The Inner Structure of Tongues

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The Blaschke Family

Given $d \in \mathbb{N}$, we will consider the generalized Blaschke products

$$B_{a,t}(z) = e^{2\pi i t} z^{d+1} \left(\frac{z-a}{1-\bar{a}z} \right)^d, \tag{1}$$

with $t \in [0,1)$ and $a \in \mathbb{C}$.

- $B_{a,t}$ leaves invariant the unit circle, since its action over $\widehat{\mathbb{C}}$ is symmetric with respect to $\mathbb{S}^1$.

Critical Orbits

$B_{a,t}$ has $4d$ critical points, counting with multiplicity:

- The fixed points $z=0$ and $z=\infty$ are critical points of multiplicity $d \geq 1$.
- $z=a$ and $z=\frac{1}{\bar{a}}$ are critical points of multiplicity $d-1$.
- There are two free critical points given by

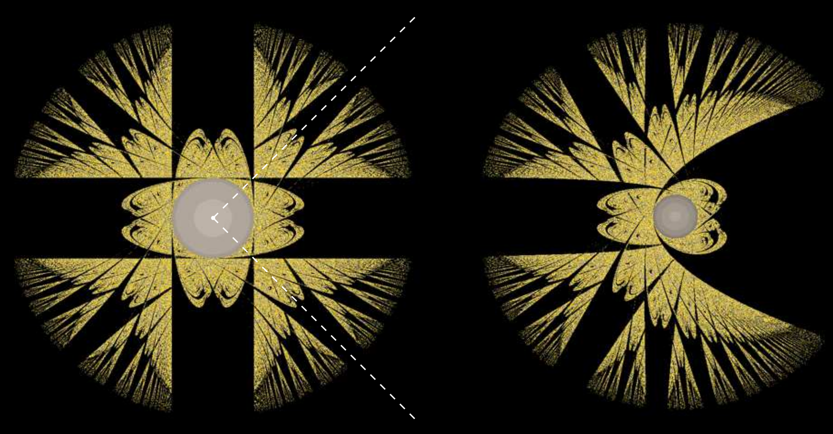
$$c_{\pm} = a \cdot \frac{2d+1+|a| \pm \sqrt{\left(|a|^2-(2d+1)^2\right)\left(|a|^2-1\right)}}{2(d+1)|a|^2}, \tag{2}$$

and their position with respect to $\mathbb{S}^1$ depend on $|a|$.

Parameter Space

In Figure 1, the parameter a is colored as follows:

- gray if the orbit of c_- tends to zero,
- black if one of the critical orbits converges to an attracting or parabolic cycle in $\mathbb{S}^1$.
- Numerical evidence suggests that a is colored yellow if one of the critical points belongs to the Julia set.



In the left, partial representation of the parameter space of $B_{a,0}$ for $d=2$. In the right, partial representation of the reduced parameter space.

Symmetries

- $B_{a,t}$ is conformally conjugate to $B_{b,0}$, where $b = ae^{2\pi i t/2d}$.
- $B_{a,0}$ is conformally conjugate to $B_{\omega a,0}$, with $\omega^{2d}=1$.
- If $a = e^{2\pi i\alpha}$, then $B_{a,0}$ is conformally conjugate to $B_{r,2d\alpha}$.

There is a one-to-one correspondence between parameters a of $B_{a,0}$ and parameters (r,α) of $B_{r,2d\alpha}$.

Degree One Circle Maps

Let $a = e^{2\pi i\alpha}$, with $r \in (1,\infty)$ and $\alpha \in (-1/4d, 1/4d]$. Consider the circle map $g_{r,\alpha} := B_{r,2d\alpha}|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$:

