Mating polynomials with Kleinian groups

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Abstract

In [1], a construction via surgery that mates a quadratic polynomial and a representation of the modular group was described. This result is an attempt at finding a bridge between the theories of rational maps and Kleinian groups, whose similarities are compiled in what is known as Sullivan's dictionary. We show that construction is continuous with respect to the parameters.

Introduction

We start the theorem of Bullet and Harvey:

Theorem (Bullet-Harvey, [1])**.** Given a quadratic polynomial $f_c(z) = z^2 + c$ wich connected Julia set and a discrete faithful representation r of $\mathrm{PSL}(2,\mathbb{Z})$ with connected regular set, there exists a $2:2$ correspondence

$$F_{a,k} = \left\{ \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k \right\}$$

and a decomposition $\overline{\mathbb{C}} = \Omega \sqcup \Lambda$ into $F_{a,k}$-invariant sets such that

- $\Omega/F_{a,k}$ is conformal equivalent to (half) the surface $\Omega(r)/r$;
- $\Lambda = \Lambda_- \sqcup \Lambda_+$, where $F_{a,k} : \Lambda_- \to \Lambda_-$ is a $2:1$ map hybrid equivalent to f_c and $F_{a,k} : \Lambda_+ \to \Lambda_+$ is a $1:2$ correspondence hybrid equivalent to f_c^{-1}.

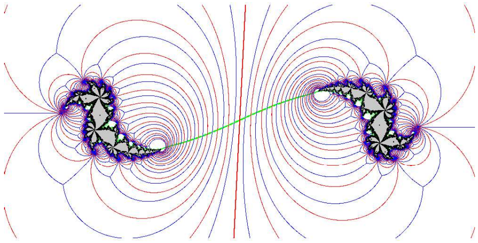

Figure 1: A mating as above.

Here, the $2:2$ correspondence is defined as an algebraic curve, but understood as a multi-valued map (with 2 images and pre-images for generic points). To explain the construction, we remind a few propeties of representations of the modular group:

- $\mathrm{PSL}(2,\mathbb{Z}) \simeq C_2 * C_3$;
- Representation $r : \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{C})$ is defined by the images σ_r, ρ_r of the two generators;
- Cross-ratio between the fixed points parametrizes the representations;
- *Discreteness locus* $\mathcal{D}$ is the set of parameters for which the representation is discrete;
- $\mathring{\mathcal{D}}$ corresponds to the representations that have connected regular set;
- They come equipped with an involution χ_r exchanging fixed points.

Bullet-Harvey Construction

For the quadratic polynomial f_c, take its Böttcher coordinate $\varphi_c : \overline{\mathbb{C}} \backslash K_c \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ conjugating f_c with z^2. Define $\tilde{U}_c := \varphi_c^{-1}(\mathbb{D}_R), U_c := f_c(\tilde{U}_c)$ and $A_c := U_c \backslash \overline{\tilde{U}_c}$, where $R > 1$ is arbitrary. Then A_c is an annulus. The map f_c induces a $2:1$ covering of the inner boundary onto the outer boundary. We also notice that the map $\varphi_c^{-1}(z) \mapsto \varphi_c^{-1}(\overline{z})$ is an orientation reversing involution of the outer boundary onto itself with two fixed points.

Given a representation $r \in \mathring{\mathcal{D}}$, one may produce a fundamental domain Δ_r that is a quadrilateral. In the quotient $\overline{\mathbb{C}}/\langle \chi_r \rangle$, the images of Δ_r under the correspondence generated by $\rho^{\pm 1}$ combine to form an annulus B_r. The branches of $\rho^{\pm 1}$ sending the inner boundary to the outer boundary of B_r glue to form a $2:1$ covering map, while the action of σ on the outer boundary is an orientation reversing involution with two fixed points.

Take a diffeomorphism $\psi : A_c \to B_r$ preserving the boundary dynamics. Letting $V_c := U_c \sqcup B_r/\psi$ and V_c' be a copy of V_c, we get a sphere Σ by gluing V_c and V_c' along the boundary via the orientation reversing involution j. We then define a $2:2$ correspondence G on Σ by:

- $f_c : \tilde{U}_c \to U_c$ (a $2:1$ map);
- $f_c^{-1} : V_c \to \tilde{V}_c$ (a $1:2$ correspondence);
- $j \circ \rho^{\pm} : A_c \to V_c \setminus \tilde{V}_c$ (a $2:2$ correspondence);
- $j : \tilde{U}_c \to \tilde{V}_c$ (a $1:1$ map).

