From Herman Rings to Herman Curves

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Herman Rings

Fix $d \in \mathbb{N}_{\geq 2}$. Let $\theta = [0; a_1, a_2, \ldots]$ be an irrational number of bounded type, i.e. it has a bound $B := \max_i a_i < \infty$.

Theorem [Z11]

The boundary of an invariant Siegel disk of a rational map $f \in \text{Rat}_d$ of rotation number θ is a $K(B,d)$-quasicircle containing a critical point.

Via Shishikura's surgery [S87], we can construct a Herman ring out of two Siegel disks, and vice versa. Consequently, the boundaries of any invariant Herman ring $\mathbf{H}$ of $f \in \text{Rat}_d$ are also $K(B, d, \text{mod}(\mathbf{H}))$-quasicircles containing a critical point. Can we remove the dependence on $\text{mod}(\mathbf{H})$?

Let $\mathcal{H}$ be the space of degree d rational maps $f \in \text{Rat}_d$ such that:

(1) the only non-repelling periodic points are superattracting fixed points 0 and ∞ with local degree $d_0 \geq 2$ and $d_\infty \geq 2$, with $d_0 + d_\infty = d+1$;
(2) f has a Herman ring $\mathbf{H}$ of rotation number θ;
(3) $\mathbf{H}$ separates 0 and ∞;
(4) critical points of f other than 0 and ∞ lie on $\partial\mathbf{H}$.

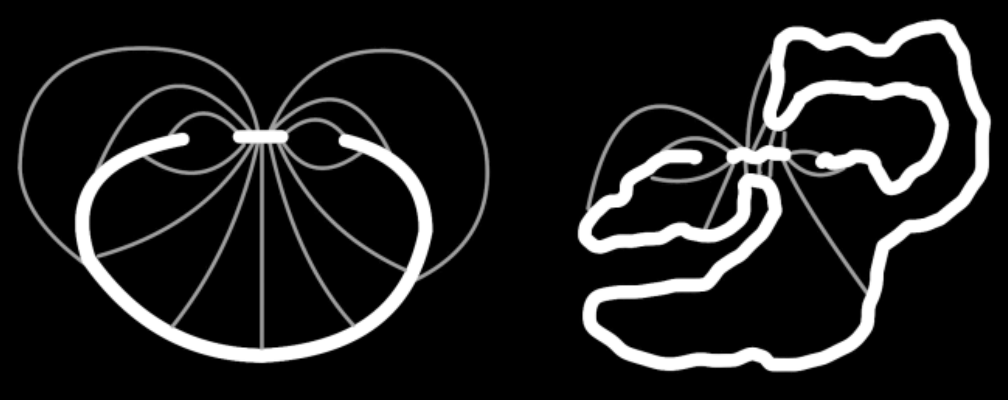
$\mathcal{H}$ consists of maps that can be obtained from Shishikura's surgery out of two polynomials with a Siegel disk satisfying a condition akin to (4). For Herman rings in this class, the dilatation is independent of the modulus!

Theorem A [L22a]

The boundary components of the Herman ring of maps in $\mathcal{H}$ are $K(B, d_0, d_\infty)$-quasicircles.

Our proof hinges on the near-degenerate machinery, inspired by [KL05, K06, DL22]. A rough sketch of the main strategy is outlined below.