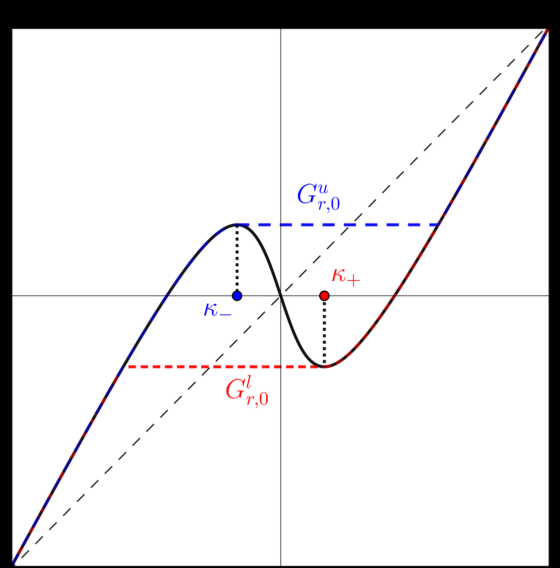
$$g_{r,\alpha}\left(e^{2\pi i x}\right) = e^{2\pi i(2d\alpha)} e^{2\pi i(d+1)x} \left(\frac{e^{2\pi i x}-r}{1-re^{2\pi i x}} \right)^d. \tag{3}$$

Let $f_r(x) = \frac{d}{\pi} \arccos\left(\frac{r-\cos(2\pi x)}{\sqrt{1-2r\cos(2\pi x)+r^2}} \right)$. The lift of $g_{r,\alpha}$ that maps 0 to $2d\alpha$ is given by

$$G_{r,\alpha}(x) = \begin{cases} x + 2d\alpha - f_r(x), & x - \lfloor x \rfloor \in [0,1/2], \\ x + 2d\alpha + f_r(x), & x - \lfloor x \rfloor \in [1/2,1). \end{cases} \tag{4}$$

Proposition 1. *The map $g_{r,\alpha}$ given in (3) is a degree one circle map. Moreover,*

(a) *for $r > 2d+1$, $g_{r,\alpha}$ is an orientation preserving diffeomorphism,*
(b) *for $r = 2d+1$, $g_{r,\alpha}$ is an orientation preserving homeomorphism,*
(c) *for $1 < r < 2d+1$, $g_{r,\alpha}$ is a continuous endomorphism.*



$G_{r,\alpha}$, $G^u_{r,\alpha}$, and $G^l_{r,\alpha}$ for $r \in (1,2d+1)$, $\alpha=0$. The points κ_+ and κ_- are the lifts of c_+ and c_-, respectively, in $[-1/2,1/2)$. Changing the value of α results in vertical translations of the graphs.

Since $g_{r,\alpha}$ is a degree one continuous circle map, then its **rotation set** is a closed interval, (Ito, 1981):

$$\varrho\left(g_{r,\alpha}\right) = \left\{ \varrho\left(x, g_{r,\alpha}\right) = \limsup_{n\to\infty} \frac{g^n_{r,\alpha}(x)-x}{n} : x \in \mathbb{S}^1 \right\} = [\varrho_l, \varrho_u],$$

where $\varrho\left(x, g_{r,\alpha}\right)$ is the rotation number of x under $g_{r,\alpha}$ and ϱ_l, ϱ_u are the rotation numbers of $G^u_{r,\alpha}$, $G^l_{r,\alpha}$, respectively, (Alsedà, Llibre, & Misiurewicz, 2000). The next Lemma is based on (Boyland, 1986).

Lemma 1. *Fix $r > 1$ and $p/q \in \mathbb{Q}$, $(p,q)=1$. Then $\varrho_l, \varrho_u : \mathbb{R} \to \mathbb{R}$ are continuous, non-decreasing and onto. Moreover, $\varrho_l^{-1}(p/q)$, $\varrho_u^{-1}(p/q)$ are non-trivial closed intervals.*

Rational Arnold Tongues

Definition 1. *For $p/q \in \mathbb{Q}$, $(p,q)=1$, the tongue $T_{p/q}$ is the set*

$$T_{p/q} = \left\{ (r,\alpha) : p/q \in \varrho\left(g_{r,\alpha}\right) \right\}. \tag{5}$$

From Lemma 1, for a fixed $r > 1$

- $\varrho_l^{-1}(p/q) = \left[\Psi_{l,p/q}, \Phi_{l,p/q}\right]$ and $\varrho_u^{-1}(p/q) = \left[\Phi_{u,p/q}, \Psi_{u,p/q}\right]$.

