNEMAN, D. AND A. TVERSKY (1979): "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, 47, 263–291.

KARNI, E. AND D. SCHMEIDLER (1991): "Atemporal Dynamic Consistency and Expected Utility Theory," *Journal of Economic Theory*, 54, 401–408.

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2009): "Recursive Smooth Ambiguity Preferences," *Journal of Economic Theory*, 144, 930–976.

KREPS, D. M. AND E. L. PORTEUS (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

KUN LIN, C. J. AND S. MARCUS (2018): "Probabilistically Distorted Risk-Sensitive Infinite-Horizon Dynamic Programming," *Automatica*, 97, 1–6.

LIPPMAN, S. A. AND J. J. MCCALL (1976): "The Economics of Job Search: A Survey," *Economic Inquiry*, 14, 155–189.

LJUNGQVIST, L. AND T. J. SARGENT (2012): *Recursive Macroeconomic Theory*, Cambridge, Massachusetts: MIT Press.

LONG, Y., J. SETHURAMAN, AND J. XUE (2021): "Equal-Quantile Rules in Resource Allocation with Uncertain Needs," *Journal of Economic Theory*, 197, 105350.

LUCAS, R. E. AND E. C. PRESCOTT (1971): "Investment Under Uncertainty," *Econometrica*, 39, 659–681.

MACCHERONI, F., M. MARINACCI, AND A. RUSTICHINI (2006a): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74, 1447–1498.

——— (2006b): "Dynamic Variational Preferences," *Journal of Economic Theory*, 128, 4–44.

MACHINA, M. J. AND D. SCHMEIDLER (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 60, 745–780.

MANSKI, C. (1988): "Ordinal Utility Models of Decision Making under Uncertainty," *Theory and Decision*, 25, 79–104.

MARINACCI, M. AND L. MONTRUCCHIO (2010): "Unique Solutions for Stochastic Recursive Utilities," *Journal of Economic Theory*, 145, 1776–1804.

MCCALL, J. J. (1970): "Economics of Information and Job Search," *Quarterly Journal of Economics*, 84, 113–126.

MENDELSON, H. (1987): "Quantile-Preserving Spread," *Journal of Economic Theory*, 42, 334–351.

MIAO, J. (2013): *Economic Dynamics: Discrete Time*, MIT Press.

MODIGLIANI, F. AND R. BRUMBERG (1954): "Utility Analysis and The Consumption Function: An Interpretation of Cross-Section Data," in *Post-Keynesian Economics*, ed. by K. K. Kurihara, New Brunswick, NJ. Rutgers University Press, 399–496.

Newey, W. K. and J. L. Powell (1987): "Asymmetric least squares estimation and testing," *Econometrica*, 55, 819–847.

Rockafellar, R. T. (1970): *Convex Analysis (Princeton Mathematical Series)*, Princeton, NJ: Princeton University Press.

Rostek, M. (2010): "Quantile Maximization in Decision Theory," *Review of Economic Studies*, 77, 339–371.

Rust, J. (1996): "Numerical Dynamic Programming in Economics," in *Handbook of Computational Economics*, ed. by H. M. Amman, D. A. Kendrick, and J. Rust, Elsevier, chap. 14, 619–729.

Sarver, T. (2018): "Dynamic Mixture-Averse Preferences," *Econometrica*, 86, 1347–1382.

Satchell, S. (2010): *Optimizing Optimization: The Next Generation of Optimization, Applications and Theory*, Academic Press.

Savage, L. J. (1954): *The Foundations of Statistics*, New York, NY: John Wiley & Sons Inc.

Schmeidler, D. (1986): "Integral Representation Without Additivity," *Proceedings of The American Mathematical Society - PROC AMER MATH SOC*, 97, 255–255.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*, Cambridge, Massachusetts: Harvard University Press.

Strzalecki, T. (2013): "Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion," *Econometrica*, 81, 1039–1074.

Tobin, J. (1969): "A General Equilibrium Approach to Monetary Theory," *Journal of Money, Credit and Banking*, 1, 15–29.

Tversky, A. and D. Kahneman (1992): "Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty*, 5, 297–323.

Wang, Z. and G. Klir (2009): *Generalized Measure Theory*, Springer.

Weil, P. (1990): "Nonexpected Utility in Macroeconomics," *The Quarterly Journal of Economics*, 105, 29–42.

Ziegel, J. F. (2016): "Coherence and elicitability," *Mathematical Finance*, 26, 901–918.