Julia-like sets in the parameter space of a family of quartic rational maps

Pedro Iván Suárez Navarro

Universidad Nacional José María Arguedas

pisuarez@unajma.edu.pe

Abstract

We report the existence of an infinite quantity of Julia-like sets in the parameter space of a family $\mathcal{F}$ of rational maps of degree four with two superattracting fixed points. We show numerically that the family $\mathcal{F}_2$, i.e, the slice of $\mathcal{F}$ such that a critical point has period two, contains the Julia-like sets and Mandelbrot-like sets that coexist in a region of the imaginary axis and in others symmetrical regions.

Introduction

We consider the family $\mathcal{F}$ of all rational mapping f of degree four with two superattracting fixed points c_1, c_2. Up to conformal conjugation it can be written as the following two-parameter family $\mathcal{F} = \{f_{\lambda,\mu}\}$

$$f_{\lambda,\mu}(z) = \lambda z^3 \frac{\mu z + \frac{1-3\mu}{2}}{\frac{3-\mu}{2} z - 1}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{C} \setminus \{0, 1\}, \tag{1}$$

where 0 and ∞ are superattarcting and 1 is a critical point. This family is the case $d = 3$ in the sequence of families

$$(z, \lambda, \mu) \mapsto \lambda z^d \frac{\mu z + \frac{1-d\mu}{d-1}}{1 - \frac{d-\mu}{d-1} z}, d \geq 2, \quad \lambda \in \mathbb{C} \setminus \{0\}, \mu \in \mathbb{C} \setminus \{0, 1\}.$$

The critical orbit is

$$0 \to 0, \infty \to \infty, 1 \to \lambda, \frac{1 - d\mu}{\mu(\mu - d)} \to \frac{(1 - \mu d)^{d+1}}{\mu^{d-1}(\mu - d)^{d+1}} \lambda.$$

In [1] pointed out the similarity of certain sets in the family $\mathcal{F}$ (for $d = 2$) with Julia-like sets. As far as we know in [3] it is shown that quasiconformal copies of quadratic Julia sets appear in a slice of the parameter space of a two-parametric family of cubic polynomials, but no evidence has been shown for other examples of rational functions where this phenomenon occurs.

The Family $\mathcal{F}_2$

We consider $\mathcal{F}_2$ the set of parameters $(\lambda, \mu) \in \mathbb{C}^2$ for which 1 is a periodic critical point with period two. In this case, we obtain a one-parameter family, where

$$\mu = \frac{-2 + \lambda + \lambda^2 + \lambda^3}{\lambda(1 + \lambda + \lambda^2 - 2\lambda^3)}.$$

This define a one-parameter family $\mathcal{F}_2 \subset \mathcal{F}$. Recall that $f_{\mu,\lambda}$ is conjugate to $f_{\frac{1}{\lambda},\frac{1}{\mu}}$ by $z \mapsto \frac{1}{z}$. For make this symmetry easier to notice (especially in figures as suggested in [1]), we replace λ by another parameter t, so that for $f_t \in \mathcal{F}_2$, f_t is conjugate to f_{-t}. Let $t = \frac{\lambda-1}{\lambda+1}$ then

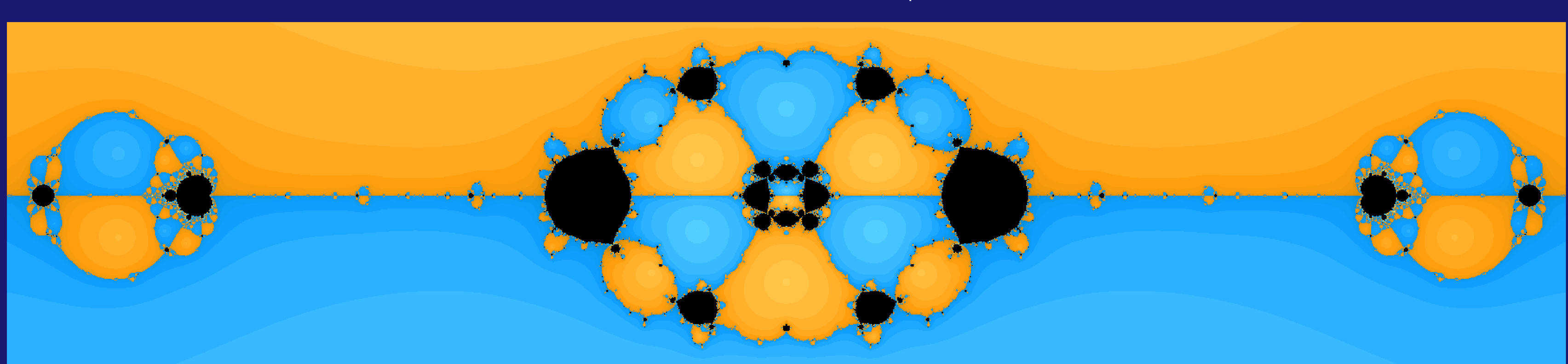

Figure 1: (rotated by 90^o). The t-parameter family $\mathcal{F}_2$.