Proposition 2. *For $\sigma = l, u$, there are functions $\Psi_{\sigma,p/q}, \Phi_{\sigma,p/q} : (1,\infty) \to \mathbb{R}$ such that $\varrho_\sigma(g_{r,\alpha}) = p/q$ iff*

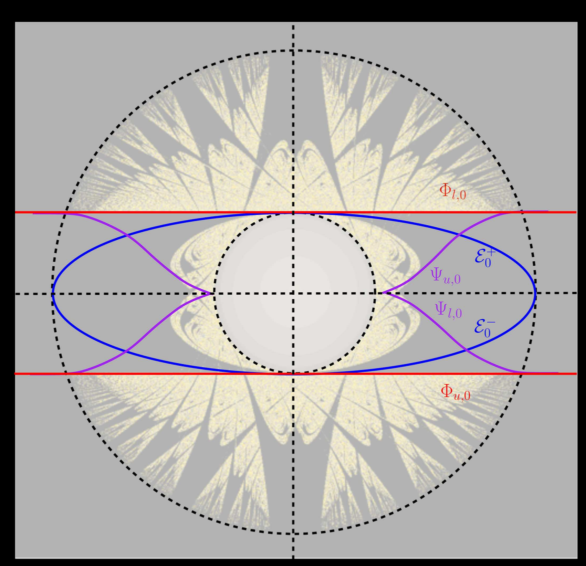
$$\Psi_{l,p/q}(r) \leq \alpha \leq \Phi_{l,p/q}(r) \text{ and } \Phi_{u,p/q}(r) \leq \alpha \leq \Psi_{u,p/q}(r).$$

The configuration of curves in the parameter plane given by Proposition 2 is the following, (Boyland, 1986)

1. $\Phi_{u,p/q} < \Phi_{l,p/q}$,
2. $\Phi_{u,p/q} \leq \Psi_{l,p/q}$, $\quad \Psi_{u,p/q} \leq \Phi_{l,p/q}$,
3. $\Psi_{l,p/q} < \Phi_{l,p/q}$, $\quad \Phi_{u,p/q} < \Psi_{u,p/q}$.

Theorem 1. *For any $p/q \in \mathbb{R} \setminus \mathbb{Z}$, $(p,q)=1$,*

$$T_{p/q} = \left\{ \begin{array}{c} (r,\alpha) \in (1,2d+1) \times (-1/4d, 1/4d] : \\ \Phi_{u,p/q}(r) \leq \alpha \leq \Phi_{l,p/q}(r). \end{array} \right\}$$



Sketch of the inner structure of the tongue T_0 given by the curves $\Psi_{l,0}$, $\Psi_{u,0}$, $\Phi_{l,0}$, $\Phi_{u,0}$, $\mathcal{E}_0^{\pm}$. The parameter space is for $d=1$.

We can provide dynamical information for parameters in the boundary of the tongue $T_{p/q}$.

Theorem 2. *$(r,\alpha) \in \partial T_{p/q}$ if and only if there is a parabolic p/q-cycle under $g_{r,\alpha}$ with multiplier $+1$.*

The curves $\Psi_{\cdot,p/q}$ and $\Phi_{\cdot,p/q}$ coincide for an specific value of r (Boyland, 1986).

Proposition 3. *For each rational p/q, there exists $r_0 \in (1,2d+1)$ such that $\Psi_{l,p/q}(r) = \Phi_{u,p/q}(r)$, and $\Psi_{u,p/q}(r) = \Phi_{l,p/q}(r)$, for $r \geq r_0$.*

There is more dynamical information inside the tongue $T_{p/q}$, as stated in the following theorem, (Boyland, 1986).

Theorem 3. *For each fixed rational p/q and $r \in (1,2d+1]$, there exists a unique $\alpha(r) = \mathcal{E}^{\pm}_{0,p/q}(r)$ such that c_+ belongs to a super attracting p/q-cycle under $g_{r,\alpha}$. Analogously for c_-.*

If we consider $\mathcal{E}^{\pm}_{0,p/q} : (1,2d+1] \to \mathbb{R}$ as functions of r, then

- $\Psi_l < \mathcal{E}^{+}_{0,p/q} < \Phi_l$,
- $\Phi_u < \mathcal{E}^{-}_{0,p/q} < \Psi_u$.