The final step is to construct a Beltrami form μ preserved by G. By the Ahlfors-Bers Theorem, $\mu = \phi^*\mu_0$ for some quasi-conformal map $\phi : \Sigma \to \overline{\mathbb{C}}$, and Weyl's Lemma concludes that $F := \phi \circ G \circ \phi^{-1}$ is a rational $2:2$ correspondence on $\overline{\mathbb{C}}$.

Results

Theorem. The map $(c, r) \in M \times \mathring{\mathcal{D}} \to (a, k) \in \mathbb{C}^2$ that associates to the pair (c, r) the mating $F_{a,k}$ of f_c and r is continuous.

The proof comes in the following steps:

1. Finding a holomorphic motion of "fundamental domains";
2. Splitting the problem into the two variables;
3. Using area arguments to show continuity when $c \in \mathring{M}$;
4. Using a result of Ahlfors to approximate $c \in \partial M$.

The holomorphic motion for the fundamental annuli A_c comes from definition, since Böttcher coordinates depend analytically on the parameter. For the "fundamental" annuli B_r, we notice that we can define (at least locally) a holomorphic motion of the domains Δ_r. Letting r_0 be the base of this motion, we can take uniformizations $\phi_r : \overline{\mathbb{C}}/\langle \chi_r \rangle \to \overline{\mathbb{C}}/\langle \chi_{r_0} \rangle$ that vary analytically with r and the image $\tilde{B}_r = \phi_r(B_r)$ is 2-covered with a single ramification point by Δ_r. Since the pair (B_{r_0}, Δ_{r_0}) has the same property, the holomorphic motion descends:

$$\begin{array}{ccccc} \Delta_{r_0} & & \longrightarrow & & B_{r_0} \\ \downarrow & & & & \vdots \\ \Delta_r & \longrightarrow & B_r & \xrightarrow{\phi_r} & \tilde{B}_r \end{array}$$

Once we have a holomorphic motion of the domains, we can follow arguments similar to the ones used in [5] to prove the continuity of the Douady-Hubbard map for analytic families of polynomial-like maps: item 3 follows in the interior of M because of the continuity of the function $c \mapsto K_c$ in the Hausdorff topology; and item 4 follows from the fact that in ∂M a quasi-conformal conjugacy is necessarily hybrid.

Conclusion

The identity representation of the modular group lives in the boundary $\partial\mathcal{D}$, where the Bullet-Harvey construction fails. Still, Bullet and Lomonaco showed in [3] that there exist matings of the modular group with *parabolic-like maps* in the family $\mathrm{Per}_1(1) = \{P_A : z \mapsto z + 1/z \pm A\}$. The set of A^2 for which P_A has connected filled Julia set is the *parabolic Mandelbrot set* M_1. Petersen and Roesch proved in [6] that there is a dynamical homeomorphism $h : M \to M_1$. One naturally conjectures:

Conjecture. If $r_n \to \mathrm{id}$ is a sequence of representations in $\mathring{\mathcal{D}}$ converging to the identity and $c \in M$, then the matings of r_n and c converge to the mating of $\mathrm{PSL}(2,\mathbb{Z})$ and $h(c)$.

References

[1] S. Bullet and W. Harvey. Mating quadratic maps with kleinian groups via quasiconformal surgery. *Electronic Research Announcements of the American Mathematical Society*, 6(3):21–30, 2000.

[2] S. Bullet and P. Haïssinsky. Pinching holomorphic correspondences. *Conformal Geometry and Dynamics of the American Mathematical Society*, 11(6):65–89, 2007.

[3] S. Bullet and L. Lomonaco. Mating quadratic maps with the modular group II. *Inventiones mathematicae*, 220:185–210, 2020.

[4] S. Bullet and C. Penrose. Mating quadratic maps with the modular group. *Inventiones mathematicae*, 115:483–511, 1994.

[5] A. Douady and J. Hubbard. On the dynamics of polynomial-like mappings. *Annales scientifique de l'École Normale Supérieure*, 18(2):287–343, 1985.

[6] Carsten Lunde Petersen and Pascale Roesch. The parabolic mandelbrot set, 2021.