Complex Dynamics in the Tropics

IMPA

Rio de Janeiro/Brazil

28 November to 2 December 2022

THE DUAL $\mathbb{R}$-TREE FOR INFINITE TOPOLOGICAL SURFACES

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Abstract

The geodesic laminations on surfaces were introduced by W. Thurston [1], which can be considered as topological objects, or as generalizations of simple closed curves on surfaces. In addition, they can be used to study group actions. In [3], Morgan and Shalen guarantee the existence of the dual $\mathbb{R}$-tree for certain measurable laminations in compact manifolds. In this work we guarantee the existence of the dual $\mathbb{R}$-tree for measurable geodesic laminations on infinite hyperbolic Riemman surfaces, with hyperbolic metric and fundamental group of the first type. In addition, we describe the point stabilizers on these actions in terms of the original laminations.

Introduction

Definition 0.1. *[5] A hyperbolic Riemann surface X, is a Riemann surface whose universal covering space $\tilde{X}$ is isomorphic to the unit disk*

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 0.2. *[8] A topological surface X is said to be infinite if the fundamental group $\pi_1(X)$ of X is infinitely generated.*

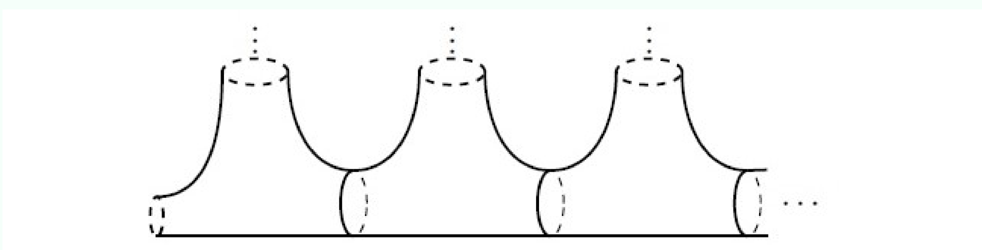

Figure 1: A flute surface [6].

Definition 0.3. *[8] Let X be an infinite hyperbolic Riemann surface with its hyperbolic metric. A geodesic lamination λ on X consists of a closed subset of X together with its foliation by simple, pairwise disjoint complete geodesics of X. By a foliation of a closed subset λ of X by geodesics we mean a decomposition of λ into a pairwise disjoint simple complete geodesics, such that each point $x \in \lambda$ has a neighborhood homeomorphic to $T \times I$ where T is homeomorphic to a closed subset of a compact geodesic arc, and I is an open interval corresponding to open arcs on geodesics.*

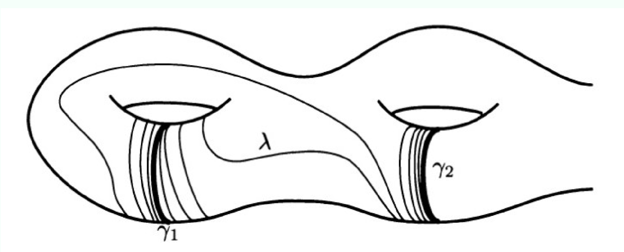

Figure 2: A geodesic lamination with finite leaves [9].

Definition 0.4. *[7] Let $V \subset M$ be a flow box for λ with $V = U \times I$. Let $X \subset I$ denote the local leaf space of λ in V. A transverse measure μ for λ in V is a Borel measure on I which has finite total mass on any compact subset of I, and which is supported on the closed set X. It is of full support if its support is exactly X.*

Definition 0.5. *[9] A measurable geodesic lamination is a pair (λ, μ) formed by a geodesic lamination λ together with an associated transverse measure μ.*

Definition 0.6. *[8] Let X be an infinite hyperbolic Riemann surface and $\Lambda(\pi_1(X))$ be the limit set of $\pi_1(X)$ on the boundary $\partial_\infty \tilde{X}$. The convex core $C(\Lambda(\pi_1(X)))$ of $\Lambda(\pi_1(X))$ is the smallest convex subset of $\tilde{X}$ that has $\Lambda(\pi_1(X))$ as its ideal boundary. The convex core $C(X)$ of X is the smallest convex subset of X that has the same homotopy type, equivalently $C(X) := C(\Lambda(\pi_1(X)))/\pi_1(X)$. Note that $\pi_1(X)$ is of the first kind if and only if $X = C(X)$.*

Theorem 0.1. *[8] Let X be an infinite hyperbolic Riemann surface equipped with its hyperbolic metric. Any geodesic lamination λ contained in the convex core $C(X)$ of X is nowhere dense. In particular, if $X = C(X)$ then any geodesic lamination λ in X is nowhere dense.*

Definition 0.7. *[4] We say that a metric space (X,d) is an $\mathbb{R}$-tree if for any $x, y \in X$ there is a unique arc from x to y and this arc is a geodesic segment.*

Results

We fix:

X: An infinite hyperbolic Riemann surface whose fundamental group $\pi_1(X)$ is of the first type;
(λ, μ): A measurable geodesic lamination with support λ in X;
$(\tilde{\lambda}, \tilde{\mu})$: An induced measurable geodesic lamination with support $\tilde{\lambda}$ in $\tilde{X}$;
$\mathcal{C}$: The set of connected components of $\tilde{X} \setminus \tilde{\lambda}$.