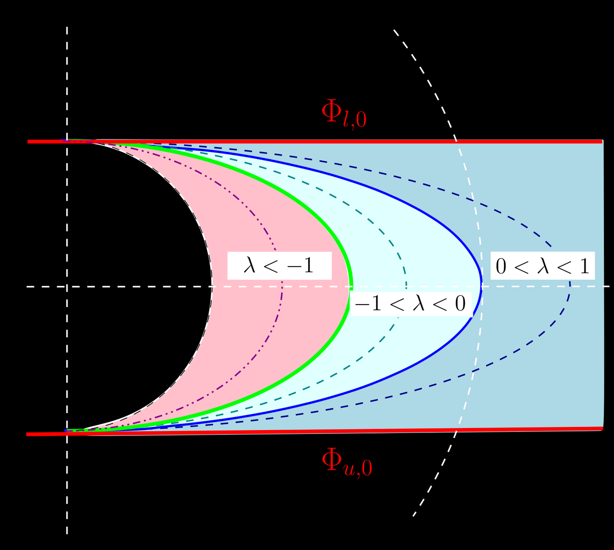
The Tongue T_0

We have more detailed information on the tongue T_0,

Theorem 4. *Let $(r,\alpha) \in T_0$ and λ the multiplier of a fixed point x_0 under $G_{r,\alpha}$. Then,*

$$\alpha(r) = \mathcal{E}^{\pm}_{\lambda,0}(r) = \pm\frac{1}{2\pi} \arccos\left(\frac{2d+1-\lambda}{2\sqrt{d(d+1-\lambda)}} \frac{\sqrt{r^2-1}}{r} \right).$$

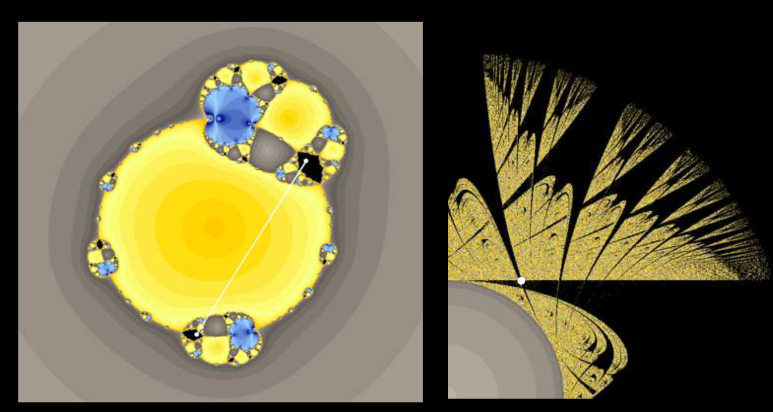
Moreover, $\mathcal{E}^{\pm}_{\lambda,0}$ is bounded by $\mathcal{E}^{-}_{1,0}$ and $\mathcal{E}^{+}_{1,0}$, for every λ.



Sketch of the inner structure of the tongue T_0 given by the multiplier curves $\mathcal{E}^{\pm}_{\lambda,0}$. In blue $\mathcal{E}^{\pm}_{0,0}$ and in green $\mathcal{E}^{\pm}_{-1,0}$.

