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Extensions of the conformal group present entropy

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Context

Given a closed and oriented topological manifold M, let

$$\mathrm{Homeo}_+(M) = \left\{ \begin{array}{l} \text{orientation preserving} \\ \text{self-homeomorphisms of } M \end{array} \right\}$$

Given $k \in \mathbb{N}$, the action of a subgroup $G \subset \mathrm{Homeo}_+(M)$ is said to be *k-transitive* if for every pair of k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ there exists some transformation $g \in G$ such that

$$q_i = g(p_i) \text{ for each } i \in \{1, \ldots, k\}.$$

When such a transformation is *unique*, the group is said to be *sharply k-transitive*.

Problem. Given M, to classify the transitive subgroups of $\mathrm{Homeo}_+(M)$ with respect to transitivity, up to their uniform closures.

The case $M = \mathbb{S}^2$

Theorem [Kerékjártó, 40's]. Let $G < \mathrm{Homeo}_+(\mathbb{S}^2)$ be compact and transitive. Then, G is conjugate to $\mathrm{Rot}(\mathbb{S}^2)$.

The conformal group

The action

$$(\pm A, z) \in \mathrm{PSL}_2(\mathbb{C}) \times \mathbb{C} \cup \{\infty\} \mapsto M_A(z) \overset{\text{def}}{=} \frac{az+b}{cz+d} \in \mathbb{C} \cup \{\infty\}$$

induces, by stereographic conjugation, the subgroup $\mathrm{Möb}(\mathbb{S}^2)$ of *Möbius transformations*.

$\mathrm{Möb}(\mathbb{S}^2)$ consists of the orientation preserving *conformal diffeomorphisms* of the 2-sphere.

Theorem [Kwakkel & Tal, –]. Let $\mathrm{Rot}(\mathbb{S}^2) < G \leq \mathrm{Diff}^1_+(\mathbb{S}^2)$. If G is sharply 3-transitive, then $G = \mathrm{Möb}(\mathbb{S}^2)$.

Question [Kwakkel & Tal, Le Roux]. Is $\mathrm{Möb}(\mathbb{S}^2)$ maximal in $\mathrm{Homeo}_+(\mathbb{S}^2)$?

Contributions

Theorem A. Let $G \leq \mathrm{Diff}^1_+(\mathbb{S}^2)$ be a proper extension of $\mathrm{Möb}(\mathbb{S}^2)$. Then, its identity component G_0 is *at least 4-transitive*.

Theorem B. Let G be as in Theorem A. Then, G contains an element f such that its restriction to the complement of four fixed points is isotopic to a pseudo-Anosov map, relative to those points. In particular, f has *strictly positive topological entropy*.

Outline of arguments

An isotopy $(f_t)_{t\in I}$ such that $0 \in I$, $f_0 = \mathrm{id}$ and $f_t \in G$ for every $t \in I$ is referred to as an *$\mathcal{I}G$-isotopy*.
Given $p \in \mathbb{S}^2$, its *trajectory* is the curve

$$\gamma_f(p) \overset{\text{def}}{=} \{f_t(p) : t \in I\}$$

If I is unbounded above, the *ω-limit* of p is the following (possibly empty) set of accumulation points:

$$\omega_f(p) \overset{\text{def}}{=} \left\{ q \in \mathcal{M} : \begin{array}{l} \text{there exists some sequence} \\ t_n \nearrow +\infty \text{ such that } f_{t_n}(p) \to q \end{array} \right\}.$$

There is an analogous notion of α-limit.

Consider the points $\mathbf{0}, \mathbf{1}$ and ∞, and let Γ be the unique meridian containing them. Then, one defines $G_2 = \mathrm{Stab}_G\{\mathbf{0}, \infty\}$ and $G_3 = \mathrm{Stab}_G\{\mathbf{0}, \mathbf{1}, \infty\}$.

Input: A proper group extension $\mathrm{Möb}(\mathbb{S}^2) \subsetneq G \subseteq \mathrm{Diff}^1_+(\mathbb{S}^2)$.

↓ Hypothesis

Extract $\hat{g} \in G_2$ having a saddle at $\mathbf{0}$.

↓ Continuous deformations by loxodromic transformations

The Extension Lemma: Construct an $\mathcal{I}G_2$-isotopy $(g_t)_{t\in[0,1]}$ from the identity to a saddle.