$$\lambda = \frac{t+1}{1-t}, \quad \mu = \frac{(t-1)(1 + 9t - 5t^2 + 3t^3)}{(1+t)(-1 + 9t + 5t^2 + 3t^3)}. \tag{2}$$

Substituting (2) in (1) we obtain a family $f_t = f_{\lambda,\mu}$ for λ, μ as in (2) given by

$$f_t(z) = -\frac{1+t}{t-1} z^3 \frac{(1+16t-14t^2+16t^3-3t^4+(-1-8t+14t^2-8t^3+3t^4)z)}{(1-8t-14t^2-8t^3-3t^4+(-1+16t+14t^2+16t^3+3t^4)z)}.$$

Results

In this section, we show some simulations with parameter space of the family f_t, $t \in \mathbb{C}$ by considering the behavior of the critical point u_t of f_t. By (2)

$$u_t = \frac{(1+t)(-1 + 9t + 5t^2 + 3t^3)(-1 - 16t + 14t^2 - 16t^3 + 3t^4)}{(t-1)(1 + 9t - 5t^2 + 3t^3)(-1 + 16t + 14t^2 + 16t^3 + 3t^4)}.$$

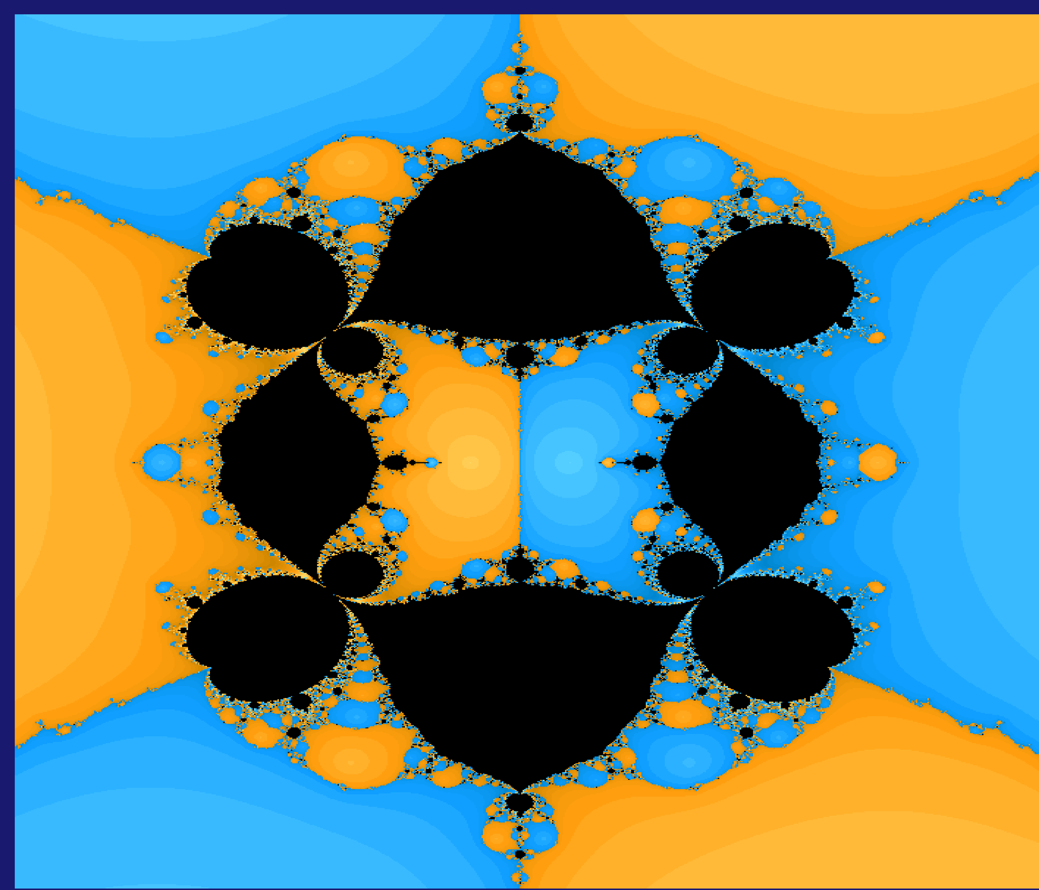

Figure 2: Zoom in the center of Figure 1

Figure 2 shows blue and orange regions that correspond to the set of parameters t such that the u_t is in the basin of 0 and ∞ respectively and the black region: u_t is in the basin of 1.