Theorem 5. *Let $m/n \in \mathbb{Q} \cap [0,1/2]$, $(m,n)=1$. Then,*

$$T_0 \cap T_{m/n} \neq \emptyset.$$



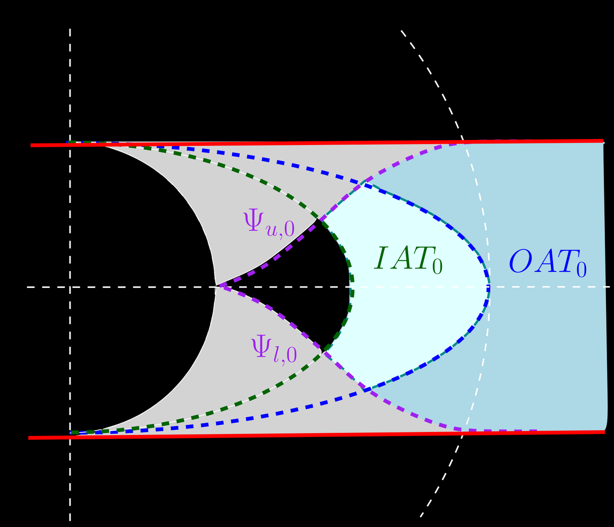
Dynamical plane of $B_{r,2d\alpha}$ with $(r,\alpha) \in \partial T_0 \cap T_{1/2}$. In yellow the basin of zero, in gray the basin of ∞, in blue the basin of a parabolic fixed point in $\mathbb{S}^1$, and in black the basin of an attracting 2-cycle with rotation number $1/2$ in $\mathbb{S}^1$.

Adjacent Parameters in T_0

Definition 2. *We define the following sets in T_0:*

$$IAT_0 = \left\{ \begin{array}{c} (r,\alpha(r)) \in (1,2d+1) \times \left(\Psi_{l,0}(r), \Psi_{u,0}(r)\right) : \\ \mathcal{E}^{+}_{-1,0}(r) \leq |\alpha(r)| \leq \mathcal{E}^{+}_{0,0}(r) \end{array} \right\}$$

$$OAT_0 = \left\{ (r,\alpha(r)) \in (1,\infty) \times \left(\Psi_{l,0}(r), \Psi_{u,0}(r)\right) : |\alpha(r)| \geq \mathcal{E}^{+}_{0,0} \right\}$$



Sketch of the regions described in Definition 2. Depicted in gray is the region where other rational tongues intersect T_0.

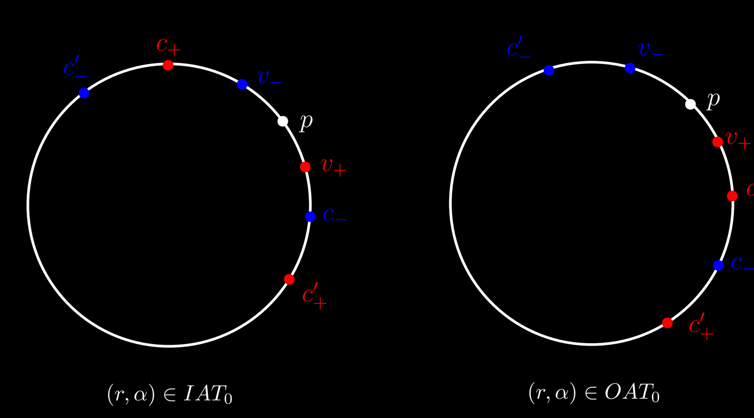
Lemma 2. *Denote as $\kappa'_{\pm}$ the cocritical points under $G_{r,\alpha}$. For every $(r,\alpha) \in IAT_0 \cup OAT_0$, there is an attracting fixed point x_0, both free critical points lie in the immediate basin of x_0, and*

1. *if $(r,\alpha) \in IAT_0$, then*

$$\kappa'_+ < \kappa_- < G_{r,\alpha}(\kappa_+) < x_0 < G_{r,\alpha}(\kappa_-) < \kappa_+ < \kappa'_-,$$

2. *if $(r,\alpha) \in OAT_0$, then*

$$\kappa'_+ < \kappa_- < \kappa_+ < G_{r,\alpha}(\kappa_+) < x_0 < G_{r,\alpha}(\kappa_-) < \kappa'_-.$$



Sketch of the configuration of points in $\mathbb{S}^1$ described in Lemma 2. The points $c_{\pm}$, $c'_{\pm}$, $v_{\pm}$ and p are the projections of $\kappa_{\pm}$, $\kappa'_{\pm}$, $G_{r,\alpha}(\kappa_{\pm})$ and x_0, respectively.

References

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Ito, R. (1981). Rotation sets are closed. In *Mathematical proceedings of the cambridge philosophical society* (Vol. 89, pp. 107–111).