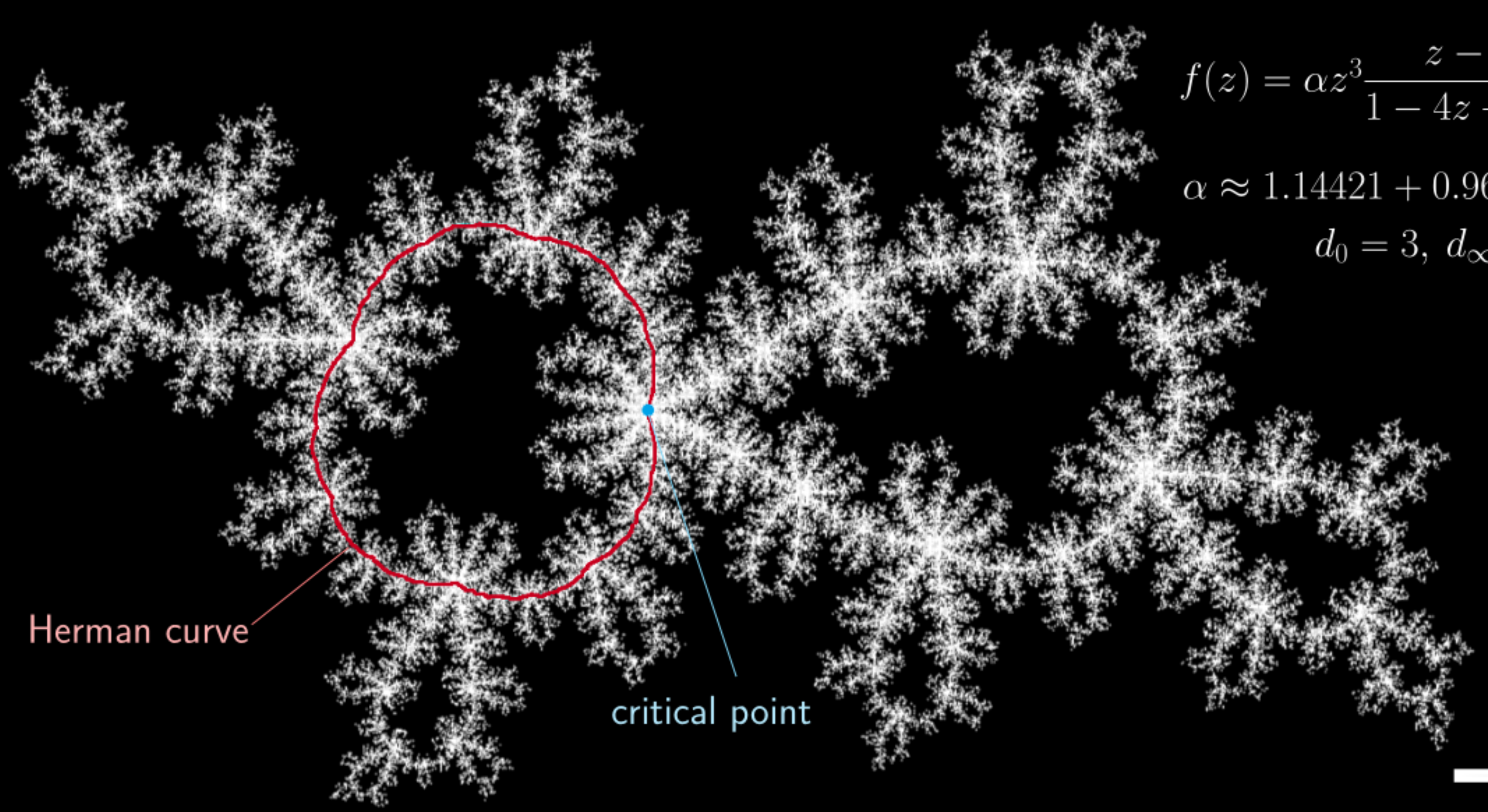
Let H be a boundary component of the Herman ring of $f \in \mathcal{H}$. Endow H with the combinatorial metric, i.e. the normalized $f|_H$-invariant metric. Let $I \subset H$ be an interval of (combinatorial) length $|I| < 0.1$. Denote by $W_{10}(I)$ the extremal width of curves connecting I and $H \backslash 10I$, where $10I$ is the interval of length $10|I|$ having the same midpoint as I.



small $W_{10}(I)$ large $W_{10}(I)$

$W_{10}(I)$ quantifies the (near-)degeneracy of H about I. Thus, it suffices to prove the following (with all bounds depending only on B, d_0, d_∞).

"If there is an interval $I \subset H$ with $|I| \ll 1$ and $W_{10}(I) = K \gg 1$, then there is another interval J with $|J| \ll 1$ and $W_{10}(J) \geq 2K$."



Examples

The Julia sets of f and g are shown above. The coefficients α and q are determined numerically such that f and g admit non-trivial Herman curves with golden mean rotation number $\theta = [0; 1, 1, \ldots]$.

Open Questions

- ▷ Is the limit of degenerating Herman rings always a Herman curve?
- ▷ Is every Herman curve a limit of degenerating Herman rings?
- ▷ Is there a Herman curve that is not a quasicircle?
- ▷ For $f \in \mathcal{X}$, $\text{area}\, J(f) = 0$? $\dim_H J(f) < 2$?

Herman Curves

An invariant Jordan curve $\mathbf{X}$ of a rational map f is a **Herman curve** if

▷ $f|_{\mathbf{X}}$ is conjugate to a rigid rotation, and
▷ $\mathbf{X}$ is not within the closure of any rotation domain.

Additionally, we call $\mathbf{X}$ a **Herman quasicircle** if it is a quasicircle. The combinatorics of $\mathbf{X}$ is encoded by the criticality and the relative combinatorial position of the critical points on $\mathbf{X}$.

Question [Eremenko]

Is there a Herman curve $\mathbf{X}$ that is non-trivial, i.e. not qc conjugate to a Blaschke product (in which case $\mathbf{X}$ is a circle) around its Julia set?

To answer this, we use Theorem A and a Thurston-type result for Herman rings [W12] to study the limits of degenerating Herman rings in $\mathcal{H}$.