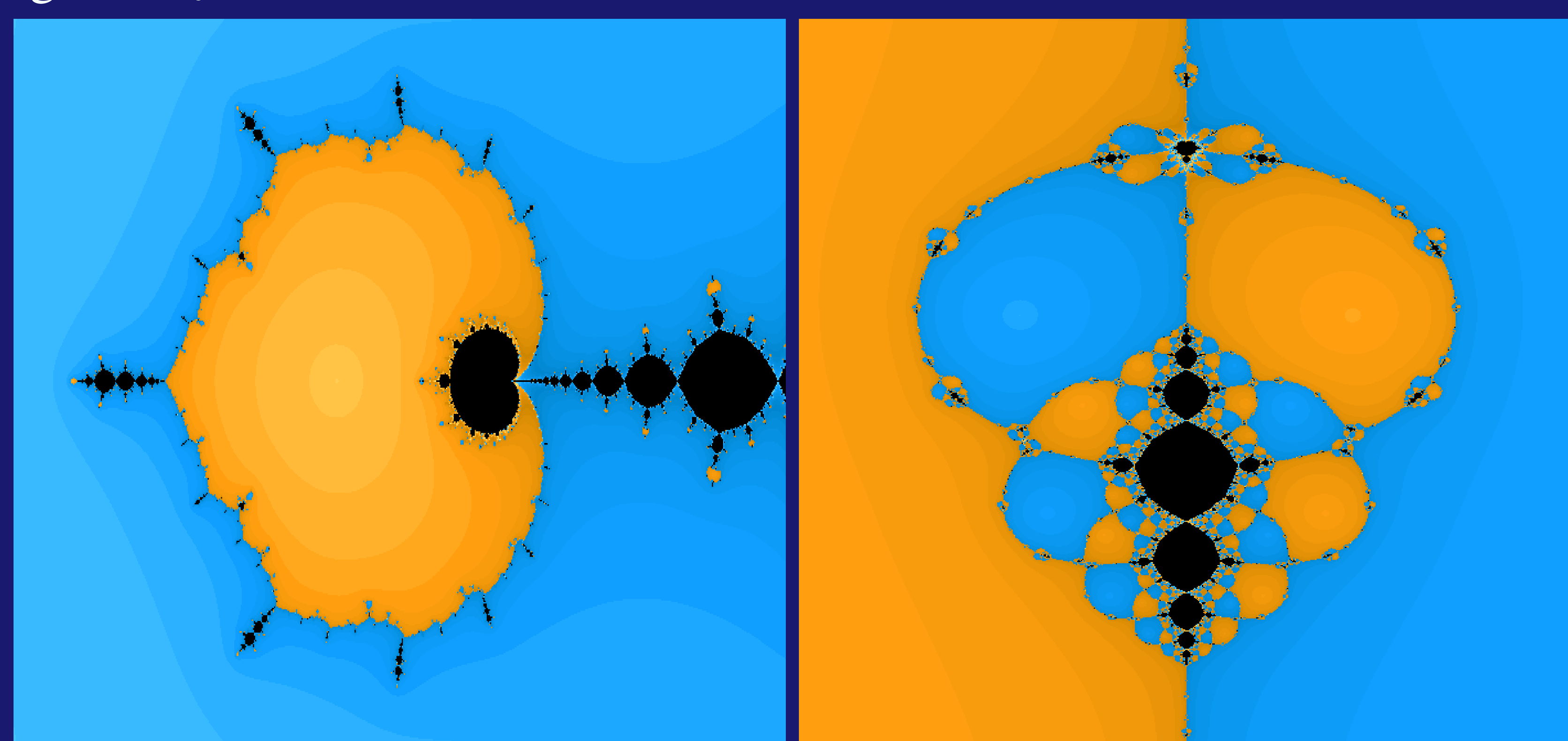

Figure 3: Zooms in Figure 2.

Figure 3 shows Julia-like sets in the center of Figure 2 localized in the imaginary axis, where also we observed Mandelbrot-like sets. On the left, we show a region symmetric with respect to the imaginary axis, that contains a Julia-like set attached along the boundary of the escape component.

Conclusion

We have explored through numerical simulations the parameter space of a family f_t to show the existence of Julia-like sets living together with Mandelbrot-like sets. Naturally, we ask ourselves: How often is this phenomenon in other parameter spaces?

References

[1] Krzysztof Barański. From newton's method to exotic basins part i: The parameter space. *Fundamenta Mathematicae*, 158(3):249–288, 1998.

[2] Bryan Boyd and Suzanne Boyd. Dynamics explorer. *Computer Software. https://sourceforge. net/projects/detool*.

[3] Xavier Buff and Christian Henriksen. Julia sets in parameter spaces. *Communications in Mathematical Physics*, 220(2):333–375, 2001.

Acknowledgment

To the Basic Sciences Department of the José María Arguedas National University, where part of this work was carried out.