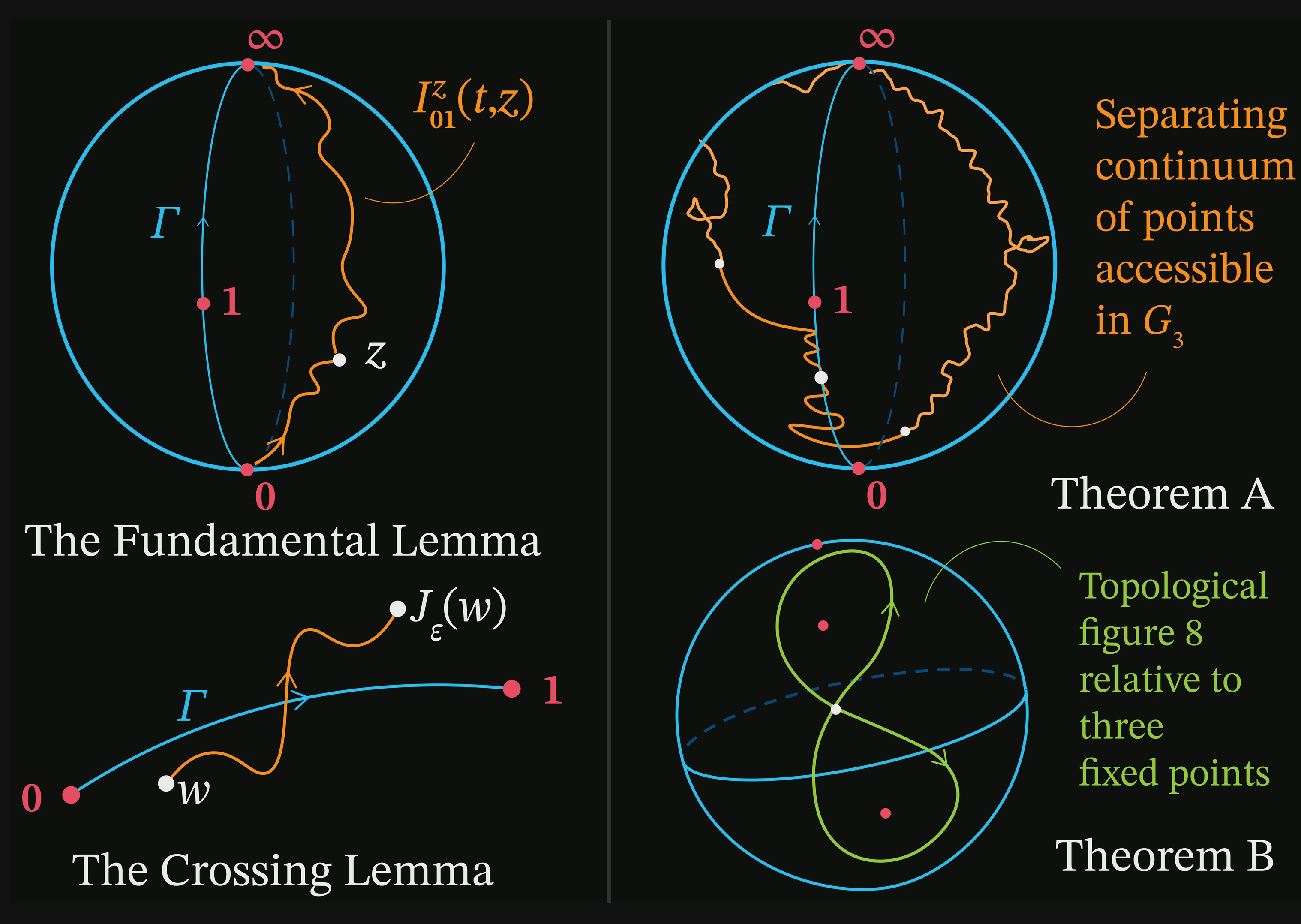
↓ Uniform cone distortion arguments and properties of Möbius transformations → **The Fundamental Lemma:** For each $z \notin \Gamma$, construct a full-time isotopy $\left(I^z_{\mathbf{ab}}(t,\cdot)\right)_{t\in\mathbb{R}}$ for which z has the two points $\mathbf{a}, \mathbf{b} \in \{\mathbf{0}, \mathbf{1}, \infty\}$ as α and ω-limits.

↓ Stability of transversal intersections → **The Crossing Lemma:** Obtain a point w to the left of Γ and an $\mathcal{I}G_3$-isotopy $(J_t)_{t\in[0,1]}$ such that $J_\varepsilon(w)$ lies to the right of Γ for some $\varepsilon > 0$.

↓ Concatenations at the levels of maps and of space

↓ Connect any two given points by arcs of trajectories of isotopies in $\mathrm{Stab}_G\{\mathbf{0}, \mathbf{1}, \infty\}$. → **Theorem A:** The group G is 4-transitive (actually, arc 4-transitive).

↓ The Nielsen-Thurston Theorem applied to a topological figure 8 → **Theorem B:** G_0 contains a map having positive topological entropy



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