Theorem B [L22a]

Every rational map in the limit space $\overline{\mathcal{H}} \backslash \mathcal{H} \subset \text{Rat}_d$ admits a Herman quasicircle. Moreover, given any combinatorics, there is a Herman quasicircle in $\overline{\mathcal{H}} \backslash \mathcal{H}$ that realises such prescribed combinatorics.

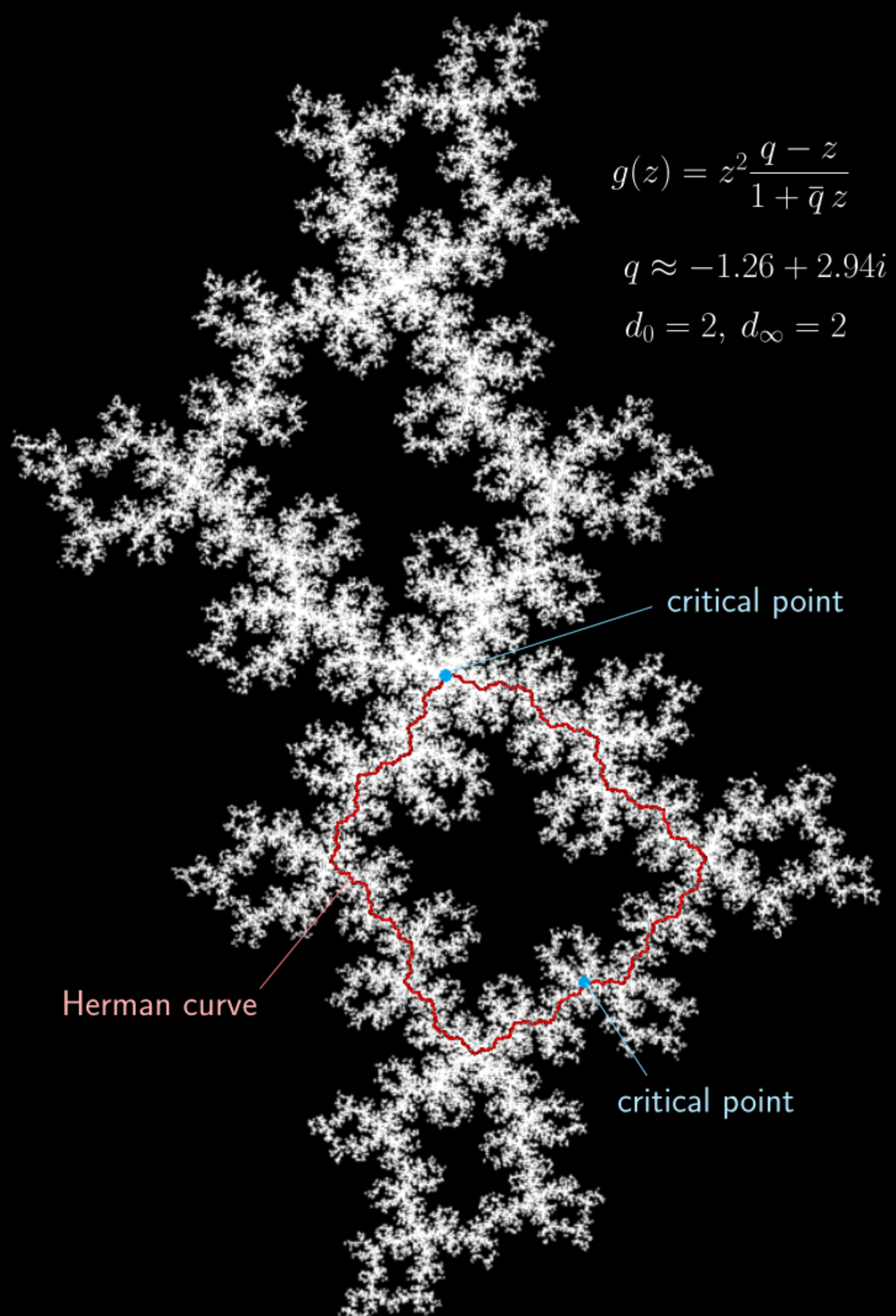
Denote by $\mathcal{X} \subset \text{Rat}_d$ the space of rational maps satisfying (1)-(4) except that $\mathbf{H}$ is replaced with a Herman quasicircle.

We adapt some ideas from [McM96] to show the absence of invariant line fields on $J(f)$ for $f \in \mathcal{X}$. By the pullback argument, we then show:

Theorem C [L22b]

Maps in $\mathcal{X}$ are combinatorially rigid.

Theorems B and C imply that every Herman curve in $\mathcal{X}$ is a limit of degenerating Herman rings, i.e. $\mathcal{X} = \overline{\mathcal{H}} \backslash \mathcal{H}$.



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