Proposition 0.1. *i) each leaf of $\tilde{\lambda}$ in the universal covering $\tilde{X}$ of X, is a closed subset of $\tilde{X}$; and ii) if p and q are points of $\tilde{X} \setminus \tilde{\lambda}$, then there is a path $w : I \to \tilde{X}$ with $w(0) = p$ and $w(1) = q$, which is transverse to the leaves of $\tilde{\lambda}$ and which crosses each leaf at most once.*

Construction of a metric:

- Let $c_0, c_1 \in \mathcal{C}$, and suppose that w and w' are transverse paths from points $x_0 \in c_0$ to $x_1 \in c_1$, and from $x'_0 \in c_0$ to $x'_1 \in c_1$ respectively with w crossing each leaf of $\tilde{\lambda}$ at most once.
- Define the function $\Omega : I \times I \to \tilde{X}$ with $\Omega|(I \times \{0\}) = w$, $\Omega|(I \times \{1\}) = w'$, $\Omega(\{0\} \times I) \subset c_0$ y $\Omega(\{1\} \times I) \subset c_1$. By the theorem 0.1, $\tilde{\lambda}$ is nowhere dense.
- Suppose that Ω is transverse to $(\tilde{\lambda}, \tilde{\mu})$. Then, Ω pulls $\tilde{\lambda}$ back to a measured lamination $\Omega^*(\tilde{\lambda}, \tilde{\mu})$ on $I \times I$.
- By i), all leaves of $\Omega^* \tilde{\lambda}$ are compact. Then, each leaf of $\Omega^* \tilde{\lambda}$ is either a circle or an arc with endpoints in $I \times \{0\}$, since w crosses each leaf of $\tilde{\lambda}$ at most once. It follows, that all the leaves of $\Omega^* \tilde{\lambda}$ which meet $\partial(I \times I)$ are either arcs running from $I \times \{0\}$ to $I \times \{1\}$ or arcs with both endpoints in $I \times \{1\}$.
- (We define a metric on $\mathcal{C}$). Let $w : I \to \tilde{X}$ be a path transverse to the leaves of $\tilde{\lambda}$, with $w(0) = x_0$ and $w(1) = x_1$. Suppose furthermore that w crosses each leaf of $\tilde{\lambda}$ at most once, and consider the function $d : \mathcal{C} \times \mathcal{C} \to [0, \infty)$ defined by $d(c_0, c_1) = \int_I w^*(\tilde{\mu})$.

Proposition 0.2. *There exist an $\mathbb{R}$-tree T and an isometric embedding $\Psi : \mathcal{C} \to T$ such that*
i) $\Psi(\mathcal{C})$ spans T (i.e., the smallest subtree of T containing $\Psi(\mathcal{C})$);
ii) Any point $t \in T - \Psi(\mathcal{C})$ is an edge point in the sense that t separates T into two components;
iii) The action of $\pi_1(X)$ on $\mathcal{C}$ extends uniquely to an action of $\pi_1(X)$ by isometries on T. Between any two such $\mathbb{R}$-trees there is an equivariant isometry commuting with the embeddings of $\mathcal{C}$.

Proof. It is guaranteed by condition. $(*)$.

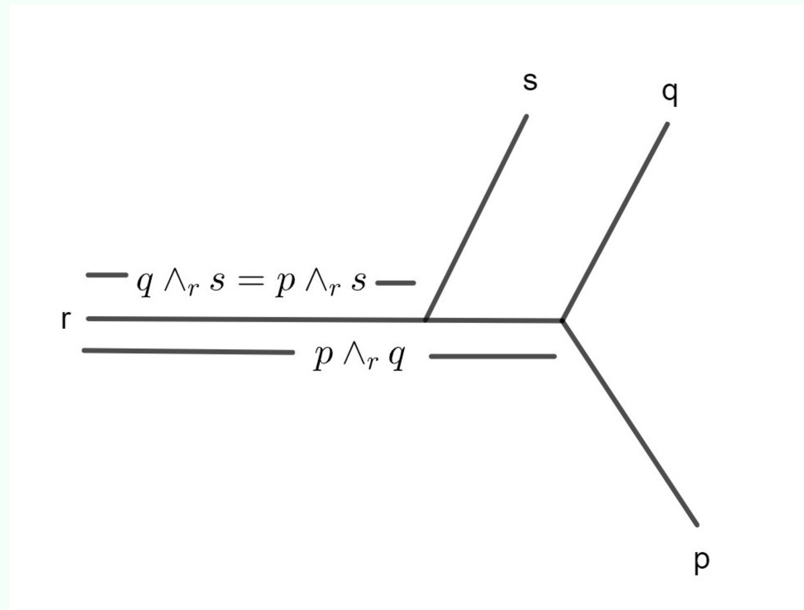

Figure 3: $(*)$ $(p \wedge_r q) > (p \wedge_r s) \Rightarrow p \wedge_r s = q \wedge_r s$.

□

If (λ, μ) satisfies the conditions i), ii), and iii) of the proposition 0.2, then the $\mathbb{R}$-tree given by the proposition 0.2, will be called the dual tree. Implicitly, the dual tree is equipped with the action given in part iii).

Theorem 0.2. *Let (λ, μ) satisfy conditions i), ii), and iii) of the proposition 0.2, and T denote the dual tree. For any $c \in C$ the stabilizer in $\pi_1(X)$ of $\Psi(c) \in T$ is equal to the stabilizer of $c \subset \tilde{X}$. For any $t \in T - \Psi(\mathcal{C})$ there is a leaf of $\tilde{\lambda}$ whose stabilizer is equal to that of t.*

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Acknowledgments

CAPES for the doctorate scholarship 88887.633197/2021-00.